On the orders of finite semisimple groups

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Abstract.   The aim of this paper is to investigate the order coincidences among the finite semisimple groups and to give a reasoning of such order coincidences through the transitive actions of compact Lie groups.

It is a theorem of Artin and Tits that a finite simple group is determined by its order, with the exception of the groups $(A_3(2), A_2(4))$ and $(B_n(q), C_n(q))$ for $n \geq 3$, q odd. We investigate the situation for finite semisimple groups of Lie type. It turns out that the order of the finite group $H(\mathbb{F}_q)$ for a split semisimple algebraic group H defined over $\mathbb{F}_q$, does not determine the group H up to isomorphism, but it determines the field $\mathbb{F}_q$ under some mild conditions. We then put a group structure on the pairs (H_1, H_2) of split semisimple groups defined over a fixed field $\mathbb{F}_q$ such that the orders of the finite groups $H_1(\mathbb{F}_q)$ and $H_2(\mathbb{F}_q)$ are the same and the groups H_i have no common simple direct factors. We obtain an explicit set of generators for this abelian, torsion-free group. We finally show that the order coincidences for some of these generators can be understood by the inclusions of transitive actions of compact Lie groups.

Keywords.   Finite semisimple groups; transitive actions of compact Lie groups; Artin's theorem.

1. Introduction

It is a theorem of Artin and Tits that two finite simple groups of the same order are isomorphic except for the pairs

$$(\mathrm{PSL}_4(\mathbb{F}_2), \mathrm{PSL}_3(\mathbb{F}_4)) \quad \text{and} \quad (\mathrm{PSO}_{2n+1}(\mathbb{F}_q), \mathrm{PSp}_{2n}(\mathbb{F}_q)) \quad \text{for } n \geq 3, q \text{ odd}.$$

This theorem was first proved by Emil Artin in 1955 for the finite simple groups that were known then [1, 2]. As new finite simple groups were discovered, Tits [9–14] verified that the above pairs are the only pairs of non-isomorphic finite simple groups of the same order. One may also look in [6] for an exposition of these proofs.

We investigate in this paper the situation for the groups of $\mathbb{F}_q$-rational points of a split semisimple algebraic group H defined over a finite field $\mathbb{F}_q$. Since the orders of the groups $H(\mathbb{F}_q)$ and $H'(\mathbb{F}_q)$ are the same if H and H' are isogenous and since the simply connected group is unique in an isogeny class, we concentrate only on simply connected groups. Since the groups $B_n(\mathbb{F}_q)$ and $C_n(\mathbb{F}_q)$ have the same order for $n \geq 3$ and for all q, we do not distinguish between them.

This paper is arranged as follows. We state some preliminary lemmas in §2 which are used in the proofs of the main theorems. Section 3 is devoted to determining the field $\mathbb{F}_q$

from the order of the group $H(\mathbb{F}_q)$. The first natural step in that direction is to obtain its characteristic. We prove that under certain mild conditions, the order of the group $H(\mathbb{F}_q)$ determines the characteristic of $\mathbb{F}_q$ (Theorem 3.2). We further prove that it eventually determines the base field (Theorem 3.3).

Theorem 3.2. *Let H_1 and H_2 be two split semisimple simply connected algebraic groups defined over finite fields $\mathbb{F}_{q_1}$ and $\mathbb{F}_{q_2}$ respectively. Let X denote the set $\{8, 9, 2^r, p\}$, where $2^r + 1$ is a prime and p is a prime of the form $2^s \pm 1$. Suppose that, for $i = 1, 2$, A_1 is not a direct factor of H_i whenever $q_i \in X$ and B_2 is not a direct factor of H_i whenever $q_i = 3$. Then, if $|H_1(\mathbb{F}_{q_1})| = |H_2(\mathbb{F}_{q_2})|$, the characteristics of $\mathbb{F}_{q_1}$ and $\mathbb{F}_{q_2}$ are equal.*

Since $|A_1(\mathbb{F}_9)| = |B_2(\mathbb{F}_2)|$, the above theorem is not true in general. We feel that this is the only counterexample, i.e., the conclusion of Theorem 3.2 is true without the hypothesis imposed there except that we must exclude the case of $H_1 = A_1$ over $\mathbb{F}_9$ and $H_2 = B_2$ over $\mathbb{F}_2$, but we have not been able to prove it.

Theorem 3.3. *Let H_1 and H_2 be two split semisimple simply connected algebraic groups defined over finite fields $\mathbb{F}_{q_1}$ and $\mathbb{F}_{q_2}$ of the same characteristic. Suppose that the order of the finite groups $H_1(\mathbb{F}_{q_1})$ and $H_2(\mathbb{F}_{q_2})$ are the same, then $q_1 = q_2$. Moreover the fundamental degrees (and their multiplicities) of the Weyl groups $W(H_1)$ and $W(H_2)$ are the same.*

Theorem 3.4. *Let H_1 and H_2 be two split semisimple simply connected algebraic groups defined over a finite field $\mathbb{F}_q$. If the finite groups $H_1(\mathbb{F}_q)$ and $H_2(\mathbb{F}_q)$ have the same order then the orders of the groups $H_1(\mathbb{F}_{q'})$ and $H_2(\mathbb{F}_{q'})$ are the same for any finite extension $\mathbb{F}_{q'}$ of $\mathbb{F}_q$.*

Thus, the question now boils down to classifying the split semisimple simply connected groups H_1, H_2 defined over a fixed field $\mathbb{F}_q$ such that the orders of the finite groups $H_1(\mathbb{F}_q)$ and $H_2(\mathbb{F}_q)$ are the same. We first characterise such pairs where each of the groups H_1 and H_2 can be written as a direct product of exactly two simple groups. We find that all such pairs can be generated by a 'nice' set of pairs, which admit a geometric reason for the order coincidence. We make further observations regarding the pairs of order coincidence at the end of §4.

These observations lead us to a natural question which we answer in the affirmative in §5. This question is about describing all the pairs (H_1, H_2) of groups defined over a fixed finite field $\mathbb{F}_q$, where the groups H_i have no common simple factor and the orders of the groups $H_1(\mathbb{F}_q)$ and $H_2(\mathbb{F}_q)$ are the same. The set of such pairs admits a structure of an abelian, torsion-free group. We determine an explicit set of generators for this group (Theorem 5.2).

Section 6 deals with a geometric reasoning for the 'nice' set of pairs of order coincidence given by transitive action of compact Lie groups. If H is a compact Lie group and H_1, H_2 are connected subgroups of H such that the natural action of H_2 on H/H_1 is transitive, then it can be seen that the split forms of $H \times (H_1 \cap H_2)$ and $H_1 \times H_2$ have the same number of $\mathbb{F}_q$-rational points for any finite field $\mathbb{F}_q$. We give such geometric reasoning for the first three pairs described in Theorem 5.2. It would be interesting to know if other pairs also admit such geometric reasoning.

2. Preliminary lemmas

We state some preliminary lemmas in this section. The first three lemmas are proved by Artin in his papers [1, 2].

Let $\Phi_n(x)$ be the nth cyclotomic polynomial and

$$\Phi_n(x, y) = y^{\varphi(n)} \Phi_n(x/y)$$

be the corresponding homogeneous form. Let a, b be integers which are relatively prime and which satisfy the inequalities

$$|a| \geq |b| + 1 \geq 2.$$

Fix a prime p which divides $a^n - b^n$ for some n. Then it is clear that p does not divide any of a and b. Let f be the order of ab^{-1} modulo p. For a natural number m, we put $\mathrm{ord}_p m = \alpha$, where p^α is the largest power of p dividing m. We call $p^{\mathrm{ord}_p m}$ as the p-contribution to m.

Lemma 2.1 (*Lemma 1 of* [1]). *With the above notations, we have the following rules*:

(1) *If p is odd*,

$$\mathrm{ord}_p \Phi_f(a, b) > 0; \quad \mathrm{ord}_p \Phi_{fp^i}(a, b) = 1 \text{ for } i \geq 1$$

and in all other cases $\mathrm{ord}_p \Phi_n(a, b) = 0$.
Therefore, we have

$$\mathrm{ord}_p(a^n - b^n) = 0, \quad if \quad f \nmid n$$
$$\mathrm{ord}_p(a^n - b^n) = \mathrm{ord}_p(a^f - b^f) + \mathrm{ord}_p n, \quad otherwise.$$

(2) *If $p = 2$, then $f = 1$*.

   (a) *If $\Phi_1(a, b) = a - b \equiv 0 \pmod 4$, then $\mathrm{ord}_2 \Phi_{2^i}(a, b) = 1$ for $i \geq 1$.*
   (b) *If $\Phi_2(a, b) = a + b \equiv 0 \pmod 4$, then $\mathrm{ord}_2 \Phi_{2^i}(a, b) = 1$ for $i = 0, 2, 3, \ldots$.*
      In all other cases $\mathrm{ord}_2 \Phi_n(a, b) = 0$.

Lemma 2.2 (§4, *eqs* (1)–(3) *of* [2]). *Let $\alpha = (a - 1)(a^2 - 1) \cdots (a^l - 1)$ for some integer $a \neq 0$ and let p_1 be a prime dividing α. Let P_1 be the p_1-contribution to α, i.e., P_1 be the highest power of p_1 dividing α and let q be a prime power. We have*:

(1) *If $a = \pm q$, then $P_1 \leq 2^l (q + 1)^l$.*
(2) *If $a = q^2$, then $P_1 \leq 4^l (q + 1)^l$.*

Lemma 2.3 (*Corollary to Lemma 2 of* [1]). *If $a > 1$ is an integer and $n > 2$ then there is a prime p which divides $\Phi_n(a)$ but no $\Phi_i(a)$ with $i < n$ unless $n = 6$ and $a = 2$.*

Lemma 2.4. *If the inequality $q^n \geq \alpha(q + 1)$, where α is a fixed positive real number, holds for a pair of positive integers (q_1, n_1), then it holds for all (q_2, n_2) satisfying $q_2 \geq q_1$ and $n_2 \geq n_1$.*

Proof. This is clear. □

Lemma 2.5. *Let H be a semisimple algebraic group defined over a finite field $\mathbb{F}_q$. If $\widetilde{H}$ denotes a connected cover of H, then $|\widetilde{H}(\mathbb{F}_q)| = |H(\mathbb{F}_q)|$.*

Proof. Let $\widetilde{H}$ be a connected cover of H. We have an exact sequence $0 \to A \to \widetilde{H} \to H \to 1$ where A is a finite abelian group. From this sequence, we get the following exact sequence of Galois cohomology sets

$$0 \to H^0(\mathbb{F}_q, A) \to \widetilde{H}(\mathbb{F}_q) \to H(\mathbb{F}_q) \to H^1(\mathbb{F}_q, A) \to H^1(\mathbb{F}_q, \widetilde{H}).$$

By Lang's Theorem (Corollary to theorem 1 of [7]), $H^1(\mathbb{F}_q, \widetilde{H}) = 0$. Since all the sets in the above sequence are finite, we have

$$|H^0(\mathbb{F}_q, A)| \cdot |H(\mathbb{F}_q)| = |\widetilde{H}(\mathbb{F}_q)| \cdot |H^1(\mathbb{F}_q, A)|.$$

Since the Galois group, $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, is procyclic and A is a finite Galois-module, its Herbrand quotient is 1, i.e., $|H^0(\mathbb{F}_q, A)| = |H^1(\mathbb{F}_q, A)|$. It follows that $|\widetilde{H}(\mathbb{F}_q)| = |H(\mathbb{F}_q)|$.
□

3. Determining the finite field

The first natural step in determining the field $\mathbb{F}_q$ is to determine its characteristic. Observe that if we have two semisimple groups H_1 and H_2 defined over finite fields $\mathbb{F}_{p_1^{r_1}}$ and $\mathbb{F}_{p_2^{r_2}}$ respectively, such that $|H_1(\mathbb{F}_{p_1^{r_1}})| = |H_2(\mathbb{F}_{p_2^{r_2}})|$ and $p_1 \neq p_2$, then either p_1 fails to give the largest contribution to the order of $H_1(\mathbb{F}_{p_1^{r_1}})$ or p_2 fails to give the largest contribution to the order of $H_2(\mathbb{F}_{p_2^{r_2}})$. Therefore we would like to obtain a description of the split semisimple algebraic groups H defined over $\mathbb{F}_{p^r}$ such that the p-contribution to the order of the group $H(\mathbb{F}_{p^r})$ is not the largest. These groups are the only possible obstructions for determining the characteristic of the base field. Since we limit ourselves to the case of simply connected groups only, every semisimple group considered in this paper is a direct product of (simply connected) simple algebraic groups. Hence we need to describe simple algebraic groups H defined over $\mathbb{F}_{p^r}$ with the property that p does not contribute the largest to the order of $H(\mathbb{F}_{p^r})$.

We remark that the main tool in the proof of the following proposition is Lemma 2.1 which is proved by Artin in [1]. Our proof of the following proposition is very much on the lines of Artin's proof of Theorem 1 in [2]. However, our result is for $H(\mathbb{F}_q)$, the groups of $\mathbb{F}_q$-rational points of a simple algebraic group H defined over $\mathbb{F}_q$ whereas Artin proved the result for finite groups that are simple. The groups $H(\mathbb{F}_q)$ that we consider here, are not always simple, because of the presence of (finite) center. Moreover, we get the counterexamples $A_1(\mathbb{F}_9)$, $A_1(\mathbb{F}_p)$ for a Fermat prime p, and $B_2(\mathbb{F}_3)$ which do not figure in Artin's list of counterexamples described in Theorem 1 of [2].

PROPOSITION 3.1

Let H be a split simple algebraic group defined over a finite field $\mathbb{F}_q$ of characteristic p. If the p-contribution to the order of the finite group $H(\mathbb{F}_q)$ is not the largest prime power dividing the order, then the group $H(\mathbb{F}_q)$ is:

(1) $A_1(\mathbb{F}_q)$ *for $q \in \{8, 9, 2^r, p\}$ where $2^r + 1$ is a Fermat prime and p is a prime of the type $2^s \pm 1$ or*
(2) $B_2(\mathbb{F}_3)$.

Moreover in all these cases, the p-contribution is the second largest prime power dividing the order of the group $H(\mathbb{F}_q)$.

We call the groups, $A_1(\mathbb{F}_q)$ and $B_2(\mathbb{F}_3)$, described above, as *counterexamples* in the remaining part of this paper.

Proof. We first recall the orders of the finite groups $H(\mathbb{F}_q)$ where H is a split simple algebraic group defined over a finite field $\mathbb{F}_q$ (see §2.9 of [3]).

$$|A_n(\mathbb{F}_q)| = q^{n(n+1)/2}(q^2 - 1)(q^3 - 1)\cdots(q^{n+1} - 1), \quad n \geq 1,$$

$$|B_n(\mathbb{F}_q)| = q^{n^2}(q^2 - 1)(q^4 - 1)\cdots(q^{2n} - 1), \quad n \geq 2,$$

$$|D_n(\mathbb{F}_q)| = q^{n(n-1)}(q^2 - 1)(q^4 - 1)\cdots(q^{2n-2} - 1)(q^n - 1), \quad n \geq 4,$$

$$|G_2(\mathbb{F}_q)| = q^6(q^2 - 1)(q^6 - 1),$$

$$|F_4(\mathbb{F}_q)| = q^{24}(q^2 - 1)(q^6 - 1)(q^8 - 1)(q^{12} - 1),$$

$$|E_6(\mathbb{F}_q)| = q^{36}(q^2 - 1)(q^5 - 1)(q^6 - 1)(q^8 - 1)(q^9 - 1)(q^{12} - 1),$$

$$|E_7(\mathbb{F}_q)| = q^{63}(q^2 - 1)(q^6 - 1)(q^8 - 1)(q^{10} - 1)(q^{12} - 1)(q^{14} - 1)$$
$$\times (q^{18} - 1),$$

$$|E_8(\mathbb{F}_q)| = q^{120}(q^2 - 1)(q^8 - 1)(q^{12} - 1)(q^{14} - 1)(q^{18} - 1)(q^{20} - 1)$$
$$\times (q^{24} - 1)(q^{30} - 1).$$

Now, let H be one of the finite simple groups listed above and let p_1 be a prime dividing the order of the finite group $H(\mathbb{F}_q)$ such that $p_1 \nmid q$. We use Lemma 2.2 to estimate P_1, the p_1-contribution to the order of $H(\mathbb{F}_q)$. Depending on the type of $H(\mathbb{F}_q)$, we put the following values of a and l in Lemma 2.2.

H	A_n	B_n, D_n	G_2	F_4	E_6	E_7	E_8
a	q	q^2	q^2	q^2	q	q^2	q^2
l	$n+1$	n	3	6	12	9	15

Now, suppose that the p-contribution to the order of the group $H(\mathbb{F}_q)$ is not the largest, i.e., the power of q that appears in the formula for $|H(\mathbb{F}_q)|$ is smaller than P_1 for some prime $p_1 \nmid q$. Then, depending on the type of the group, we get following inequalities from Lemma 2.2:

$$A_n : P_1 \leq 2^{n+1}(q+1)^{n+1} \implies q^{n/2} < 2(q+1),$$

$$B_n : P_1 \leq 4^n(q+1)^n \implies q^n < 4(q+1),$$

$$D_n : P_1 \leq 4^n(q+1)^n \implies q^{n-1} < 4(q+1),$$

$$G_2 : P_1 \leq 4^3(q+1)^3 \implies q^2 < 4(q+1),$$

$$F_4 : P_1 \leq 4^6(q+1)^6 \implies q^4 < 4(q+1),$$

$$E_6 : P_1 \leq 2^{12}(q+1)^{12} \implies q^3 < 2(q+1),$$

$$E_7 : P_1 \leq 4^9(q+1)^9 \implies q^7 < 4(q+1),$$

$$E_8 : P_1 \leq 4^{15}(q+1)^{15} \implies q^8 < 4(q+1).$$

In all the cases where the above inequalities of the type $q^m < \alpha(q+1)$ do not hold, we get that p contributes the largest to the order of $H(\mathbb{F}_q)$. Observe that the last four inequalities,

i.e., the inequalities corresponding to the groups F_4, E_6, E_7 and E_8 do not hold for $q = 2$ and hence by Lemma 2.4 they do not hold for any $q \geq 2$. Thus, for $H = F_4, E_6, E_7$ and E_8, the p-contribution to $|H(\mathbb{F}_q)|$ is always the largest prime power dividing $|H(\mathbb{F}_q)|$.

Similarly we obtain the following table of the pairs of positive integers (q, n) where the remaining inequalities fail. Then using Lemma 2.4, we know that for all (q', n') with $q' \geq q$ and $n' \geq n$, the contribution of the characteristic to the order of the finite group $H(\mathbb{F}_q)$ is the largest. Therefore, we are left with the cases for (q', n') such that $q' < q$ or $n' < n$, which are to be checked. The adjoining table shows the groups $H(\mathbb{F}_q)$ which are to be checked.

A_n	$q = 2$,	$n \geq 6$	$A_3(\mathbb{F}_2), A_3(\mathbb{F}_3), A_3(\mathbb{F}_4), A_3(\mathbb{F}_5),$
	$q = 3, 4, 5$,	$n \geq 4$	$A_4(\mathbb{F}_2), A_5(\mathbb{F}_2),$
	$q \geq 7$,	$n \geq 3$	$A_1(\mathbb{F}_q), A_2(\mathbb{F}_q) \; \forall q,$
B_n	$q = 2$,	$n \geq 4$	$B_2(\mathbb{F}_2), B_2(\mathbb{F}_3), B_2(\mathbb{F}_4),$
	$q = 3, 4$,	$n \geq 3$	$B_3(\mathbb{F}_2),$
	$q \geq 5$,	$n \geq 2$	
D_n	$q = 2$,	$n \geq 5$	$D_4(\mathbb{F}_2),$
	$q \geq 3$,	$n \geq 4$	
G_2	$q \geq 5$		$G_2(\mathbb{F}_2), G_2(\mathbb{F}_3), G_2(\mathbb{F}_4).$

In all the cases other than $A_1(\mathbb{F}_q)$ and $A_2(\mathbb{F}_q)$, we can do straightforward calculations and check that p contributes the largest to the order of every group except for $B_2(\mathbb{F}_3)$. In the case of $B_2(\mathbb{F}_3)$, the prime 3 indeed fails to give the largest contribution, however it gives the second largest contribution to the order of the group. The cases of A_1 and A_2 over a general finite field $\mathbb{F}_q$ are done in a different way.

We first deal with the case of the group $A_2(\mathbb{F}_q)$. Recall that

$$|A_2(\mathbb{F}_q)| = q^3(q^2 - 1)(q^3 - 1) = q^3(q^2 + q + 1)(q + 1)(q - 1)^2.$$

Let $p_1 \nmid q$ be a prime dividing the order of $A_2(\mathbb{F}_q)$ and let P_1 be the contribution of p_1 to $|A_2(\mathbb{F}_q)|$. Let f denote the order of q modulo p_1. If $f \neq 1$ then it is clear that the p_1-contribution to the order of $A_2(\mathbb{F}_q)$ is not more than q^3. If $f = 1$ and $p_1 \neq 2$ or 3, then by Lemma 2.1, P_1 divides $(q - 1)^2$ which is less than q^3. If $f = 1$ and $p_1 = 2$ or 3, then P_1 divides either $3(q - 1)^2, 2(q - 1)^2$ or $4(q + 1)$. Thus, if q^3 is not the largest prime power dividing $|A_2(\mathbb{F}_q)|$, then $q^3 < P_1$ for some prime $p_1 \neq p$ and hence we have

$$q^3 < 3(q - 1)^2, \quad 2(q - 1)^2 \quad \text{or} \quad 4(q + 1).$$

Again as above, we observe that none of the above inequalities are satisfied by $q \geq 3$, and then we check that the 2-contribution to the order of $A_2(\mathbb{F}_2)$ is the largest one.

Now, for the group $A_1(\mathbb{F}_q)$, we observe that for any prime $p_1 \nmid q$, the p_1-contribution to the order of $A_1(\mathbb{F}_q)$ divides $q^2 - 1 = (q + 1)(q - 1)$. We make two cases here depending on q being odd or even.

If q is odd, both $q + 1$ and $q - 1$ are even. The 2-contribution to one of the numbers $q + 1$ and $q - 1$ is 2, and the other number then must be a power of 2 if q is not the largest prime power dividing $|A_1(\mathbb{F}_q)|$. If $q + 1$ is a power of 2, then q is necessarily a prime of the form $q = p = 2^s - 1$, a Mersenne prime. However, if $q - 1$ is a power of 2 then the only possibilities for q are that q is a Fermat prime, $q = p = 2^s + 1$, or $q = 9$.

If q is even, $q - 1$ and $q + 1$ are both odd and hence they do not have any common prime factor. If q is not the largest prime power dividing $|A_1(\mathbb{F}_q)|$, then the largest prime

power dividing $|A_1(\mathbb{F}_q)|$ must be $q + 1$. Let $q = 2^r$ and $P_1 = p_1^s = 2^r + 1$. Here p_1 is odd, and hence by Lemma 2.3, if $s > 2$, there is a prime divisor of $p_1^s - 1$ which does not divide $p_1 - 1$, a contradiction. If $s = 2$, $2^r = p_1^2 - 1$. Then both $p_1 \pm 1$ are powers of two and hence we obtain that $p_1 = 3$ and $q = 2^3 = 8$. If $s = 1$, $p_1 = 2^r + 1$, a Fermat prime. $\qquad\square$

Thus, if $H(\mathbb{F}_q)$ is not one of the counterexamples described in the above proposition, then the characteristic of $\mathbb{F}_q$ contributes the largest to the order of $H(\mathbb{F}_q)$. Since every (simply connected) semisimple algebraic group is a direct product of (simply connected) simple algebraic groups, we get that whenever a finite semisimple group $H(\mathbb{F}_q)$ does not have any of the above counterexamples as direct factors, then the characteristic of $\mathbb{F}_q$ contributes the largest to the order of $H(\mathbb{F}_q)$.

Theorem 3.2. *Let H_1 and H_2 be two split semisimple simply connected algebraic groups defined over finite fields $\mathbb{F}_{q_1}$ and $\mathbb{F}_{q_2}$ respectively. Let X denote the set $\{8, 9, 2^r, p\}$ where $2^r + 1$ is a Fermat prime and p is a prime of the type $2^s \pm 1$. Suppose that for $i = 1, 2$, A_1 is not one of the direct factors of H_i whenever $q_i \in X$ and B_2 is not a direct factor of H_i whenever $q_i = 3$. Then, if $|H_1(\mathbb{F}_{q_1})| = |H_2(\mathbb{F}_{q_2})|$, the characteristics of $\mathbb{F}_{q_1}$ and $\mathbb{F}_{q_2}$ are equal.*

Proof. This is clear. $\qquad\square$

Now, we come to the main theorem of this section. Recall that if H is a split semisimple algebraic group of rank n defined over a finite field $\mathbb{F}_q$, then the order of $H(\mathbb{F}_q)$ is given by the formula,

$$|H(\mathbb{F}_q)| = q^N (q^{d_1} - 1)(q^{d_2} - 1) \cdots (q^{d_n} - 1),$$

where $d_1, d_2, \ldots, d_n$ are the degrees of the basic invariants of $W(H)$, the Weyl group of H and $N = \sum_i (d_i - 1)$ (§2.9 of [3]). Now onwards, we call d_i as the *degrees* of $W(H)$, the Weyl group of H. Observe that for every split simple algebraic group H, the integer 2 always occurs as a degree of $W(H)$ with multiplicity one (§3.7 of [5]). Therefore the multiplicity of the integer 2 among the degrees of $W(H)$ determines the number of simple direct factors of the group H. We remark here that the degrees d_i of $W(H)$ may appear with multiplicities.

Theorem 3.3. *Let H_1 and H_2 be two split semisimple simply connected algebraic groups defined over finite fields $\mathbb{F}_{q_1}$ and $\mathbb{F}_{q_2}$ of the same characteristic. Suppose that the order of the finite groups $H_1(\mathbb{F}_{q_1})$ and $H_2(\mathbb{F}_{q_2})$ are the same, then $q_1 = q_2$. Moreover the fundamental degrees (and the multiplicities) of the Weyl groups $W(H_1)$ and $W(H_2)$ are the same.*

Proof. Let p be the characteristic of the fields $\mathbb{F}_{q_1}$ and $\mathbb{F}_{q_2}$, and let $q_1 = p^{t_1}, q_2 = p^{t_2}$. Let the orders of the finite groups $H_1(\mathbb{F}_{q_1})$ and $H_2(\mathbb{F}_{q_2})$ be given by

$$
\begin{aligned}
|H_1(\mathbb{F}_{q_1})| &= (q_1)^r \, (q_1^{r_1} - 1)(q_1^{r_2} - 1) \cdots (q_1^{r_n} - 1) \\
&= (p^{t_1})^r ((p^{t_1})^{r_1} - 1)((p^{t_1})^{r_2} - 1) \cdots ((p^{t_1})^{r_n} - 1) \\
|H_2(\mathbb{F}_{q_2})| &= (q_2)^s (q_2^{s_1} - 1) \, (q_2^{s_2} - 1) \cdots (q_2^{s_m} - 1) \\
&= (p^{t_2})^s ((p^{t_2})^{s_1} - 1) \, ((p^{t_2})^{s_2} - 1) \cdots ((p^{t_2})^{s_m} - 1)
\end{aligned}
$$

As remarked above, the integers r_i and s_j are the respective degrees of the Weyl groups $W(H_1)$ and $W(H_2)$. Moreover the rank of the group H_1 is n and that of H_2 is m. Further, we have

$$r = \sum_{i=1}^{n} (r_i - 1) \quad \text{and} \quad s = \sum_{j=1}^{m} (s_j - 1).$$

Since $|H_1(\mathbb{F}_{q_1})| = |H_2(\mathbb{F}_{q_2})|$, we have that

$$t_1 r = t_2 s$$

and

$$\prod_{i=1}^{n} ((p^{t_1})^{r_i} - 1) = \prod_{j=1}^{m} ((p^{t_2})^{s_j} - 1). \tag{3.1}$$

Assume that $r_1 \leq r_2 \leq \cdots \leq r_n$ and $s_1 \leq s_2 \leq \cdots \leq s_m$. We treat both the products in eq. (3.1) as polynomials in p and factor them into the cyclotomic polynomials in p.

Let us assume for the time being that $p \neq 2$, so that we can apply Lemma 2.3 to conclude that the cyclotomic polynomials appearing on both sides of eq. (3.1) are the same with the same multiplicities. Observe that on the left-hand side (LHS) the highest order cyclotomic polynomial is $\Phi_{t_1 r_n}(p)$ whereas such a polynomial on the right-hand side (RHS) is $\Phi_{t_2 s_m}(p)$. Since the cyclotomic polynomials appearing on both sides are the same, we have that $t_1 r_n = t_2 s_m$. Thus, the polynomial $p^{t_1 r_n} - 1$, which is same as the polynomial $p^{t_2 s_m} - 1$, can be cancelled from both sides of eq. (3.1). Continuing in this way we get that $t_1 r_{n-k} = t_2 s_{m-k}$ for all k. This implies in particular that $m = n$. Further,

$$t_1 r = t_2 s \implies \sum t_1 r_i - t_1 n = \sum t_2 s_j - t_2 n.$$

But, by the above observation, this gives us that $t_1 n = t_2 n$ and hence $t_1 = t_2$, i.e., $q_1 = q_2$. Thus, the fields $\mathbb{F}_{q_1}$ and $\mathbb{F}_{q_2}$ are isomorphic.

Now, it also follows that $r_i = s_i$ for all i, i.e., the degrees of the corresponding Weyl groups are the same.

Now, let $p = 2$. So, we have the equation

$$(2^{t_1})^r \prod_{i=1}^{n} ((2^{t_1})^{r_i} - 1) = (2^{t_2})^s \prod_{j=1}^{m} ((2^{t_2})^{s_j} - 1). \tag{3.2}$$

The only possible obstruction to the desired result in this case comes from $\Phi_6(2) = 3$ and $\Phi_2(2) = 3$. Moreover, if $\Phi_6(2)$ divides the LHS of the equation but not the RHS, then it is clear that $(2^3 - 1)(2^2 - 1)^2$ divides the RHS with the same power as that of $2^6 - 1$ in the LHS. Other than these polynomials, all the factors of type $2^l - 1$ occur on both sides with the same multiplicities.

Since $s_j > 1$ for all j, $t_2 = 1$, i.e., $q_2 = 2$ and the possible values for q_1 are 2, 2^2 and 2^3, since t_1 divides 6. We prove the result in only one case, $q_1 = 2$, as other cases can be handled by a similar reasoning. If $q_1 = 2$, eq. (3.2) becomes

$$2^r \prod_{i=1}^{n} (2^{r_i} - 1) = 2^s \prod_{j=1}^{m} (2^{s_j} - 1).$$

But then $r = s$ and hence $\sum_i (r_i - 1) = \sum_j (s_j - 1)$. Now, the term $(2^6 - 1)$ contributes 5 to r whereas the term $(2^3 - 1)(2^2 - 1)^2$ contributes only 4 to s. As other factors are same on both the sides, this is a contradiction. Hence the factor $(2^6 - 1)$ in LHS of eq. (3.2) must be adjusted by the same factor in the RHS. Then we get that $m = n$ and $r_i = s_i$ for all i. $\square$

Theorem 3.4. *Let H_1 and H_2 be two split semisimple simply connected algebraic groups defined over a finite field $\mathbb{F}_q$. If the orders of the finite groups $H_1(\mathbb{F}_q)$ and $H_2(\mathbb{F}_q)$ are same then the orders of $H_1(\mathbb{F}_{q'})$ and $H_2(\mathbb{F}_{q'})$ are the same for any finite extension $\mathbb{F}_{q'}$ of $\mathbb{F}_q$.*

Proof. Let H_1, H_2 be split semisimple algebraic groups defined over $\mathbb{F}_q$. By Theorem 3.3, we have that if $|H_1(\mathbb{F}_q)| = |H_2(\mathbb{F}_q)|$ then the degrees of the Weyl groups $W(H_1)$ and $W(H_2)$ are the same with the same multiplicities. Then the formulae for the orders of the groups $H_1(\mathbb{F}_q)$ and $H_2(\mathbb{F}_q)$ are the same as polynomials in q. Hence the orders of the groups $H_1(\mathbb{F}_{q'})$ and $H_2(\mathbb{F}_{q'})$ are the same for any finite extension $\mathbb{F}_{q'}$ of $\mathbb{F}_q$. $\square$

4. Order coincidences

We fix a finite field $\mathbb{F}_q$ and all algebraic groups considered in this section are assumed to be defined over $\mathbb{F}_q$.

In this section, we concentrate on the pairs of split semisimple groups (H_1, H_2), such that the orders of the groups $H_1(\mathbb{F}_q)$ and $H_2(\mathbb{F}_q)$ are the same. We want to characterise all possible pairs of order coincidence (H_1, H_2) and to understand the reason behind the coincidence of these orders. This section and the next one are devoted towards characterising all pairs of order coincidence and we discuss a geometric reasoning of these order coincidences in the last section.

We know by Theorem 3.3, that for such a pair (H_1, H_2), the degrees of the corresponding Weyl groups, $W(H_1)$ and $W(H_2)$, must be the same with the same multiplicities. For a Weyl group W, we denote the collection of degrees of W by $d(W)$. We now make the following easy observations which follow from the basic theory of the Weyl groups [5].

Remark 4.1. Let H_1 and H_2 be two split semisimple algebraic groups over a finite field $\mathbb{F}_q$ such that the groups $H_1(\mathbb{F}_q)$ and $H_2(\mathbb{F}_q)$ have the same order. Then we have:

(1) The rank of the group H_1 is the same as the rank of H_2.
(2) The number of direct simple factors of the groups H_1 and H_2 is the same.
(3) If one of the groups, say H_1, is simple, then so is H_2 and in that case H_1 is isomorphic to H_2.

(We remind the reader once again that we do not distinguish between the groups of type B_n and C_n.)

The next natural step would be to look at the order coincidences in the case of groups each having two simple factors. We characterise such pairs in the following theorem.

Theorem 4.2. *Let H_1 and H_2 be split semisimple simply connected algebraic groups each being a direct product of exactly two simple algebraic groups. Assume that H_1 and H_2 do not have any common simple direct factor. Then the pairs (H_1, H_2) such that $|H_1(\mathbb{F}_q)| = |H_2(\mathbb{F}_q)|$ are exhausted by the following list:*

(1) $(A_{2n-2}B_n, A_{2n-1}B_{n-1})$ *for $n \geq 2$, with the convention that $B_1 = A_1$,*

(2) $(A_{n-2}D_n, A_{n-1}B_{n-1})$ *for* $n \geq 4$,
(3) $(B_{n-1}D_{2n}, B_{2n-1}B_n)$ *for* $n \geq 2$, *with the convention that* $B_1 = A_1$,
(4) (A_1A_5, A_4G_2),
(5) (A_1B_3, B_2G_2),
(6) (A_1D_6, B_5G_2),
(7) (A_2B_3, A_3G_2) *and*
(8) (B_3^2, D_4G_2).

Proof. Let $H_1 = H_{1,1} \times H_{1,2}$ and $H_2 = H_{2,1} \times H_{2,2}$ where $H_{i,j}$ are split simple algebraic groups. We denote $W(H_i)$ by W_i and $W(H_{i,j})$ by $W_{i,j}$.

Since the orders of the groups $|H_1(\mathbb{F}_q)|$ and $|H_2(\mathbb{F}_q)|$ are the same, by Theorem 3.3, the degrees of the Weyl groups W_1 and W_2 are the same with the same multiplicities. Moreover for $i = 1, 2$, we have $W_i = W_{i,1} \times W_{i,2}$.

Let n be the maximum of the degrees of W_1. Then it is the largest of the maxima of the degrees of $W_{1,j}$ for $j = 1, 2$. Suppose that n is the maximum degree of $W_{1,1}$. Then depending on n, we have the following choices for the group $H_{1,1}$:

$n = 2$	A_1,	$n = 4$	A_3, B_2,
$n = 6$	A_5, B_3, D_4, G_2,	$n = 12$	$A_{11}, B_6, D_7, F_4, E_6$,
$n = 18$	A_{17}, B_9, D_{10}, E_7,	$n = 30$	$A_{29}, B_{15}, D_{16}, E_8$,
		$n = 2m$	
n odd	A_{n-1},	$n \notin \{2, 4, 6, 12, 18, 30\}$	A_{2m-1}, B_m, D_{m+1},

The general philosophy of the proof is as follows:

Once we fix n, we have a finite set of choices for $H_{1,1}$ and $H_{2,1}$. Then we fix one choice each for $H_{1,1}$ and $H_{2,1}$, and compare the degrees of $W_{1,1}$ and $W_{2,1}$. Since $H_{1,1} \neq H_{2,1}$ the collections $d(W_{1,1})$ and $d(W_{2,1})$ are different. The degrees of $W_{1,1}$ that do not occur in $d(W_{2,1})$ must occur in the collection $d(W_{2,2})$ and similarly the degrees of $W_{2,1}$ that do not occur in $d(W_{1,1})$ must occur in the collection $d(W_{1,2})$. Moreover the degrees of $W_{1,2}$ and $W_{2,2}$ are bounded above by n. This gives us further finitely many choices for the groups $H_{1,2}$ and $H_{2,2}$. For these choices, we simply verify the equality of the collections $d(W_1)$ and $d(W_2)$. If the collections are equal, we get a coincidence of orders (H_1, H_2).

As a sample, we do the case of $n = 4$ to illustrate the above philosophy.

Let us assume that $H_{1,1} = A_3$ and $H_{2,1} = B_2$. Then we have

$$d(W_{1,1}) = \{2, 3, 4\} \subseteq d(W_1) \quad \text{and} \quad d(W_{2,1}) = \{2, 4\} \subseteq d(W_2).$$

Thus, 3 is a degree of $W_{2,2}$ and the maximum of the degrees of $W_{2,2}$ is less than or equal to 4, hence $H_{2,2} = A_2$. Then, since $d(W_1) = d(W_2)$, the only possibility for the collection $d(W_{1,2})$ is $\{2\}$ and we get $H_{1,2} = A_2$. This gives us the order coincidence (A_1A_3, A_2B_2). $\square$

Observe that in the above theorem, we have three infinite families of pairs,

$$(A_{2n-2}B_n, A_{2n-1}B_{n-1}), (A_{n-2}D_n, A_{n-1}B_{n-1})$$

and

$$(B_{n-1}D_{2n}, B_{2n-1}B_n).$$

If we consider the following pairs given by the first two infinite families:

$$(H_1, H_2) = (A_{2n-2}B_n, A_{2n-1}B_{n-1})$$

and

$$(H_3, H_4) = (A_{2n-2}D_{2n}, A_{2n-1}B_{2n-1}),$$

then

$$(H_1H_4, H_2H_3) = (A_{2n-2}A_{2n-1}B_nB_{2n-1}, A_{2n-1}A_{2n-2}B_{n-1}D_{2n}).$$

This implies that $(B_{2n-1}B_n, B_{n-1}D_{2n})$ is also a pair of order coincidence and this is precisely our third infinite family! Thus, the third infinite family of order coincidences can be obtained from the first two infinite families.

Similarly if we consider

$$(H_1, H_2) = (A_2D_4, A_3B_3) \quad \text{and} \quad (H_3, H_4) = (A_2B_3, A_3G_2),$$

then we get the pair (B_3^2, D_4G_2) from the pair (H_2H_3, H_1H_4).

Similarly we observe that

$$(A_1B_3, B_2G_2) \text{ can be obtained from } (A_1A_3, A_2B_2) \text{ and } (A_2B_3, A_3G_2),$$

$$(A_1A_5, A_4G_2) \text{ can be obtained from } (A_4B_3, A_5B_2) \text{ and } (A_1B_3, B_2G_2),$$

$$(A_1D_6, B_5G_2) \text{ can be obtained from } (A_4D_6, A_5B_5) \text{ and } (A_1A_5, A_4G_2).$$

We record our observation as a remark below.

Remark 4.3. All the pairs of order coincidence described in Theorem 4.2 can be obtained from the following pairs:

(1) $(A_{2n-2}B_n, A_{2n-1}B_{n-1})$ for $n \geq 2$, with the convention that $B_1 = A_1$,
(2) $(A_{n-2}D_n, A_{n-1}B_{n-1})$ for $n \geq 4$, and
(3) (A_2B_3, A_3G_2).

These pairs are quite special, in the sense that they admit a geometric reasoning for the coincidence of orders. We describe it in the last section.

If we do not restrict ourselves to the groups having exactly two simple factors, then we also find the following pairs (H_1, H_2) involving other exceptional groups:

$$(A_1B_4B_6, B_2B_5F_4), (A_4G_2A_8B_6, A_3A_6B_5E_6), (A_1B_7B_9, B_2B_8E_7),$$

and

$$(A_1B_4B_7B_{10}B_{12}B_{15}, \ B_3B_5B_8B_{11}B_{14}E_8).$$

One now asks a natural question whether these four pairs, together with the pairs described in Remark 4.3, generate all possible pairs of order coincidence. We make this question more precise in the next section and answer it in the affirmative.

5. On a group structure on pairs of groups of equal order

Fix a finite field $\mathbb{F}_q$. Let $\mathcal{A}$ be the set of ordered pairs (H_1, H_2) where H_1 and H_2 are split semisimple algebraic groups defined over the field $\mathbb{F}_q$ such that the orders of the finite groups $H_1(\mathbb{F}_q)$ and $H_2(\mathbb{F}_q)$ are the same. We define an equivalence relation on

$\mathcal{A}$ by saying that an element $(H_1, H_2) \in \mathcal{A}$ is related to $(H'_1, H'_2) \in \mathcal{A}$, denoted by $(H_1, H_2) \sim (H'_1, H'_2)$, if and only if there exist two split semisimple algebraic groups H and K defined over $\mathbb{F}_q$ such that

$$H'_1 \times K = H_1 \times H \quad \text{and} \quad H'_2 \times K = H_2 \times H.$$

It can be checked that $\sim$ is an equivalence relation. We denote the set of equivalence classes in $\mathcal{A}$ given by $\sim$, $\mathcal{A}/\sim$, by $\mathcal{G}$ and the equivalence class of an element $(H_1, H_2) \in \mathcal{A}$ is denoted by $[(H_1, H_2)]$. This set $\mathcal{G}$ describes all pairs of order coincidence (H_1, H_2) where the split semisimple (simply connected) groups H_i do not have any common direct simple factor.

We put a binary operation on $\mathcal{G}$ given by

$$[(H_1, H_2)] \circ [(H'_1, H'_2)] = [(H_1 \times H'_1, H_2 \times H'_2)].$$

It is easy to see that the above operation is a well-defined modulo, the equivalence that we have introduced. The set $\mathcal{G}$ is obviously closed under $\circ$ which is an associative operation. The equivalence class $[(H, H)]$ acts as the identity and $[(H_1, H_2)]^{-1} = [(H_2, H_1)]$. Thus $\mathcal{G}$ is an abelian, torsion-free group. Since the first two infinite families described in Remark 4.3 are independent, the group $\mathcal{G}$ is not finitely generated.

Let $\mathcal{G}'$ be the subgroup of $\mathcal{G}$ generated by following elements.

(1) $\mathsf{B}_n = [(A_{2n-2}B_n, A_{2n-1}B_{n-1})]$, for $n \geq 2$, with the convention that $B_1 = A_1$,
(2) $\mathsf{D}_n = [(A_{n-2}D_n, A_{n-1}B_{n-1})]$, for $n \geq 4$,
(3) $\mathsf{G}_2 = [(A_2B_3, A_3G_2)]$,
(4) $\mathsf{F}_4 = [(A_1B_4B_6, B_2B_5F_4)]$,
(5) $\mathsf{E}_6 = [(A_4G_2A_8B_6, A_3A_6B_5E_6)]$,
(6) $\mathsf{E}_7 = [(A_1B_7B_9, B_2B_8E_7)]$,
(7) $\mathsf{E}_8 = [(A_1B_4B_7B_{10}B_{12}B_{15}, B_3B_5B_8B_{11}B_{14}E_8)]$.

(For a group such as B_n, we use B_n to denote a pair (H_1, H_2) in which B_n appears as a group of the largest degree.)

Lemma 5.1. *Let n be a positive integer. Let H_1 and H_2 be split, simply connected, simple algebraic groups such that the Weyl groups $W(H_1)$ and $W(H_2)$ have the same highest degree and it is equal to n. Then there is an element in $\mathcal{G}'$ which can be represented as the equivalence class of a pair (K_1, K_2) such that for $i = 1, 2$, H_i is one of the simple factors of K_i and for any other simple factor H'_i of K_i the highest degree of $W(H'_i)$ is less than n.*

Proof. We prove this lemma by explicit calculations. If n is odd or $n = 2$, there is nothing to prove as there is only one group, A_{n-1}, with n as the highest degree.

For $n = 4$, the groups A_3 and B_2 are the only groups with 4 as the highest degree and $\mathsf{B}_2 = [(A_2B_2, A_3B_1)]$ is an element of the group $\mathcal{G}'$ where A_3 and B_2 appear as factors on either sides and all other simple groups that appear have highest degree less than 4.

If $n = 2m$ for $m > 2$ and $m \notin \{3, 6, 9, 15\}$, then A_{2m-1}, B_m and D_{m+1} are the only groups with n as the highest degree. Consider following elements of $\mathcal{G}'$:

$$\mathsf{B}_m = [(A_{2m-2}B_m, A_{2m-1}B_{m-1})],$$
$$\mathsf{D}_{m+1} = [(A_{m-1}D_{m+1}, A_mB_m)]$$

and

$$\mathsf{D}_{m+1} \circ \mathsf{B}_m = [(A_{m-1}A_{2m-2}D_{m+1}, \; A_m A_{2m-1} B_{m-1})].$$

The element B_m contains the simple groups A_{2m-1} and B_m on its either sides and other simple groups appearing in B_m have highest degree less than $2m$. Similarly the elements D_{m+1} and $\mathsf{D}_{m+1} \circ \mathsf{B}_m$ are the required elements of $\mathcal{G}'$ for the pairs $\{D_{m+1}, B_m\}$ and $\{A_{2m-1}, D_{m+1}\}$.

Now, we consider the cases when $n \in \{6, 12, 18, 30\}$. These cases involve exceptional groups.

For $n = 6$, the groups A_5, B_3, D_4 and G_2 are the only groups with 6 as the highest degree. We have following elements of $\mathcal{G}'$ for the corresponding pairs.

$$\mathsf{B}_3 = [(A_4 B_3, A_5 B_2)] \text{ for the pair } \{B_3, A_5\},$$
$$\mathsf{D}_4 = [(A_2 D_4, A_3 B_3)] \text{ for the pair } \{D_4, B_3\},$$
$$\mathsf{G}_2 = [(A_2 B_3, A_3 G_2)] \text{ for the pair } \{B_3, G_2\},$$
$$\mathsf{D}_4 \circ \mathsf{G}_2 = [(A_2^2 D_4, A_3^2 G_2)] \text{ for the pair } \{D_4, G_2\},$$
$$\mathsf{B}_3 \circ \mathsf{D}_4 = [(A_2 A_4 D_4, A_3 A_5 B_2)] \text{ for the pair } \{D_4, A_5\},$$
$$\mathsf{G}_2 \circ \mathsf{B}_3^{-1} = [(A_2 A_5 B_2, A_3 A_4 G_2)] \text{ for the pair } \{A_5, G_2\}.$$

In the same way, we give the following elements of the group $\mathcal{G}'$ for all possible simple groups having highest degree 12, 18 and 30.

Element of $\mathcal{G}'$	Pair	Element of $\mathcal{G}'$	Pair
		$n = 12$	
$\mathsf{B}_6 = [(A_{10}B_6, A_{11}B_5)]$	$\{B_6, A_{11}\}$,	$\mathsf{D}_7 = [(A_5 D_7, A_6 B_6)]$	$\{D_7, B_6\}$,
$\mathsf{F}_4 = [(A_1 B_4 B_6, B_2 B_5 F_4)]$	$\{B_6, F_4\}$,	$\mathsf{E}_6 = [(A_4 G_2 A_8 B_6,$	$\{B_6, E_6\}$
		$A_3 A_6 B_5 E_6)]$	
$\mathsf{B}_6 \circ \mathsf{D}_7$	$\{D_7, A_{11}\}$,	$\mathsf{D}_7 \circ \mathsf{F}_4$	$\{D_7, F_4\}$,
$\mathsf{B}_6^{-1} \circ \mathsf{F}_4$	$\{A_{11}, F_4\}$,	$\mathsf{D}_7 \circ \mathsf{E}_6$	$\{D_7, E_6\}$,
$\mathsf{B}_6^{-1} \circ \mathsf{E}_6$	$\{A_{11}, E_6\}$,	$\mathsf{F}_4^{-1} \circ \mathsf{E}_6$	$\{F_4, E_6\}$.
		$n = 18$	
$\mathsf{B}_9 = [(A_{16}B_9, A_{17}B_8)]$	$\{B_9, A_{17}\}$,	$\mathsf{D}_{10} = [(A_8 D_{10}, A_9 B_9)]$	$\{D_{10}, B_9\}$,
$\mathsf{E}_7 = [(A_1 B_7 B_9, B_2 B_8 E_7)]$	$\{B_9, E_7\}$,	$\mathsf{B}_9^{-1} \circ \mathsf{E}_7$	$\{A_{17}, E_7\}$,
$\mathsf{B}_9 \circ \mathsf{D}_{10}$	$\{D_{10}, A_{17}\}$,	$\mathsf{D}_{10} \circ \mathsf{E}_7$	$\{D_{10}, E_7\}$.
		$n = 30$	
$\mathsf{B}_{15} = [(A_{28}B_{15}, A_{29}B_{14})]$	$\{B_{15}, A_{29}\}$,	$\mathsf{B}_{15}^{-1} \circ \mathsf{E}_8$	$\{A_{29}, E_8\}$,
$\mathsf{D}_{16} = [(A_{14}D_{16}, A_{15}B_{15})]$	$\{D_{16}, B_{15}\}$,	$\mathsf{B}_{15} \circ \mathsf{D}_{16}$	$\{D_{16}, A_{29}\}$,
$\mathsf{E}_8 = [(A_1 B_4 B_7 B_{10} B_{12} B_{15},$	$\{B_{15}, E_8\}$,	$\mathsf{D}_{16} \circ \mathsf{E}_8$	$\{D_{16}, E_8\}$.
$B_3 B_5 B_8 B_{11} B_{14} E_8)]$			

This completes the proof of the lemma. □

Theorem 5.2. *The groups $\mathcal{G}$ and $\mathcal{G}'$ are the same. In other words, the group $\mathcal{G}$ is generated by the following elements*:

(1) $[(A_{2n-2}B_n, A_{2n-1}B_{n-1})]$ *for* $n \geq 2$, *with the convention that* $B_1 = A_1$,
(2) $[(A_{n-2}D_n, A_{n-1}B_{n-1})]$ *for* $n \geq 4$,
(3) $[(A_2B_3, A_3G_2)]$,
(4) $[(A_1B_4B_6, B_2B_5F_4)]$,
(5) $[(A_4G_2A_8B_6, A_3A_6B_5E_6)]$,
(6) $[(A_1B_7B_9, B_2B_8E_7)]$,
(7) $[(A_1B_4B_7B_{10}B_{12}B_{15}, B_3B_5B_8B_{11}B_{14}E_8)]$.

Proof. Let $[(H_1, H_2)] \in \mathcal{G}$. By Theorem 3.3, the fundamental degrees of the Weyl groups $W(H_1)$ and $W(H_2)$ are the same with the same multiplicities. Let n be the highest degree of $W(H_1)$ which is the same as the highest degree of $W(H_2)$. For $i = 1, 2$, let K_i be one of the simple factors of H_i such that n is the highest degree of $W(K_i)$. Then by previous lemma, there exists an element $[(H_1', H_2')] \in \mathcal{G}'$ such that K_i are the simple factors of H_i' and the other simple factors of H_i' have highest degree less than n. Thus, the element $[(H_1H_2', H_2H_1')]$ is an element of the group $\mathcal{G}$ and the multiplicity of K_i on either side of this element is now reduced by 1. This way, we cancel all the simple factors having n as the highest degree and then the result is obtained by induction. $\qquad\square$

6. Transitive actions of compact Lie groups

Here we explain how a transitive action of compact Lie groups is related to the coincidence of orders. The exposition is based on Chapter 2, page 121 of [4].

Suppose H is a compact simply connected Lie group acting transitively on a compact manifold $X = H/H_1$ with H_1 connected. Suppose that H_2 is a closed connected Lie subgroup of H and that the action of H on X when restricted to H_2 remains transitive. Then $X = H/H_1 = H_2/(H_1 \cap H_2)$. By looking at the homotopy exact sequence for the fibration $1 \rightarrow H' \rightarrow H \rightarrow H/H' \rightarrow 1$ for any closed subgroup H' of H,

$$\pi_1(H') \rightarrow \pi_1(H) \rightarrow \pi_1(H/H') \rightarrow \pi_0(H'),$$

we find that H/H' is simply connected if and only if H' is connected. Therefore $X = H/H_1$ is simply connected and hence if $X = H_2/(H_1 \cap H_2)$ with H_2 simply connected, $H_1 \cap H_2$ is connected.

We now assume that there is an analogue of the action of H on $X = H/H_1$ over finite fields, which we now take to be all defined over $\mathbb{F}_q$. By Lang's theorem (Corollary to Theorem 1 of [7]) if H_1 is connected then

$$|(H/H_1)(\mathbb{F}_q)| = \frac{|H(\mathbb{F}_q)|}{|H_1(\mathbb{F}_q)|}.$$

Therefore for the equality of spaces H/H_1 and $H_2/(H_1 \cap H_2)$, with $H_1, H_2, H_1 \cap H_2$ connected, we find that

$$|H(\mathbb{F}_q)| \cdot |(H_1 \cap H_2)(\mathbb{F}_q)| = |H_1(\mathbb{F}_q)| \cdot |H_2(\mathbb{F}_q)|.$$

Thus transitive action of compact Lie groups gives rise to coincidence of orders of finite semisimple groups.

We call an ordered 3-tuple (H, H_1, H_2), as discussed above, a triple of *inclusion of transitive actions*. We first classify all such triples of inclusion of transitive actions and explain the geometric reasoning behind the order coincidence for the first three pairs described in Theorem 5.2. We note some observations.

Remark 6.1. Let (H, H_1, H_2) be a triple of inclusion of transitive actions, where H, H_1 and H_2 are compact Lie groups such that H_1 is a subgroup of H and the natural action of H_1 on H/H_2 is transitive. Then

(1) $H = H_1 H_2$ (Lemma 4.1, page 138 of [4]) and
(2) either H_1 or H_2 has the same maximal exponent as the maximal exponent of the group H (Corollary 2, page 143 of [4]).

(We recall that a natural number a is an exponent of a compact Lie group H if and only if $a + 1$ is a degree of the Weyl group of the split form of H.)

Therefore to classify the inclusions among the transitive actions, equivalently to determine the triples (H, H_1, H_2) of inclusion of transitive actions, it would be desirable to classify the subgroups of a given Lie group of the maximal exponent. We restrict ourselves to the case when H is a simple Lie group.

Theorem 6.2 [8]. *Let H be a connected simple compact Lie group and H_1 be a compact Lie subgroup of H of maximal exponent. Then, the pairs $H_1 \subseteq H$ are exhausted by the following list*:

$$Sp_n \subset SU_{2n} \quad (n > 1), \quad G_2 \subset SO_7, \quad SO_{2n-1} \subset SO_{2n} \quad (n > 3),$$
$$\text{Spin}_7 \subset SO_8, \quad G_2 \subset SO_8, \quad F_4 \subset E_6.$$

Observe that the subgroup $H_1 \subset H$ is automatically a simple group. Now, we classify the triples (H, H_1, H_2), of inclusion of transitive actions, where H is a simple Lie group.

Theorem 6.3 [8]. *The triples (H, H_1, H_2) of inclusion of transitive actions where the group H is simple are the following ones*:

H	H_1	H_2	$H_1 \cap H_2$	H	H_1	H_2	$H_1 \cap H_2$
SU_{2n} $(n \geq 2)$	Sp_n	SU_{2n-1}	Sp_{n-1}	SO_{4n} $(n \geq 2)$	SO_{4n-1}	Sp_n	Sp_{n-1}
SO_7	G_2	SO_6 SO_5	SU_3 SU_2	SO_{16}	SO_{15}	Spin_9	Spin_7
SO_{2n} $(n \geq 4)$	SO_{2n-1}	SU_n	SU_{n-1}	SO_8	Spin_7	SO_7 SO_6 SO_5	G_2 SU_3 SU_2

Observe that the exponents of the groups $H \times (H_1 \cap H_2)$ and $H_1 \times H_2$ are the same in all the above cases. Hence $(H \times (H_1 \cap H_2), H_1 \times H_2)$ is a pair of order coincidence for us. The pairs described in Remark 4.3 occur in the above descriptions. We can, in fact, give an explicit description of the inclusion among the transitive actions corresponding to the pairs given in Remark 4.3.

The groups O_n, U_n and Sp_n act on the spaces $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{H}^n$, respectively, in a natural way. By restricting this action to the corresponding spheres, we get that the groups O_n, U_n and Sp_n act transitively on the spheres S^{n-1}, S^{2n-1} and S^{4n-1}, respectively. By fixing a point in each of the spheres, we get the corresponding stabilizers as $O_{n-1} \subset O_n$, $U_{n-1} \subset U_n$ and $Sp_{n-1} \subset Sp_n$.

By treating the space $\mathbb{C}^n = \mathbb{R}^{2n}$, we obtain an inclusion of transitive actions $U_n \subset O_{2n}$, with both the groups acting transitively on S^{2n-1}. Since S^{2n-1} is connected, the actions of $SU_n \subset U_n$ and $SO_{2n} \subset O_{2n}$ on S^{2n-1} remain transitive. Thus we get an inclusion of actions $SU_n \subset SO_{2n}$ and the corresponding stabilisers are $SU_{n-1} \subset SO_{2n-1}$. Thus, we get a triple $(SO_{2n}, SU_n, SO_{2n-1})$ or equivalently we get a pair of order coincidence as $(D_n A_{n-2}, \ A_{n-1} B_n)$.

Similarly, by treating $\mathbb{H}^n$ as $\mathbb{C}^{2n}$ and repeating the above arguments, we get the inclusion of transitive actions $Sp_n \subset SU_{2n}$, acting on the sphere S^{4n-1}, with $Sp_{n-1} \subset SU_{2n-1}$ as the corresponding stabilisers. This gives us the triple $(SU_{2n}, Sp_n, SU_{2n-1})$ and the pair of order coincidence $(A_{2n-1} B_{n-1}, B_n A_{2n-2})$.

Thus, we get the two infinite families described in Remark 4.3. The remaining pair of order coincidence, $(A_1 B_3, B_2 G_2)$, can be obtained in a similar way by considering the natural inclusion $G_2 \subset SO_7$. These groups act transitively on the sphere S^6 and the corresponding stabilisers are $SU_3 \subset SO_6$. We observe that the split form of SO_6 is isomorphic to SL_4, and therefore the triple (SO_7, G_2, SO_6) gives us $(A_2 B_3, A_3 G_2)$ as the corresponding pair of order coincidence.

Remark 6.4. It would be interesting to know if the pairs (4) to (7) of Theorem 5.2 involving exceptional groups are also obtained in this geometric way.

Acknowledgements

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