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New criteria to identify spectrum

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Abstract. In this paper we give some new criteria for identifying the components of a probability measure, in its Lebesgue decomposition. This enables us to give new criteria to identify spectral types of self-adjoint operators on Hilbert spaces, especially those of interest.

Keywords. Lebesgue decomposition; spectral theory; self-adjoint operators; wavelet transforms; spectral type.

1. Introduction

Let us briefly motivate our interest in determining the spectral type of a self-adjoint operator.

Let μ be a probability measure on the real line $\mathbf{R}$. It is well-known that this measure has a Lebesgue decomposition $\mu = \mu_{\mathrm{ac}} + \mu_{\mathrm{sc}} + \mu_{\mathrm{pp}}$, where μ_{ac} is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}$, μ_{sc} is singular with respect to Lebesgue measure, and has no atomic part, i.e. $\mu_{\mathrm{sc}}(\{x\}) = 0$ for all $x \in \mathbf{R}$, and μ_{pp} is purely atomic.

This decomposition of a probability measure has important applications in the theory of a self-adjoint operator H on a (separable) Hilbert space $\mathcal{H}$. Associated with H is the spectral measure $E(\cdot)$. The spectral theorem states that we have

$$\langle u, Hu\rangle = \int_{\mathbf{R}} \lambda d\langle u, E(\lambda)u\rangle.$$

If $\|u\| = 1$, then $d\langle u, E(\cdot)u\rangle$ is a probability measure, which is supported on the spectrum $\sigma(H)$ of H. The Lebesgue decomposition of probability measures leads to an orthogonal decomposition of the Hilbert space

$$\mathcal{H} = \mathcal{H}_{\mathrm{ac}} \oplus \mathcal{H}_{\mathrm{sc}} \oplus \mathcal{H}_{\mathrm{pp}}.$$

Each subspace is the closure of vectors u, such that $d\langle u, E(\cdot)u\rangle$ is purely absolutely continuous, etc. The subspaces reduce the operator H, such that $H_{|\mathcal{H}_{\mathrm{ac}}}$ is a self-adjoint operator on $\mathcal{H}_{\mathrm{ac}}$, etc. In the case of absolutely continuous and singular continuous parts, one defines the corresponding parts of the spectrum to be those of the restricted operators. In the point spectrum case one usually takes $\sigma_{\mathrm{pp}}(H)$ to be the set of eigenvalues of H, in order to handle the case, when the operator has a dense set of eigenvalues. The spectrum of the operator restricted to $\mathcal{H}_{\mathrm{pp}}$ is then the closure of this set.

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The spectral types of an operator H, which is the Hamiltonian of a quantum mechanical system, is related to the dynamics of the system, although the relation is by no means simple. The relation comes from the representation of the time evolution operator e^{-itH} as

$$\langle u, e^{-itH}u\rangle = \int_{\mathbf{R}} e^{-it\lambda} d\langle u, E(\lambda)u\rangle.$$

In some quantum mechanical systems (e.g. atoms and molecules) the absolutely continuous part is related to the scattering states, since $\langle u, e^{-itH}u\rangle$ tends to zero for $u \in \mathcal{H}_{ac}$ (a consequence of the Riemann–Lebesgue lemma), and the eigenvalues of H are related to the bound states. In many of these systems one expects that the singular continuous component is absent, and many techniques have been developed to prove this type of result. In solid state physics the situation is somewhat different, and here one has a wider variety of spectral types.

These applications have motivated us to seek new criteria for identifying the spectral type of a self-adjoint operator.

The components of a probability measure can be identified via a transform of the measure. Two of these are well-known, viz. the Fourier transform and the Borel transform. In this paper we address the question of identifying the components using a more general transform. We give results using a general approximate identity, and an associated continuous wavelet transform.

Concerning the literature, the connection between an approximate identity and the continuous wavelet transform was discussed by Holschneider [1], while wavelet coefficients of fractal measures were studied by Strichartz [4]. In the theory of self-adjoint operators finer decomposition of spectra with respect to Hausdorff measures was first used by Last [2] and general criteria for recovering a measure from its Borel transform was done by Simon [3].

2. The criteria

We need to introduce conditions on our function ψ. Several of these can be relaxed in some of the results. We use the standard notation $\langle x\rangle = (1+x^2)^{1/2}$.

Assumption 2.1. Assume that $\psi \in C^1(\mathbf{R})$, $\psi(0) = 1$, ψ is even, and there exist $C > 0$ and $\delta > 1$, such that

$$|\psi(x)| + |x\psi'(x)| \leq C\langle x\rangle^{-\delta}, \quad x \in \mathbf{R}. \tag{2.1}$$

We set $A_\psi = \int_{\mathbf{R}} \psi(x)\mathrm{d}x$ and assume that $A_\psi \neq 0$.

In the sequel we always impose this assumption on ψ. We introduce the notation

$$\psi_a(x) = \psi(x/a) \quad \text{and} \quad \tilde{\psi}_a(x) = \frac{1}{a}\psi_a(x), \quad a > 0. \tag{2.2}$$

In particular, the family $\{A_\psi^{-1}\tilde{\psi}_a\}$ is an approximate identity. Let μ be a probability measure on $\mathbf{R}$ in what follows, with Lebesgue decomposition $\mu = \mu_s + \mu_{ac}$. Let f be a function. We recall that the convolution $(f * \mu)(x) = \int f(x-y)\mathrm{d}\mu(y)$ is defined, when the integral converges. Since ψ is bounded, the convolution $\psi_a * \mu$ is defined for all $a > 0$.

For $0 \le \alpha \le 1$ we define

$$(d_\alpha \mu)(x) = \lim_{\varepsilon \downarrow 0} \frac{\mu((x-\varepsilon, x+\varepsilon))}{(2\varepsilon)^\alpha}, \tag{2.3}$$

whenever the limit on the right-hand side exists.

We can now state the results. We first give results based on ψ_a and $\tilde{\psi}_a$, and then on an associated continuous wavelet transform.

Theorem 2.2. *Let μ be a probability measure. Then we have as follows*:

1. *Let ψ satisfy Assumption* 2.1. *Then for every continuous function f of compact support, the following is valid.*

$$\lim_{a\to 0} \int (\tilde{\psi}_a * \mu)(x) f(x) \mathrm{d}x = A_\psi \int f(x) \mathrm{d}\mu(x).$$

2. $\lim_{a\to 0} (\psi_a * \mu)(x) = \mu(\{x\})$.
3. *Assume $0 < \alpha \le 1$ and $(d_\alpha \mu)(x)$ finite. Then we have*

$$\lim_{a\to 0} a^{-\alpha} (\psi_a * \mu)(x) = c_\alpha (d_\alpha \mu)(x), \tag{2.4}$$

where $c_\alpha = \int_0^\infty \alpha 2^\alpha y^{\alpha-1} \psi(y) \mathrm{d}y$.

Remark 2.3.

(1) Equation (2.4) implies that if μ is purely singular, then the limit of $\tilde{\psi}_a * \mu(x)$ is zero almost everywhere with respect to the Lebesgue measure, since the derivative $(d_1\mu)(x) = 0$ almost everywhere for purely singular μ.

(2) If x is not in the topological support of μ, then for each $0 \le \alpha \le 1$,

$$\lim_{a\to 0} a^{-\alpha} \psi_a * \mu(x) = 0.$$

Our next theorem says a bit more and the first part is analogous to Wiener's theorem and its extension by Simon [3].

Theorem 2.4. *Let μ be a probability measure. Then for any bounded interval (c, d) the following are valid.*

1. *Let*

$$C = \int_{\mathbf{R}} |\psi(x)|^2 \mathrm{d}x,$$

then

$$\lim_{a\to 0} \frac{1}{a} \int_c^d |(\psi_a * \mu)(x)|^2 \mathrm{d}x$$
$$= C \left(\sum_{x\in(c,d)} \mu(\{x\})^2 + \frac{1}{2}[\mu(\{c\})^2 + \mu(\{d\})^2] \right). \tag{2.5}$$

2. *For* $0 < p < 1$, *we have*

$$\lim_{a\to 0}\int_c^d |(\tilde{\psi}_a * \mu)(x)|^p \mathrm{d}x = |A_\psi|^p \int_c^d \left|\frac{\mathrm{d}\mu_{\mathrm{ac}}}{\mathrm{d}x}(x)\right|^p \mathrm{d}x. \tag{2.6}$$

This theorem has the following corollary.

COROLLARY 2.5

Let μ *be a probability measure. Then we have the following results*:

1. μ *has no point part in* $[c, d]$, *if and only if*

$$\liminf_{a\to 0}\frac{1}{a}\int_c^d |(\psi_a * \mu)(x)|^2 \mathrm{d}x = 0. \tag{2.7}$$

2. *If* μ *has no absolutely continuous part in* (c, d), *if and only if for some* $p, 0 < p < 1$,

$$\liminf_{a\to 0}\int_c^d |(\tilde{\psi}_a * \mu)(x)|^p \mathrm{d}x = 0. \tag{2.8}$$

Now to state the results in terms of the continuous wavelet transform, we introduce

$$h(x) = \psi(x) + x\psi'(x). \tag{2.9}$$

Under Assumption 2.1 we clearly have

$$|h(x)| \le C\langle x\rangle^{-\delta}, \tag{2.10}$$

with the δ from the assumption. Integration by parts and eq. (2.9) imply that h satisfies the admissibility condition for a continuous wavelet, i.e. $\int_{-\infty}^{\infty} h(x)\mathrm{d}x = 0$.

Thus we can define the continuous wavelet transform of a probability measure μ as

$$W_h(\mu)(b, a) = \frac{1}{a}\int_{-\infty}^{\infty} h((b-y)/a)\mathrm{d}\mu(y). \tag{2.11}$$

The connection between the approximate identity and this transform is

$$-a\frac{\partial}{\partial a}(\tilde{\psi}_a * \mu)(b) = W_h(\mu)(b, a). \tag{2.12}$$

This result follows from

$$-a\frac{\partial}{\partial a}\left(\frac{1}{a}\psi\left(\frac{x}{a}\right)\right) = \frac{1}{a}\left(\psi\left(\frac{x}{a}\right) + \frac{x}{a}\psi'\left(\frac{x}{a}\right)\right),$$

and the definitions.

We have the following analogue of Theorem 2.2.

Theorem 2.6. *Let* μ *be a probability measure. Then we have the following results*:

1. *We have*

$$\lim_{\varepsilon\downarrow 0}\varepsilon\int_{\varepsilon}^{\infty} W_h(\mu)(b, a)\frac{\mathrm{d}a}{a} = \mu(\{b\}). \tag{2.13}$$

2. *Let* $0 < \alpha \leq 1$. *Assume that* $(d_\alpha \mu)(b)$ *exists. Then*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{1-\alpha} \int_\varepsilon^\infty W_h(\mu)(b,a) \frac{\mathrm{d}a}{a} = c_\alpha (d_\alpha \mu)(b), \tag{2.14}$$

where c_α *was defined in Theorem* 2.2.

Remark 2.7. We note that for $0 < \alpha < 1$ we can replace $\int_\varepsilon^\infty$ by $\int_\varepsilon^M$ for any $M > 0$ (see the proof of the Theorem).

We also have the following analogue of Theorem 2.4(1).

Theorem 2.8. *Let* μ *be a probability measure. Then for any bounded interval* (c, d) *we have the following result. Let*

$$C_h = \int_{\mathbf{R}} |h(x)|^2 \mathrm{d}x.$$

Then we have

$$\lim_{a \downarrow 0} \int_c^d |W_h(\mu)(b,a)|^2 \mathrm{d}b$$
$$= C_h \left(\sum_{x \in (c,d)} \mu(\{x\})^2 + \frac{1}{2}(\mu(\{c\})^2 + \mu(\{d\})^2) \right). \tag{2.15}$$

Even when the quantity $(d_\alpha \mu)(x)$ does not exist, it is possible to say something on the wavelet transforms, to cover the cases of measures which are not supported on the sets where such limits exist. Set

$$C^\alpha_{\mu,\psi}(x) = \limsup_{a \to 0} \frac{\psi_a * \mu}{a^\alpha}(x) \quad \text{and} \quad D^\alpha_\mu(x) = \limsup_{\varepsilon \to 0} \frac{\mu((x-\varepsilon, x+\varepsilon))}{(2\varepsilon)^\alpha}.$$

Then we have the following theorem.

Theorem 2.9. *Let* μ *be a probability measure, and let* ψ *satisfy Assumption* 2.1. *Then* $C^\alpha_{\mu,\psi}(x)$ *is finite for any* x, *whenever* $D^\alpha_\mu(x)$ *is finite for the same* x, *and, if* ψ *is non-negative, they are both finite or both infinite.*

Remark 2.10. The above theorem implies that if $\limsup_{a \to 0} |(\tilde{\psi}_a * \mu)(x)| < \infty$ for all $x \in (c, d)$, then there is no singular part of μ supported in (c, d).

Finally as an application of the above theorems we consider $\mathcal{H}$ to be a separable Hilbert space and A a self-adjoint operator. Then we have the following theorem.

Theorem 2.11. *Suppose* A *is a self-adjoint operator on* $\mathcal{H}$. *Consider a function* ψ *satisfying Assumption* 2.1. *Then*

1. λ *is in the point spectrum of* A, *if for some* $f \in \mathcal{H}$, $\|f\| = 1$,

$$\lim_{a \to 0} \langle f, \psi_a(A - \lambda) f \rangle = 0.$$

2. *Let $B \subset \mathbf{R}$ be a Borel set of positive Lebesgue measure. Then $B \cap \sigma_{\mathrm{ac}}(A) \neq \emptyset$, if for some $f \in \mathcal{H}$, $\|f\| = 1$,*

$$\lim_{a\to 0} \langle f, \tilde{\psi}_a(A-\lambda)f\rangle \neq 0, \quad \text{for a.e. } \lambda \in B.$$

3. *The point spectrum of A in (c,d) is empty, if and only if for some orthonormal basis $\{f_n\}$, of $\mathcal{H}$, one has for every n,*

$$\liminf_{a\to 0} \frac{1}{a}\int_c^d |\langle f_n, \psi_a(A-\lambda)f_n\rangle|^2 \mathrm{d}\lambda = 0.$$

4. *The absolutely continuous spectrum of A in (c,d) is empty, if and only if for some orthonormal basis $\{f_n\}$ of $\mathcal{H}$, one has for every n and some $0 < p < 1$,*

$$\liminf_{a\to 0} \int_c^d \left|\frac{1}{a}\langle f_n, \psi_a(A-\lambda)f_n\rangle\right|^p \mathrm{d}\lambda = 0.$$

3. Proofs

Throughout the computations below the letter C denote a constant, whose value may vary from line to line.

Proof of Theorem 2.2.

Part (1): Since f is a continuous function of compact support and ψ_a is bounded for each $a > 0$, $f(x)\psi_a(x-y)$ is absolutely integrable and the integral is uniformly bounded in $y \in \mathbf{R}$. Therefore, by an application of Fubini, a change of variable $x \to ax + y$ and dominated convergence theorem, in that order, it follows that

$$\begin{aligned}
\lim_{a\to 0}\int \mathrm{d}x f(x)(\tilde{\psi}_a * \mu)(x) &= \lim_{a\to 0}\int \mathrm{d}x f(x)\int \tilde{\psi}_a(x-y)\mathrm{d}\mu(y)\\
&= \lim_{a\to 0}\int \mathrm{d}\mu(y)\int f(x)\tilde{\psi}_a(x-y)\mathrm{d}x\\
&= \lim_{a\to 0}\int \mathrm{d}\mu(y)\int f(ax+y)\psi(x)\mathrm{d}x\\
&= \int \mathrm{d}\mu(y)\int \left(\lim_{a\to 0} f(ax+y)\right)\psi(x)\mathrm{d}x\\
&= \int f(y)\mathrm{d}\mu(y)\cdot\int \psi(x)\mathrm{d}x.
\end{aligned}$$

Part (2): This is a direct consequence of the definition of the integral noting pointwise that we have

$$\lim_{a\to 0}\psi_a(x) = \begin{cases} 0, & \text{if } x \neq 0,\\ 1, & \text{if } x = 0.\end{cases}$$

We also need to use the dominated convergence theorem to interchange the limit and the integral.

Part (3): Let Φ_μ denote the distribution function of μ. Then we have

$$\begin{aligned}\frac{1}{a^\alpha}\int_{\mathbf{R}}\psi_a(x-y)\mathrm{d}\mu(y) &= -\frac{1}{a^\alpha}\int_{\mathbf{R}}\frac{\mathrm{d}}{\mathrm{dy}}\psi((x-y)/a)\Phi_\mu(y)\mathrm{d}y\\ &= \frac{1}{a^\alpha}\int_{\mathbf{R}}\psi'(y)\Phi_\mu(x-ay)\mathrm{d}y\\ &= -\int_0^\infty \psi'(y)(2y)^\alpha\frac{\Phi_\mu(x+ay)-\Phi_\mu(x-ay)}{(2ay)^\alpha}\mathrm{d}y,\end{aligned} \tag{3.1}$$

where in the first step we used integration by parts, in the next step we used changed variables and in the last step we used the oddness of ψ' to split the integral into positive and negative half-lines and multiplied by appropriate powers.

We observe that

$$(d_\alpha\mu)(x) = \lim_{a\to 0}\frac{\Phi_\mu(x+ay)-\Phi_\mu(x-ay)}{(2ay)^\alpha}$$

for each $y\in\mathbf{R}$, and is finite by assumption. Furthermore, the function $(\Phi_\mu(x+ay)-\Phi_\mu(x-ay))(2ay)^{-\alpha}$ is a bounded measurable function, and due to (2.1) we can take the limits inside the integral sign in (3.1) and use the dominated convergence theorem.

Now doing an integration by parts gives the value of the integral as stated in the theorem.

Proof of Theorem 2.4.

Part (1): We have

$$\frac{1}{a}\int_c^d |\psi_a*\mu(x)|^2\mathrm{d}x = \iint \mathrm{d}\mu(y_1)\mathrm{d}\mu(y_2)\int_c^d \mathrm{d}x\frac{1}{a}\overline{\psi_a(x-y_1)}\psi_a(x-y_2).$$

Since the function ψ_a is bounded, the interval (c,d) is bounded, and μ is a probability measure, the right-hand side integral converges absolutely, so we used Fubini to interchange integrals to get the equality above. Let

$$h_a(y_1,y_2) = \int_c^d \mathrm{d}x\frac{1}{a}\overline{\psi_a(x-y_1)}\psi_a(x-y_2).$$

Suppose $y_1\neq y_2$, then using the bound $|\psi(x)|\le C\langle x\rangle^{-\delta}$, we see that the bound

$$\begin{aligned}|h_a(y_1,y_2)| &\le \frac{C}{a}\int_{-\infty}^\infty \langle (x+y_2-y_1)/a\rangle^{-\delta}\langle x/a\rangle^{-\delta}\mathrm{d}x\\ &= \frac{C}{a}\left(\int_{|x|\le|y_1-y_2|/2}+\int_{|x|\ge|y_1-y_2|/2}\right)(\cdots)\mathrm{d}x\\ &\le \frac{Ca^\delta}{|y_1-y_2|^\delta}\int_{-\infty}^\infty\langle x/a\rangle^{-\delta}\mathrm{d}(x/a)\\ &\le \frac{Ca^\delta}{|y_1-y_2|^\delta}\end{aligned}$$

is valid. It follows that $\lim_{a\to 0} h_a(y_1, y_2) = 0$ for $y_1 \neq y_2$. It remains to consider $y_1 = y_2$. This is done by noting that

$$h_a(y_1, y_1) = \int_c^d \frac{1}{a}|\psi_a(x - y_1)|^2 \mathrm{d}x = \int_{(c-y_1)/a}^{(d-y_1)/a} |\psi(x)|^2 \mathrm{d}x,$$

from which taking limits, we obtain the stated value for the coefficient, either C or $C/2$, based on whether $c < y_1 < d$ or $y_1 = c, d$, using the evenness of ψ. Now to complete the proof, we note the estimate

$$|h_a(y_1, y_2)| \leq C \int_{\mathbf{R}} \langle x/a\rangle^{-\delta} \mathrm{d}(x/a) \leq C_0,$$

where the constant C_0 is independent of a, y_1, and y_2. Thus the proof is completed using the dominated convergence theorem.
Part (2): We adapt the arguments in [3] to the case at hand. We split the measure in three components: $\mu = \mu_1 + \mu_2 + \mu_3$. Here $\mathrm{d}\mu_1 = (1 - \chi_{[c-1,d+1]})\mathrm{d}\mu$, $\mathrm{d}\mu_2 = g\mathrm{d}x$ with $g \in L^1([c-1, d+1])$, and μ_3 is purely singular, and supported on $[c-1, d+1]$. We have for $x \in [c, d]$ the estimate

$$|(\tilde{\psi}_a * \mu_1)(x)| \leq C \int_{\mathbf{R}\setminus[c-1,d+1]} a^{-1}\langle (x-y)/a\rangle^{-\delta} \mathrm{d}\mu_1(y) \leq Ca^{\delta-1}.$$

We now look at the μ_2 part. We have, for $0 < p < 1$, by the reverse Hölder inequality

$$\int_c^d |(\tilde{\psi}_a * g)(x) - A_\psi g(x)|^p \mathrm{d}x$$
$$\leq \left(\int_c^d |(\tilde{\psi}_a * g)(x) - A_\psi g(x)|\mathrm{d}x\right)^p (d-c)^{1-p},$$

which implies that $\tilde{\psi}_a * g \to A_\psi g$ in $L^p((c, d))$, $0 < p \leq 1$.

Now we will show that the singular part μ_3 does not contribute to the limit. So assume that μ_3 is purely singular and that its support S is contained in $[c-1, d+1]$. Since μ_3 is singular, by the definition of support, S satisfies $\mu_3(\mathbf{R}\setminus S) = 0$ and $|S| = 0$, with $|\cdot|$ denoting the Lebesgue measure. By the regularity of the Lebesgue measure, given an $\varepsilon > 0$, there is an open set $O \subset (c-2, d+2)$, such that $S \subset O$, with $|O\setminus S| < \varepsilon$. We also have $|O| \leq |O\setminus S| + |S| < \varepsilon$. For the same ε, since the measure μ_3 is regular, we also have a compact $K \subset S$, such that $\mu_3(S\setminus K) < \varepsilon$. In addition, since $K \subset S$, and S has Lebesgue measure zero, K also has Lebesgue measure zero.

The above reverse Hölder inequality gives

$$\begin{aligned}\int_c^d |(\tilde{\psi}_a * \mu_3)(x)|^p \mathrm{d}x &= \int_O |(\tilde{\psi}_a * \mu_3)(x)|^p \mathrm{d}x \\ &\quad + \int_{(c,d)\setminus O} |(\tilde{\psi}_a * \mu_3)(x)|^p \mathrm{d}x \\ &\leq |O|^{1-p}\mu_3((c,d))^p \|\psi\|_1^p \\ &\quad + |d-c|^{1-p}\left(\int_{(c,d)\setminus O} |(\tilde{\psi}_a * \mu_3)(x)|\mathrm{d}x\right)^p\end{aligned}$$

$$\leq C\varepsilon^{1-p} + |d-c|^{1-p}$$
$$\times \left(\int_{(c,d)\setminus O} |(\tilde{\psi}_a * \mu_3)(x)| \mathrm{d}x \right)^p .$$

Now consider a bounded continuous function h which is 1 on $(c,d)\setminus O$, and 0 on K.

Then using Assumption 2.1 that $|\psi(x)| \leq C\langle x\rangle^{-\delta}$, and setting $\phi(x) = \langle x\rangle^{-\delta}$,

$$\begin{aligned}\int_{(c,d)\setminus O} |(\tilde{\psi}_a * \mu_3)(x)|\mathrm{d}x &\leq \int_{(c,d)\setminus O} \frac{1}{a}\int_{\mathbf{R}} |\psi_a(x-y)|\mathrm{d}\mu_3(y)\mathrm{d}x \\ &\leq C\int_{(c,d)\setminus O} \frac{1}{a}\int_{\mathbf{R}} \langle (x-y)/a\rangle^{-\delta}\mathrm{d}\mu_3(y)\mathrm{d}x \\ &\leq C\int_{(c,d)\setminus O} h(x)(\tilde{\phi}_a * \mu_3)(x)\mathrm{d}x.\end{aligned}$$

The function ϕ satisfies Assumption 2.1, so Theorem 2.2(1) is applicable with ψ replaced by ϕ. Therefore the last term, which has positive integrand, converges to $\int_{(c,d)\setminus O} h(x)\mathrm{d}\mu(x)$ as a goes to zero, which is bounded by $\int_{(c,d)\setminus K} \mathrm{d}\mu(x)$,

$$\int_{(c,d)\setminus O} h(x)d\mu(x) \leq \mu((c,d)\setminus K) \leq \mu((c,d)\setminus S) + \mu(S\setminus K) < \varepsilon,$$

using the facts that $\mu((c,d)\setminus S) = 0$ and $\mu(S\setminus K) < \varepsilon$.

Using the inequality $(a+b+c)^p \leq a^p + b^p + c^p$ for $0 < p < 1$ and non-negative numbers a, b, c, we have

$$\begin{aligned}\int_c^d |(\tilde{\psi}_a * \mu)(x) - A_\psi g(x)|^p \mathrm{d}x \leq &\int_c^d |(\tilde{\psi}_a * \mu_1)(x)|^p \mathrm{d}x \\ &+ \int_c^d |(\tilde{\psi}_a * \mu_2)(x) - A_\psi g(x)|^p \mathrm{d}x \\ &+ \int_c^d |(\tilde{\psi}_a * \mu_3)(x)|^p \mathrm{d}x.\end{aligned}$$

Putting the above estimates together and using ε arbitrary, one gets

$$\lim_{a\to 0}\int_c^d |(\tilde{\psi}_a * \mu)(x) - A_\psi g(x)|^p \mathrm{d}x = 0.$$

Now the spaces $L^p((c,d))$, $0 < p < 1$, are metric spaces with the metric $d(f,g) = \|f-g\|_p^p$. It then follows from the triangle inequality for this metric that

$$\lim_{a\to 0}\int_c^d |(\tilde{\psi}_a * \mu)(x)|^p \mathrm{d}x = |A_\psi|^p \int_c^d |g(x)|^p \mathrm{d}x.$$

Since $g = \mathrm{d}\mu_{\mathrm{ac}}/\mathrm{d}x$, the result follows.

Proof of Theorem 2.6. Let $0 < \varepsilon < M < \infty$. It follows from (2.12) that we have

$$\int_\varepsilon^M W_h(\mu)(b,a)\frac{\mathrm{d}a}{a} = (\tilde{\psi}_\varepsilon * \mu)(b) - (\tilde{\psi}_M * \mu)(b).$$

The results now follow from Theorem 2.2. □

Proof of Theorem 2.8. The proof is entirely analogous to the proof of Theorem 2.4, replacing ψ by h and adjusting the powers of a. □

Proof of Theorem 2.9. Consider the case when $D_\mu^\alpha(x)$ is finite for some x and for some fixed α. Then for any $0 < y < 1$, $\mu(x - y, x + y) \leq C|y|^\alpha$ for some finite constant C. So, using the last line in eq. (3.1) and estimating the right-hand side, one has, by Assumption 2.1,

$$\left|\frac{1}{a^\alpha}(\psi_a * \mu)(x)\right| \leq C\int_0^\infty |\psi'(y)|(2y)^\alpha \mathrm{d}y \leq C\int_0^\infty \langle y\rangle^{-\delta}|y|^{-1+\alpha}\mathrm{d}y < \infty.$$

Now taking the lim sup of the left-hand side the finiteness of $C_{\mu,\psi}^\alpha$ follows.

On the other hand, since ψ is positive continuous with $\psi(0) = 1$, there is a $\beta > 0$ such that $\psi(y) > 1/2$, $-\beta < y < \beta$. Using this and the evenness of ψ,

$$\begin{aligned}\frac{1}{a^\alpha}(\psi_a * \mu)(x) &= \frac{1}{a^\alpha}\int \psi_a(x - y)\mathrm{d}\mu(y) = \int \psi(y/a)\mathrm{d}\mu(y + x)\\ &\geq \frac{1}{a^\alpha}\int_{-\beta a}^{\beta a} \frac{1}{2}\mathrm{d}\mu(y + x)\\ &\geq \frac{1}{2a^\alpha}[\mu(x + a\beta) - \mu(x - a\beta)],\end{aligned}$$

where $\psi \geq 0$ is used to get the first inequality. The above inequalities immediately imply that since β is fixed, $D_\mu^\alpha(x) = \infty$ implies the same for $C_{\mu,\psi}^\alpha(x)$. □

Proof of Theorem 2.11. Parts (1) and (2) are a direct application of Theorems 2.2(2) and (3) respectively. Parts (3) and (4) are a direct application of Corollaries 2.5(1) and (2) respectively.

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