Proc. Indian Acad. Sci. (Math. Sci.) Vol. 115, No. 2, May 2005, pp. 191–200.

L^p-continuity for Calderón–Zygmund operator

Q X YANG

Department of Mathematics, Wuhan University, 430072 Hubei, China
E-mail: yangqi99@public.wh.hb.cn

MS received 22 July 2004; revised 10 December 2004

Abstract. Given a Calderón–Zygmund (C–Z for short) operator T, which satisfies Hörmander condition, we prove that: if T maps all the characteristic atoms to WL^1, then T is continuous from L^p to $L^p(1 < p < \infty)$. So the study of strong continuity on arbitrary function in L^p has been changed into the study of weak continuity on characteristic functions.

Keywords. C–Z operator; characteristic atoms; WL^1; Hardy–Littlewood maximal operator; *-maximal operator.

1. Principal theorem

In this paper, $L^p(1 < p < \infty)$ continuity is obtained without assumption on L^2 continuity, but with a continuity which is much more weaker than the continuity from L^2 to $WL^2 - T$ is continuous from characteristic atoms to WL^1 and no information about its adjoint is assumed; and so, an analysis problem is changed into a geometric problem.

Let $B(u, t)$ be a ball with center u and radius t. A linear operator T, which is continuous from $S(R^n)$ to $S'(R^n)$, corresponds to a kernel distribution $K(x, y)$.

DEFINITION 1

One calls T a C–Z operator or $T \in HCZ$, if T satisfies the following four conditions:

(i) Size condition:

$$\sup_{x,r} \int_{r\le|x-y|\le 2r} \{|K(x, y)| + |K(y, x)|\}\mathrm{d}y < \infty. \tag{1.1}$$

(ii) Hörmander regular condition:

$$\sup_{x,x'} \int_{|x-y|\ge 2|x-x'|} \{|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')|\}\mathrm{d}y < \infty. \tag{1.2}$$

(iii) T1 condition:

$$T1 \in \mathrm{BMO}, \quad T^*1 \in \mathrm{BMO}. \tag{1.3}$$

(iv) Weak bounded condition:

$$|\langle Tf, g\rangle| \leq ct^n(\|f\|_\infty + t\|\nabla f\|_\infty)(\|g\|_\infty + t\|\nabla g\|_\infty),$$
$$\forall u \in R^n, t > 0, B(u,t), f, g \in C_0^1(B(u,t)). \tag{1.4}$$

Before, L^p continuity is often obtained under the assumption of L^2 continuity. Whether an operator in Definition 1.1 is continuous on $L^2(R^n)$ or not is a very difficult open problem (see [8]). In this paper, L^p continuity is obtained under a condition which is more weaker than L^2 continuity – we suppose only that the given operator is continuous from characteristic atoms to WL^1.

In this paper, $|F|$ denotes the measure of set F. Let E be a cube and let N_E be the biggest integer such that $2^{N_E}|E| \leq 1$. First, we introduce some definitions about atoms.

DEFINITION 2

(i) One calls $a(x)$ an atom on E or $a(x) \in A_1(E)$ or $a(x) \in A_1$, if $\operatorname{supp} a(x) \subset E$, $\|a(x)\|_\infty \leq 2^{N_E}$ and $\int a(x)\mathrm{d}x = 0$.
(ii) One calls $a(x)$ a characteristic atom or $a(x) \in A_1^0(E)$ or $a(x) \in A_1^0$, if there exist two sub-cubes G and H in E which do not intersect each other such that $a(x) = \frac{1}{|E|}(\chi_F(x) - \chi_G(x))$, where F is a subset of cube H and $|F| = |G|$.
(iii) One calls $a(x)$ a special atom or $a(x) \in A(E)$ or $a(x) \in A$, if $a(x) = \frac{1}{|E|}(\chi_F(x) - \chi_G(x))$, where F and G are two sub-cubes in E such that $|F \cap G| = 0$ and $|F| = |G|$.

In fact, $a(x) \in A_1$ is the usual ∞-atom in Hardy space, $a(x) \in A$ is defined first in [7] for Besov space $B_1^{0,1}$. The characteristic atom set A_1^0 is composed by characteristic functions and $A \subsetneqq A_1^0 \subsetneqq A_1$. Now we present a definition of Lorentz space WL^1.

DEFINITION 3

One calls $f(x) \in WL^1(E)$, if $\forall\lambda > 0, \lambda|\{x: |f(x)| > \lambda\} \cap E| < \infty$.

Whatever $E = R^n$ or not, sometimes, one denotes $WL^1(E) = WL^1$. It is known that WL^1 is not a Banach space and it is only a completed metric space, because its norm does not satisfy triangle inequality and this brings some difficulties in the study of continuity.

The principal theorem in this paper is the following theorem.

Theorem 1.1. *Given* $1 < p < \infty$. *If* $T \in HCZ$, *then the following two conditions are equivalent*:

$$T: L^p \to L^p. \tag{1.5}$$

$$T: A_1^0 \to WL^1. \tag{1.6}$$

The real analysis books discuss interpolation theorem (see [1,9]). Further, several years after I have proved our Theorem 1.1 (see [12]), somebody told me that Journé in 1983 proved in [6] the following theorem.

Theorem 1.2. $\forall 1 < p < \infty$, *if* $\|T\|_{H^1\to L^1} + \|T\|_{L^\infty\to \mathrm{BMO}} < \infty$, *then* $T: L^p \to L^p$.

Applying the above Theorem 1.2, the principal Theorem 1.1 can be decomposed to the following two theorems.

Theorem 1.3. *If* $T \in HCZ$, *then* (i) $T\colon A_1 \to WL^1$ *implies* $T\colon A_1 \to L^1$ *and* (ii) $T\colon A_1^0 \to WL^1$ *implies* $T\colon A_1^0 \to L^1$.

Theorem 1.4. *If* $T \in HCZ$, *then* $T\colon A_1^0 \to WL^1$ *implies* $T^*\colon A_1 \to WL^1$.

The proof for Theorems 1.3 and 1.4 will be given in §§3 and 4. Here, we apply the above three theorems to prove Theorem 1.1.

If $T \in HCZ$ and T satisfies condition (1.6), by applying Theorem 1.4, T^* satisfies condition (1.9) below, and hence T^* satisfies condition (1.6). Then one applies another time Theorem 1.4 for T^*, one gets $T, T^*\colon A_1 \to WL^1$.

Further, applying (i) of Theorem 1.3, one gets $T, T^*\colon A_1 \to L^1$. That is to say, $T, T^*\colon H^1 \to L^1$. Since the dual space of H^1 is BMO, one applies Theorem 1.2, $\forall 1 < p < \infty$, $T\colon L^p \to L^p$.

Furthermore, (1.5) implies that T satisfies (1.7) below, and so T satisfies (1.6). This finishes the proof of Theorem 1.1.

Remark 1.1. We indicate here, that so much work has been done [3–5,7,10,11] which deals with the continuity of C–Z operators since the famous $T1$ theorem of David and Journé [2]. Note that the following conditions are more and more weaker:

$$T\colon H^1 \to L^1. \tag{1.7}$$

$$T\colon L^1 \to WL^1. \tag{1.8}$$

$$T\colon A_1 \to WL^1. \tag{1.9}$$

$$T\colon A \to WL^1. \tag{1.10}$$

(1) A famous conjecture is: If $T \in HCZ$, then T is continuous from L^2 to L^2; Meyer [8] calls it C–Z conjecture on Hörmander condition. According to a famous result in [7], if $T \in HCZ$, then T satisfies condition (1.10); but we do not know, under the condition that $T \in HCZ$, whether (1.6) is stronger than (1.10) or not. Hence C–Z conjecture rests still open.
(2) The conclusion in Theorem 1.3 depends only on one single side Hörmander condition.
(3) As a C–Z operator T, it is known that L^2 continuity implies that T satisfies all the conditions from (1.5) to (1.10). In contrast, although Journé proved Theorem 1.2, in our Theorem 1.1, we do not suppose any condition on the adjoint operator and we suppose only a geometric condition (1.6) on the operator itself, which is weaker than condition (1.7) or condition (1.8). In fact, there is no characteristic atom decomposition for H^1, hence (1.6) is weaker than (1.9). Furthermore, WL^1 is not a Banach space, so (1.9) is weaker than (1.7) or (1.8).

There are too many constants in this paper, C may be different at each occurrence; but when a constant depends on some quantity, this constant will be specified.

2. Preliminaries

In this section, we establish some results about the relations among sets, WL^1 continuity, approximation of operators and maximal operators. Let M be Hardy–Littlewood maximal operator and $0 < \delta < 1$.

First, we know that Meyer has proved the following two lemmas in chapter 7 of [8]. For arbitrary Borel set B and $\forall f(x)$, denote $\theta = \sup_{\lambda>0} \lambda|\{x\colon |f(x)| > \lambda\} \cap B|$. Then one has:

Lemma 2.1. $\int_B |f(x)|^\delta \mathrm{d}x \le C(n,\delta)|B|^{1-\delta}\theta^\delta$.

Proof. Let $E_k = \{x \in B, |f(x)| > 2^k\}$ and k_B the biggest integer satisfying $2^k|B| \le \theta$. If $k < k_B$, then blow up $|E_k|$ to $|B|$; if $k \ge k_B$, then blow up $|E_k|$ to $2^{-k}\theta$. Hence one has

$$\int_B |f(x)|^\delta \mathrm{d}x \le C\sum_{-\infty}^{+\infty} 2^{k\delta}|E_k| \le C|B|\sum_{-\infty}^{k_B-1} 2^{k\delta} + C\theta\sum_{k_B}^{+\infty} 2^{-k(1-\delta)}$$
$$= C|B|2^{k_B\delta} + C\theta 2^{-k_B(1-\delta)} \le C(n,\delta)|B|^{1-\delta}\theta^\delta.$$

Lemma 2.2. *If* $E_N^\delta = \{x\colon M|f|^\delta(x) > 2^{N\delta}\}$, *then* $|E_N^\delta| \le C2^{-N\delta}\int_{E_N^\delta} |f(x)|^\delta \mathrm{d}x$.

Proof. Let $g(x) = f(x)|_{E_N^\delta}$, then

$$\{x\colon M|g|^\delta(x) > 2^{N\delta}\} = \{x\colon M|f|^\delta(x) > 2^{N\delta}\}.$$

Further, for arbitrary function g, one has $|\{x\colon M|g|^\delta(x) > 2^{N\delta}\}| \le C_n 2^{-N\delta}\|g\|_{L^\delta}^\delta$. Hence $|E_N^\delta| \le C2^{-N\delta}\int_{E_N^\delta} |f(x)|^\delta \mathrm{d}x$. □

For arbitrary ball or cube B, denote $\tilde{B}$ a ball or a cube with the same center and double diameter. Then one can find the following result in [7].

Lemma 2.3.

(i) $\forall T \in HCZ$, *there exists constant* $C, \forall u \in R^n$, *for arbitrary ball or cube* B *with center* u, *one has*: $\forall x \notin \tilde{B}$, $|T\chi_B(x)| \le \frac{C|B|}{|x-u|}$.
(ii) *If an operator* T *satisfies conditions* (1.1) *and* (1.2), *then the following two conditions are equivalent:*

$$T \text{ satisfies conditions (1.3) and (1.4).} \tag{2.1}$$

$$\|T\chi_B(x)\|_{L^1(\tilde{B})} + \|T^*\chi_B(x)\|_{L^1(\tilde{B})} \le C|B|, \forall \text{ ball or cube } B. \tag{2.2}$$

Now, let us consider the action of approximation operators on atoms. Let $M_\delta f(x) = (M|f|^\delta(x))^{1/\delta}$ and let $K_\varepsilon(x,y) = K(x,y)|_{|x-y|\ge\varepsilon}$ be the kernel distribution of T_ε. Then, one has

Lemma 2.4. *If* $T \in HCZ$, *then for arbitrary cube* E, *we have*

(i) $\forall a(x) \in A(E)$, *one has*

$$|T_\varepsilon a(x)| \le CM_\delta Ta(x) + C2^{N_E}. \tag{2.3}$$

(ii) $\forall a(x) \in A_1(E)$ *and* T *satisfies condition* (1.9), (2.3) *is still true.*
(iii) $\forall a(x) \in A_1^0(E)$ *and* T *satisfies condition* (1.6), (2.3) *is still true.*

Proof. The proof of the above three conclusions in Lemma 2.4 is similar, and so one proves only (ii). Let B_ε be the ball $\{s\colon |x-s| \le \frac{\varepsilon}{2}\}$ and $\tilde{B}_\varepsilon$ the ball with the same center and twice the radius and let $\tilde{B}_\varepsilon^c = R^n \backslash \tilde{B}_\varepsilon$. Then we define $f_2(x)$ as follows: (a) f_2 is zero on $\tilde{B}_\varepsilon$ and (b) $f_2(s) = a(s)$ outside $\tilde{B}_\varepsilon$. Further, we decompose $a - f_2$ into two functions f_1 and f_I: f_1 is a function whose support is on $\tilde{B}_\varepsilon$ and whose integral is zero, f_I is a constant function on $\tilde{B}_\varepsilon$ and zero outside it. Hence $a(x)$ is decomposed into three functions and $a = f_1 + f_I + f_2$.

Since $|f_2| \le C2^{N_E}$ and $|Tf_2(s) - Tf_2(x)| = |\int_{|x-y|\ge\varepsilon} K(x,y)a(y)\mathrm{d}y - \int_{|x-y|\ge\varepsilon} K(s,y)a(y)\mathrm{d}y| \le \int_{|x-y|\ge\varepsilon} |K(x,y) - K(s,y)||a(y)|\mathrm{d}y$, according to Hörmander condition (1.2), $\forall s \in B_\varepsilon$, one has: $|Tf_2(x) - Tf_2(s)| \le C2^{N_E}$. Since $Ta(s) = Tf_1(s) + Tf_I(s) + Tf_2(s)$, then: $|Tf_2(x)| \le |Ta(s)| + |Tf_1(s)| + |Tf_I(s)| + C2^{N_E}$.

One makes δ order integration for s on B_ε and gets

$$|Tf_2(x)| \le CM_\delta Ta(x) + C\varepsilon^{-(n/\delta)}\left(\int_{B_\varepsilon} |Tf_1|^\delta \mathrm{d}s\right)^{1/\delta} + C\varepsilon^{-(n/\delta)}\left(\int_{B_\varepsilon} |Tf_I|^\delta \mathrm{d}s\right)^{1/\delta} + C2^{N_E}. \tag{2.4}$$

Applying Lemma 2.3, one gets $\|Tf_I\|_{WL^1(\tilde{B}_\varepsilon)} \le C\varepsilon^n 2^{N_E}$. Since $\varepsilon^{-n}2^{-N_E} f_1$ is an atom, by assumption, $\|Tf_1\|_{WL^1} \le C\varepsilon^n 2^{N_E}$. Applying then Lemma 2.1, one gets

$$\int_{B_\varepsilon} |Tf_1|^\delta \mathrm{d}s + \int_{B_\varepsilon} |Tf_I|^\delta \mathrm{d}s \le C|\varepsilon|^{(1-\delta)n}\varepsilon^{n\delta}2^{\delta N_E} = C\varepsilon^n 2^{\delta N_E}. \tag{2.5}$$

By (2.4) and (2.5), we have $|Tf_2(x)| \le CM_\delta Ta(x) + C2^{N_E}$.

Since $Tf_2(x) = T_\varepsilon a(x)$, one gets (2.3).

In the end of this section, we consider $*$-maximal operator of T. Let $T_* a(x) = \sup_{\varepsilon>0} |T_\varepsilon a(x)|$, then one has:

Lemma 2.5. *If* $T \in HCZ$, *then for arbitrary cube* E, *one has*:

(i) *If* T *is satisfying condition* (1.9), *then there exists constant* C, $\forall a(x) \in A_1(E)$, *and one has*:

$$|Ta(x)| \le |T_* a(x)| + C2^{-N_E}. \tag{2.6}$$

(ii) *If* T *is satisfying condition* (1.6), *then there exists constant* C, $\forall a(x) \in A_1^0(E)$, *and* (2.6) *is still true.*

Proof. The proof of (ii) is similar to that of (i), so one proves only (i). If $T \in HCZ$, then applying Lemmas 2.3 and 2.4, the operators $\{T_\varepsilon\}_\varepsilon$ satisfy the condition (2.2) uniformly. Then there exist an operator T_0 satisfying (1.1), (1.2) and (2.2), and a subsequence ε_j

which converges to 0 such that $T_{\varepsilon_j} \to T_0$ in the sense of norm of (1.1), (1.2) and (2.2). According to Lemma 2.3, $T_0 \in HCZ$.

Further, choosing two test functions such that their supports are disjoint, one knows that the kernel distribution of $T - T_0$ vanishes out of the diagonal $x = y$. Then there exists a L^∞ function $m(x)$ such that, for arbitrary cube E and F and for arbitrary function $f(x) \in L^\infty(F)$, one has

$$\langle Tf(x), \chi_E(x)\rangle = \lim\langle T_{\varepsilon_j} f(x), \chi_E(x)\rangle + \langle m(x)f(x), \chi_E(x)\rangle. \tag{2.7}$$

Finally, if $T \in HCZ$ is satisfying condition (1.9), by (2.7), there exists constant C such that, for arbitrary cube E and $\forall a(x) \in A_1(E)$, (2.6) is true.

3. Upgrade of regularity

As to a C–Z operator, usually, $H^1 \to L^1$ continuity is obtained by L^2 continuity; but the L^2 continuity is often established by a fixed decomposition (continuous or discrete) of the operator. But when the regularity of $K(x, y)$ is weakened to Hörmander condition (1.2), it is difficult to revert to the operator itself from the operators which have been decomposed [8,11,12]. Further, when we try to establish operator's continuity under wavelet basis, the continuity from H^1 to L^1 often needs a much stronger weak regularity than from L^2 to L^2 [11]. In this section, one proves Theorem 1.3 through upgrading WL^1 continuity to L^1 continuity. Since the proofs for (i) and (ii) are similar, one proves only (i).

Now one proves (i) of Theorem 1.3 in three steps.

(1) $\forall$ atom $a(x)$ on cube E, $\forall N \geq N_E$, the following inequality is true:

$$2^N|\{T_* a(x) > 2^N\}| \leq C. \tag{3.1}$$

Applying Lemma 2.4, one has

$$|T_\varepsilon a(x)| \leq CM_\delta Ta(x) + C2^{N_E}.$$

That is to say, $|T_* a(x)| \leq CM_\delta Ta(x) + C2^{N_E}$.

Let $E_N^\delta = \{x\colon M|Ta|^\delta(x) > 2^{N\delta}\}$. According to Lemma 2.2, one has

$$|E_N^\delta| \leq C2^{-N\delta}\int_{E_N^\delta} |Ta(x)|^\delta \mathrm{d}x.$$

Since $\|Ta(x)\|_{WL^1} \leq C$, applying Lemma 2.1, one gets

$$\int_{E_N^\delta} |Ta(x)|^\delta \mathrm{d}x \leq C(n, \delta)|E_N^\delta|^{1-\delta}.$$

Combining the last two inequalities, one gets $2^N|E_N^\delta| \leq C$. So (3.1) is true.

(2) There exists a constant C, $\forall$ atom $a(x)$ on cube E and $\forall N > N_E$, the following inequality is true:

$$|\{x\colon T_* a(x) > 2^{N+2}\}| \leq C2^{N_E - N}|\{x\colon T_* a(x) > 2^N\}|. \tag{3.2}$$

$\forall x \in R^n$ and let $\|x\|$ be the maximum value of the coordinates. Let $E_N = |\{x\colon T_* a(x) > 2^N\}|$. Then there exist the biggest dyadic cubes E_N^l such that $|E_N^l \cap$

$E_N^{l'}| = 0$ for $l \neq l'$ and $E_N = \cup_l E_N^l$. Let β_l be the center of E_N^l and d_l the length of E_N^l and let $\tilde{E}_N^l$ be the dyadic cube which contains E_N^l with double length. Hence there exists a point α_l in $\tilde{E}_N^l$, but $\alpha_l \notin E_N$. Hence $|T_* a(\alpha_l)| \leq 2^N$ and $\|\alpha_l - \beta_l\| \leq \frac{3\sqrt{2}}{2} d_l$. Let F_N^l be the set of x belonging to E_N^l such that $|T_* a(x)| > 2^{N+2}$. In order to prove (3.2), it is sufficient to prove

$$|F_N^l| \leq C 2^{N_E - N} |E_N^l|, \quad \forall N > C + N_E. \tag{3.3}$$

Fix l, denote by $E_N^{l,*}$ the cube with the same center but with 12 times diameter of E_N^l. Let $a_N^l = \frac{1}{|E_N^{l,*}|} \int_{E_N^{l,*}} a(x)\mathrm{d}x$. To prove (3.3), one decomposes $a(x)$ into three functions: $a_p(x) = a_N^l \chi_{E_N^{l,*}}(x)$, $a_1(x) = (a(x) - a_N^l)\chi_{E_N^{l,*}}(x)$ and $a_c = a(x)(1 - \chi_{E_N^{l,*}}(x))$. Let $B_x = B(x, \beta_l, \varepsilon, d_l) = \{y\colon |x - y| \geq \varepsilon \text{ and } \|y - \beta_l\| \geq 6d_l\}$. For $u \in R^n$ and $t > 0$, let $B(u,t) = \{y\colon |y - u| < t\}$ and $B(u,t)^c = \{y\colon |y - u| \geq t\}$.

For $a_c(x)$, one has

$$\begin{aligned}
|T_* a_c(x) - T_* a_c(\alpha_l)| &\leq \sup_{\varepsilon>0} |T_\varepsilon a_c(x) - T_\varepsilon a_c(a_l)| \\
&= \sup_{\varepsilon>0} \left| \int_{B(x,\beta_l,\varepsilon,d_l)} K(x,y)a(y)\mathrm{d}y \right. \\
&\quad \left. - \int_{B(\alpha_l,\beta_l,\varepsilon,d_l)} K(\alpha_l,y)a(y)\mathrm{d}y \right| \\
&= \sup_{\varepsilon>0} I^\varepsilon.
\end{aligned}$$

$\forall x \in E_N^l$, one considers two cases:

(†) If $|\varepsilon| \leq (6 - \frac{3\sqrt{2}}{2})d_l$, then $B_x = B_{\alpha_l} = \{y\colon \|y - \beta_l\| \geq 6d_l\}$. Hence $I^\varepsilon = |\int (K(x,y) - K(\alpha_l,y))a_c(y)\mathrm{d}y| \leq \int |K(x,y) - K(\alpha_l,y)||a_c(y)|\mathrm{d}y$. Applying Hörmander condition (1.2), one gets $|I^\varepsilon| \leq C 2^{N_E}$.

(‡) If $|\varepsilon| > (6 - \frac{3\sqrt{2}}{2})d_l$, then the symmetry difference of B_x and B_{α_l} is $SD = B_1 \cup B_2$ where $B_1 = B(x,\varepsilon)^c \cap B(\beta_l, 6d_l) \cap B(\alpha_l, \varepsilon)$ and $B_2 = B(\alpha_l,\varepsilon)^c \cap B(\beta_l, 6d_l)$ $\cap B(x,\varepsilon)$. Hence

$$\begin{aligned}
I^\varepsilon &\leq \left| \int_{B_x} (K(x,y) - K(\alpha_l,y))a(y)\mathrm{d}y \right| + \int_{SD} |K(\alpha_l,y)||a(y)|\mathrm{d}y \\
&= I_1 + I_2.
\end{aligned}$$

As to I_1, let $C_1 = \{y\colon |x-y| \geq 2|x-\alpha_l|\}$ and $C_2 = \{y\colon (\frac{3\sqrt{2}}{2} - \frac{3}{4})|x-\alpha_l| \leq |x-y| \leq 2|x-\alpha_l|\}$. Then we have $B_x \subset C_1 \cup C_2$ and $I_1 \leq \int_{C_1} |K(x,y) - K(\alpha_l,y)||a(y)|\mathrm{d}y + \int_{C_2} |K(x,y) - K(\alpha_l,y)||a(y)|\mathrm{d}y$. We apply Hörmander condition (1.2) to the first part and apply size condition (1.1) to the second part, we get $I_1 \leq C2^{N_E}$. As to I_2, we apply simply size condition (1.1), and get $I_2 \leq C2^{N_E}$. Hence one has $|I^\varepsilon| \leq C2^{N_E}$.

Combining the conclusions in the above two cases (†) and (‡), one has

$$|T_* a_c(x) - T_* a_c(\alpha_l)| \leq C 2^{N_E}. \tag{3.4}$$

Now one proves

$$|T_*a_c(\alpha_l)| \le |T_*a(\alpha_l)| + C2^{N_E}. \tag{3.5}$$

First, according to the definition of $T_*a_c(\alpha_l)$, one has

$$T_*a_c(\alpha_l) = \sup_{\varepsilon>0}\left|\int_{\substack{|\alpha_l - y| \ge \varepsilon \\ \|y-\beta_l\| \ge 12d_l}} K(\alpha_l, y)a(y)\mathrm{d}y\right| = \sup_{\varepsilon>0} I^\varepsilon.$$

When $\varepsilon \ge d_l$, one has

$$I^\varepsilon \le \left|\int_{\substack{|\alpha_l - y| \ge \varepsilon \\ \|y-\beta_l\| \ge 12d_l}} K(\alpha_l, y)a(y)\mathrm{d}y - \int_{|\alpha_l - y|\ge\varepsilon} K(\alpha_l, y)a(y)\mathrm{d}y\right| + T_*a(\alpha_l).$$

Reasoning as above, one has, $I^\varepsilon \le T_*a(\alpha_l) + C2^{N_E}$.

When $\varepsilon < d_l$, one has

$$I^\varepsilon = \left|\int_{\|\beta_l - y\|\ge 8d_l} K(\alpha_l, y)a(y)\mathrm{d}y\right|.$$

Reasoning as above, one has, $I^\varepsilon \le T_*a(\alpha_l) + C2^{N_E}$. Hence (3.5) is true.

Combining (3.4) and (3.5), one gets

$$\forall x \in E_N^l, \quad |T_*a_c(x)| \le 2^N + C2^{N_E}. \tag{3.6}$$

According to Lemma 2.3, the WL^1 norm of $Ta_p(x)$ is not greater than $C2^{N_E}|E_N^l|$. According to the assumption (i) of Theorem 1.3, the WL^1 norm of $Ta_1(x)$ is not greater than $C2^{N_E}|E_N^l|$. Applying (3.1), $\forall N \ge N_E$, the following inequality is true:

$$2^N|\{x\colon T_*(a_1 + a_p)(x) > 2^N\}| \le C2^{N_E}|E_N^l|. \tag{3.7}$$

Let $E_l' = \{x \in E_N^l, T_*(a_1 + a_p)(x) > 2^{N+1}\}$, then $|E_l'| \le C2^{N_E - N}|E_N^l|$. Since $2^{N+1} > C2^{N_E}|E_N^l|$, by (3.6), $F_N^l \subset E_l'$, hence (3.3) is true.

(3) Repeat applying (3.2), one gets

$$\sum_{N\ge C+N_E} 2^N|\{x\colon T_*a(x) > 2^N\}| < \infty.$$

Furthermore, $\forall N \ge C + N_E$, and applying Lemma 2.5, one gets

$$\{x\colon |Ta(x)| > 2^N\} \subset \{x\colon T_*a(x) > 2^{N+1}\}. \tag{3.8}$$

Hence one gets

$$\sum_{N\ge C+N_E} 2^N|\{x\colon Ta(x) > 2^N\}| < \infty. \tag{3.9}$$

Let E^d be the cube which has the same center as E but whose edge has double length and $E^c = R^n\backslash E^d$. Applying then Hörmander condition (1.2), $Ta(x) \in L^1(E^c)$ one gets

$$\sum_{N< C+N_E} 2^N|\{x\colon Ta(x) > 2^N\}| < \infty. \tag{3.10}$$

By (3.9) and (3.10), $Ta(x) \in L^1$. Hence $T\colon A_1 \to L^1$.

4. From characteristic atom to atom

For arbitrary cube E_N, let E_N^k be the cube with the same center as E_N but with 2^k times length. In this section, one proves Theorem 1.4 by contradiction.

If Theorem 1.4 is not true, then $\forall N > 0$, there exists cube E_N, atom $a_N(x) \in A_1(E_N)$ and integer S_N such that $2^{S_N}|F_N| \geq N$, where $F_N = \{T^* a_N(x) > 2^{S_N}\} \cap E_N^1$. By contradiction hypothesis,

$$|\langle T^* a_N(x), \chi_{F_N}(x)\rangle| \geq N. \tag{4.1}$$

One chooses a cube G_N which is contained in $E_N^3 - E_N^2$ and with the same measure of F_N. Let $b_N(x) = \frac{1}{|E_N|}(\chi_{G_N}(x) - \chi_{F_N}(x))$. Applying then condition (1.1), one has

$$|\langle T^* a_N(x), \chi_{G_N}(x)\rangle| \leq C. \tag{4.2}$$

Hence one has

$$|\langle T^* a_N(x), b_N(x)\rangle| \geq \frac{N - C}{|E_N|}. \tag{4.3}$$

According to the assumption of Theorem 1.4, $Tb_N(x) \in WL^1$. Applying then (ii) of Theorem 1.3, one gets that $Tb_N(x) \in L^1$. Hence one has

$$|\langle Tb_N(x), a_N(x)\rangle| \leq C\|a_N(x)\|_\infty.$$

Since $|\langle T^* a_N(x), b_N(x)\rangle| = |\langle Tb_N(x), a_N(x)\rangle|$, then we have

$$|\langle T^* a_N(x), b_N(x)\rangle| \leq C\|a_N(x)\|_\infty. \tag{4.4}$$

Since $\|a_N(x)\|_\infty |E_N| \leq C$, then there is a contradiction between (4.3) and (4.4).

Acknowledgements

This research is supported by the NSFC of China (No. 10001027), the innovation fund of Wuhan University and the subject construction fund of Mathematics and Statistic School, Wuhan University.

References

[1] Cheng M D, Deng D G and Long R L, Real analysis (in Chinese) (Beijing: High education publish house) (1993) pp. 271–279

[2] David G and Journé J L, A boundedness criterion for generalized Calderón–Zygmund operators, *Ann. Math.* **120** (1984) 371–397

[3] Deng D G, Yan L X and Yang Q X, Blocking analysis and T(1) theorem, *Science in China* **41** (1998) 801–808

[4] Deng D G, Yan L X and Yang Q X, On Hörmander condition, *Chin. Sci. Bull.* **42** (1997) 1341–1345

[5] Han Y S and Hofman S, T1 Theorem for Besov and Triebel–Lizorkin spaces, *Transaction of the American Mathematical Society* **337** (1993) 839–853

[6] Journé J L, *Lecture Notes in Math.* **994** (1983) 43–45

[7] Meyer Y, La minimalité de l'espace de Besov $B_1^{0,1}$ et la continuité des opérateurs definis par des intégrales singulières, *Monografias de Matematicas* (Universidad autónoma de Madrid) Vol. 4

[8] Meyer Y, Ondelettes et opérateurs, I et II (Paris: Hermann) (1991–1992)

[9] Stein E M, Harmonic analysis–real variable methods, orthogonality, and integrals (Princeton University Press) (1993)

[10] Yabuta K, Generalizations of Calderón–Zygmund operators, *Studia Math.* **82** (1985) 17–31

[11] Yang Q X, Decomposition in blocks at the level of coefficients and T(1) Theorem on Hardy space, *J. Zhejiang University Science* **1** (2002) 94–99

[12] Yang Q X, Wavelet and distribution (Beijing: Beijing Science and Technology Press) (2002)