Proc. Indian Acad. Sci. (Math. Sci.) Vol. 115, No. 2, May 2005, pp. 167–190.

Some BMO estimates for vector-valued multilinear singular integral operators

LIU LANZHE

College of Mathematics and Computer, Changsha University of Science and Technology, Changsha 410077, People's Republic of China
E-mail: lanzheliu@263.net

MS received 4 April 2004; revised 19 December 2004

Abstract. In this paper, we prove some BMO end-point estimates for some vector-valued multilinear operators related to certain singular integral operators.

Keywords. Vector-valued multilinear operator; singular integral operators; BMO space.

1. Introduction and notations

Let $b \in \mathrm{BMO}(R^n)$ and T be the Calderón–Zygmund singular integral operator. The commutator $[b, T]$ generated by b and T is defined as $[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x)$. By using a classical result of Coifman *et al* [8], we know that the commutator $[b, T]$ is bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo [1] proves a similar result when T is replaced by the fractional integral operator. However, it was observed that the commutator is not bounded, in general, from $H^p(R^n)$ to $L^p(R^n)$ for $0 < p \le 1$ [13–15]. In [11], the boundedness properties of the commutator for the extreme values of p are obtained. Also, in [2], Chanillo studies some commutators generated by a very general class of pseudo-differential operators and proves the boundedness on $L^p(R^n)$ $(1 < p < \infty)$ for the commutators, and note that the conditions on the kernel of the singular integral operator arise from a pseudo-differential operator. As the development of singular integral operators and their commutators, multilinear singular integral operators have been well-studied. It is known that multilinear operator, as a non-trivial extension of the commutator, is of great interest in harmonic analysis and has been widely studied by many authors [3–7]. In [9], the weighted $L^p(p > 1)$-boundedness of the multilinear operator related to some singular integral operators is obtained and in [3], the weak (H^1, L^1)-boundedness of the multilinear operator related to some singular integral operators is obtained. The main purpose of this paper is to establish the BMO end-point estimates for some vector-valued multilinear operators related to certain singular integral operators.

First, let us introduce some notations [10,16]. Throughout this paper, $Q = Q(x, r)$ will denote a cube of R^n with sides parallel to the axes and centered at x and having side length. For a locally integrable function f and non-negative weight function w, let $w(Q) = \int_Q w(x)\mathrm{d}x$, $f_{w,Q} = w(Q)^{-1}\int_Q f(x)w(x)\mathrm{d}x$ and $f^{\#}(x) = \sup_{x\in Q} w(Q)^{-1}\int_Q |f(y) - f_{w,Q}|w(x)\mathrm{d}y$. f is said to belong to BMO(w) if $f^{\#} \in L^{\infty}(w)$ and define $\|f\|_{\mathrm{BMO}(w)} = \|f^{\#}\|_{L^{\infty}(w)}$. We denote $\mathrm{BMO}(w) = \mathrm{BMO}(R^n)$ and $\|f\|_{\mathrm{BMO}} = \|f^{\#}\|_{L^{\infty}}$

if $w = 1$. It is well-known that [12]

$$\|f\|_{\mathrm{BMO}(w)} \approx \sup_{Q} \inf_{c \in C} w(Q)^{-1} \int_{Q} |f(x) - c| w(x) \mathrm{d}x.$$

We also define the weighted central BMO space by CMO(w), which is the space of those functions $f \in L_{\mathrm{loc}}(R^n)$ such that

$$\|f\|_{\mathrm{CMO}(w)} = \sup_{d>1} w(Q(0,d))^{-1} \int_{Q} |f(y) - f_{w,Q}| w(y) \mathrm{d}y < \infty.$$

We denote the Muckenhoupt weights by A_p for $1 \leq p < \infty$ [10], that is,

$$A_p = \left\{ 0 < w \in L^1_{\mathrm{loc}}(R^n) \colon \sup_{Q} \left(\frac{1}{|Q|} \int_{Q} w(x) \mathrm{d}x \right) \right.$$
$$\left. \times \left(\frac{1}{|Q|} \int_{Q} w(x)^{-1/(p-1)} \mathrm{d}x \right)^{p-1} < \infty \right\}, \quad 1 < p < \infty,$$

$$A_1 = \left\{ 0 < w \in L^1_{\mathrm{loc}}(R^n) \colon \sup_{x \in Q} \frac{w(Q)}{|Q|} \leq C w(x), \text{a.e.} \right\}$$

and

$$A_\infty = \bigcup_{1 \leq p < \infty} A_p.$$

DEFINITION

(1) Let $0 < \delta < n$ and $1 < p < n/\delta$. We shall call $B^{\delta}_{p}(R^n)$ the space of those functions f on R^n such that

$$\|f\|_{B^{\delta}_{p}} = \sup_{r>1} r^{-n(1/p - \delta/n)} \|f \chi_{Q(0,r)}\|_{L^p} < \infty.$$

(2) Let $1 < p < \infty$ and w be a non-negative weight function on R^n. We shall call $B_p(w)$ the space of that function f on R^n such that

$$\|f\|_{B_p(w)} = \sup_{r>1} [w(Q(0,r))]^{-1/p} \|f \chi_{Q(0,r)}\|_{L^p(w)} < \infty.$$

2. Theorems

In this paper, we will study a class of vector-valued multilinear operators related to some singular integral operators, whose definitions are the following.

Fix $\varepsilon > 0$ and $\delta \geq 0$. Let $T \colon S \to S'$ be a linear operator and there exists a locally integrable function $K(x,y)$ on $R^n \times R^n \setminus \{(x,y) \in R^n \times R^n \colon x = y\}$ such that

$$T_\delta(g)(x) = \int_{R^n} K(x,y) g(y) \mathrm{d}y$$

for every bounded and compactly supported function g, where K satisfies:

$$|K(x,y)| \le C|x-y|^{-n+\delta}$$

and

$$|K(y,x)-K(z,x)|+|K(x,y)-K(x,z)| \le C|y-z|^{\varepsilon}|x-z|^{-n-\varepsilon+\delta}$$

if $2|y-z| \le |x-z|$. Let m_j be the positive integers $(j=1,\dots,l)$, $m_1+\cdots+m_l=m$ and A_j be the functions on R^n $(j=1,\dots,l)$. For $1<r<\infty$, the vector-valued multilinear operator associated with T is defined as

$$|T_\delta^A(f)(x)|_r = \left(\sum_{i=1}^{\infty} |T_\delta^A(f_i)(x)|^r\right)^{1/r},$$

where

$$T_\delta^A(f_i)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j;x,y)}{|x-y|^m} K(x,y) f_i(y)\,\mathrm{d}y$$

and

$$R_{m_j+1}(A_j;x,y) = A_j(x) - \sum_{|\alpha|\le m_j} \frac{1}{\alpha!} D^\alpha A_j(y)(x-y)^\alpha.$$

Set

$$|T_\delta(f)(x)|_r = \left(\sum_{i=1}^{\infty} |T(f_i)(x)|^r\right)^{1/r} \quad \text{and} \quad |f|_r = \left(\sum_{i=1}^{\infty} |f_i(x)|^r\right)^{1/r}.$$

We write $T_\delta = T$, $|T_\delta|_r = |T|_r$ and $|T_\delta^A|_r = |T^A|_r$ if $\delta = 0$.

Note that when $m=0$, T_δ^A is just the multilinear commutators of T_δ and A [13–15]. In this paper, we will prove the BMO estimates for the vector-valued multilinear operators $|T_\delta^A|_r$ and $|T^A|_r$.

Now we state our results as follows.

Theorem 1. *Let* $1<r<\infty, 0<\delta<n, 1<p<n/\delta$ *and* $D^\alpha A_j \in \mathrm{BMO}(R^n)$ *for all* α *with* $|\alpha| = m_j$ *and* $j=1,\dots,l$. *Suppose that* $|T_\delta|_r$ *maps* $L^s(R^n)$ *continuously into* $L^t(R^n)$ *for any* $s,t \in (1,+\infty]$ *with* $1<s<n/\delta$ *and* $1/t = 1/s - \delta/n$. *Then*

(a) $|T_\delta^A|_r$ *maps* $L^{n/\delta}(R^n)$ *continuously into* $\mathrm{BMO}(R^n)$, *that is*

$$\||T_\delta^A(f)|_r\|_{\mathrm{BMO}} \le C\||f|_r\|_{L^{n/\delta}}.$$

(b) $|T_\delta^A|_r$ *maps* $B_p^\delta(R^n)$ *continuously into* $\mathrm{CMO}(R^n)$, *that is*

$$\||T_\delta^A(f)|_r\|_{\mathrm{CMO}} \le C\||f|_r\|_{B_p^\delta}.$$

Theorem 2. *Let* $1<r<\infty, 1<p<\infty$ *and* $D^\alpha A_j \in \mathrm{BMO}(R^n)$ *for all* α *with* $|\alpha| = m_j$ *and* $j=1,\dots,l$.

(i) *If* $w \in A_\infty$ *and that* $|T|_r$ *is bounded on* $L^s(w)$ *for any* $1 < s \le \infty$ *and* $w \in A_\infty$, *then* $|T^A|_r$ *maps* $L^\infty(w)$ *continuously into* BMO(w), *that is,*

$$\||T^A(f)|_r\|_{\mathrm{BMO}(w)} \le C\||f|_r\|_{L^\infty(w)};$$

(ii) *If* $w \in A_1$ *and that* $|T|_r$ *is bounded on* $L^s(w)$ *for any* $1 < s \le \infty$ *and* $w \in A_1$, *then* $|T^A|_r$ *maps* $B_p(w)$ *continuously into* CMO(w), *that is,*

$$\||T^A(f)|_r\|_{\mathrm{CMO}(w)} \le C\||f|_r\|_{B_p(w)}.$$

3. Proofs of theorems

To prove the theorems, we need the following lemmas.

Lemma 1 [6]. *Let* A *be a function on* R^n *and* $D^\alpha A \in L^q(R^n)$ *for all* α *with* $|\alpha| = m$ *and some* $q > n$. *Then*

$$|R_m(A; x, y)| \le C|x-y|^m \sum_{|\alpha|=m} \left(\frac{1}{|\tilde{Q}(x,y)|}\int_{\tilde{Q}(x,y)} |D^\alpha A(z)|^q \mathrm{d}z\right)^{1/q},$$

where $\tilde{Q}(x, y)$ *is the cube centered at* x *and having side length* $5\sqrt{n}|x-y|$.

Lemma 2. *Let* $w \in A_\infty$, *then* BMO$(w) =$ BMO(R^n).

The proof of the lemma follows from [12] and the John–Nirenberg Lemma for BMO [10].

Proof of Theorem 1(*a*). It is only to prove that there exists a constant C_Q such that

$$\frac{1}{|Q|}\int_Q ||T_\delta^A(f)(x)|_r - C_Q|\mathrm{d}x \le C\||f|_r\|_{L^{n/\delta}}$$

holds for any cube Q. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$. Let $\tilde{Q} = 5\sqrt{n}Q$ and $\tilde{A}_j(x) = A_j(x) - \sum_{|\alpha|=m_j} \frac{1}{\alpha!}(D^\alpha A_j)_{\tilde{Q}} x^\alpha$, then $R_{m_j}(A_j; x, y) = R_{m_j}(\tilde{A}_j; x, y)$ and $D^\alpha \tilde{A}_j = D^\alpha A_j - (D^\alpha A_j)_{\tilde{Q}}$ for $|\alpha| = m_j$. We split $f = g + h = \{g_i\} + \{h_i\}$ for $g_i = f_i\chi_{\tilde{Q}}$ and $h_i = f_i\chi_{R^n\setminus\tilde{Q}}$. Write

$$\begin{aligned}
T_\delta^A(f_i)(x) &= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^m} K(x,y) f_i(y)\mathrm{d}y \\
&= \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j+1}(\tilde{A}_j; x, y)}{|x-y|^m} K(x,y) h_i(y)\mathrm{d}y \\
&\quad + \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m} K(x,y) g_i(y)\mathrm{d}y \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!}\int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1}\tilde{A}_1(y) K(x,y) g_i(y)\mathrm{d}y
\end{aligned}$$

$$- \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{A}_2(y) K(x, y) g_i(y) \mathrm{d}y$$

$$+ \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!}$$

$$\times \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{A}_1(y) D^{\alpha_2} \tilde{A}_2(y)}{|x-y|^m} K(x, y) g_i(y) \mathrm{d}y,$$

then, by the Minkowski's inequality,

$$\frac{1}{|Q|} \int_Q ||T_\delta^A(f)(x)|_r - |T_\delta^{\tilde{A}}(h)(x_0)|_r| \mathrm{d}x$$

$$\leq \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} |T_A(f_i)(x) - T_{\tilde{A}}(h_i)(x_0)|^r \right)^{1/r} \mathrm{d}x$$

$$\leq \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left| \int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m} K(x, y) g_i(y) \mathrm{d}y \right|^r \right)^{1/r} \mathrm{d}x$$

$$+ \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left| \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \right.\right.$$

$$\left.\left. \times \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \tilde{A}_1(y) K(x, y) g_i(y) \mathrm{d}y \right|^r \right)^{1/r} \mathrm{d}x$$

$$+ \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left| \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \right.\right.$$

$$\left.\left. \times \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{A}_2(y) K(x, y) g_i(y) \mathrm{d}y \right|^r \right)^{1/r} \mathrm{d}x$$

$$+ \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left| \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1! \alpha_2!} \right.\right.$$

$$\left.\left. \times \int_{R^n} \frac{(x-y)^{\alpha_1+\alpha_2} D^{\alpha_1} \tilde{A}_1(y) D^{\alpha_2} \tilde{A}_2(y)}{|x-y|^m} K(x, y) g_i(y) \mathrm{d}y \right|^r \right)^{1/r} \mathrm{d}x$$

$$+ \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left| T_\delta^{\tilde{A}}(h_i)(x) - T_\delta^{\tilde{A}}(h_i)(x_0) \right|^r \right)^{1/r} \mathrm{d}x$$

$$:= \mathrm{I}_1 + \mathrm{I}_2 + \mathrm{I}_3 + \mathrm{I}_4 + \mathrm{I}_5.$$

Now, let us estimate $\mathrm{I}_1, \mathrm{I}_2, \mathrm{I}_3, \mathrm{I}_4$ and I_5, respectively. First, for $x \in Q$ and $y \in \tilde{Q}$, by Lemma 1, we get

$$R_{m_j}(\tilde{A}_j; x, y) \leq C|x-y|^{m_j} \sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}}.$$

Thus, by the $(L^{n/\delta}, L^\infty)$-boundedness of $|T_\delta|_r$, we get

$$\begin{aligned}
\mathrm{I}_1 &\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) \frac{1}{|Q|} \int_Q |T_\delta(g)(x)|_r \mathrm{d}x \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) \||T_\delta(g)|_r\|_{L^\infty} \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) \||f|_r\|_{L^{n/\delta}}.
\end{aligned}$$

For I_2, by the (L^p, L^q)-boundedness of T_δ for $1/q = 1/p - \delta/n$, $n/\delta > p > 1$ and the Hölder's inequality, we get

$$\begin{aligned}
\mathrm{I}_2 &\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{\mathrm{BMO}} \sum_{|\alpha_1|=m_1} \frac{1}{|Q|} \int_Q |T_\delta(D^{\beta_1} \tilde{A}_1 g)(x)|_r \mathrm{d}x \\
&\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{\mathrm{BMO}} \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{R^n} |T_\delta(D^{\alpha_1} \tilde{A}_1 g)(x)|_r^q \mathrm{d}x \right)^{1/q} \\
&\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{\mathrm{BMO}} \sum_{|\alpha_1|=m_1} |Q|^{-1/q} \left(\int_{R^n} |D^{\alpha_1} \tilde{A}_1(x) g(x)|_r^p \mathrm{d}x \right)^{1/p} \\
&\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{\mathrm{BMO}} \\
&\quad \times \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{\tilde{Q}} |D^{\alpha_1} A_1(x) - (D^{\alpha_1} A_1)_{\tilde{Q}}|^q \mathrm{d}x \right)^{1/q} \||f|_r\|_{L^{n/\delta}} \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) \||f|_r\|_{L^{n/\delta}}.
\end{aligned}$$

For I_3, similar to the proof of I_2, we get

$$\mathrm{I}_3 \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) \||f|_r\|_{L^{n/\delta}}.$$

Similarly, for I_4, choose $1 < p < n/\delta$ and $q, t_1, t_2 > 1$ such that $1/q = 1/p - \delta/n$ and $1/t_1 + 1/t_2 + p\delta/n = 1$. We obtain, by the Hölder's inequality,

$$\begin{aligned}
\mathrm{I}_4 &\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|} \int_Q |T_\delta(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 g)(x)|_r \mathrm{d}x \\
&\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{R^n} |T_\delta(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 g)(x)|_r^q \mathrm{d}x \right)^{1/q} \\
&\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1/q} \left(\int_{R^n} |D^{\alpha_1}\tilde{A}_1(x) D^{\alpha_2}\tilde{A}_2(x) g(x)|_r^p \mathrm{d}x \right)^{1/p} \\
&\le C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{\tilde{Q}} |D^{\alpha_1}\tilde{A}_1(x)|^{pt_1} \mathrm{d}x \right)^{1/pt_1} \\
&\quad \times \left(\frac{1}{|Q|} \int_{\tilde{Q}} |D^{\alpha_2}\tilde{A}_2(x)|^{pt_2} \mathrm{d}x \right)^{1/pt_2} \| |f|_r \|_{L^{n/\delta}} \\
&\le C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) \| |f|_r \|_{L^{n/\delta}}.
\end{aligned}$$

For I_5, we write

$$\begin{aligned}
&T_\delta^{\tilde{A}}(h_i)(x) - T_\delta^{\tilde{A}}(h_i)(x_0) \\
&= \int_{R^n} \left(\frac{K(x,y)}{|x-y|^m} - \frac{K(x_0,y)}{|x_0-y|^m} \right) \prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, y) h_i(y) \mathrm{d}y \\
&\quad + \int_{R^n} \left(R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; x_0, y) \right) \frac{R_{m_2}(\tilde{A}_2; x, y)}{|x_0-y|^m} K(x_0,y) h_i(y) \mathrm{d}y \\
&\quad + \int_{R^n} \left(R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y) \right) \frac{R_{m_1}(\tilde{A}_1; x_0, y)}{|x_0-y|^m} K(x_0,y) h_i(y) \mathrm{d}y \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[\frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} K(x,y) \right. \\
&\quad \left. - \frac{R_{m_2}(\tilde{A}_2; x_0, y)(x_0-y)^{\alpha_1}}{|x_0-y|^m} K(x_0,y) \right] D^{\alpha_1}\tilde{A}_1(y) h_i(y) \mathrm{d}y \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[\frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} K(x,y) \right. \\
&\quad \left. - \frac{R_{m_1}(\tilde{A}_1; x_0, y)(x_0-y)^{\alpha_2}}{|x_0-y|^m} K(x_0,y) \right] D^{\alpha_2}\tilde{A}_2(y) h_i(y) \mathrm{d}y
\end{aligned}$$

$$+\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}K(x,y)\right.$$

$$\left.-\frac{(x_0-y)^{\alpha_1+\alpha_2}}{|x_0-y|^m}K(x_0,y)\right]D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)h_i(y)\mathrm{d}y$$

$$=\mathrm{I}_5^{(1)}+\mathrm{I}_5^{(2)}+\mathrm{I}_5^{(3)}+\mathrm{I}_5^{(4)}+\mathrm{I}_5^{(5)}+\mathrm{I}_5^{(6)}.$$

By Lemma 1 and the following inequality [16]

$$|b_{Q_1}-b_{Q_2}|\le C\log(|Q_2|/|Q_1|)\|b\|_{\mathrm{BMO}}\quad \text{for } Q_1\subset Q_2,$$

we know that, for $x\in Q$ and $y\in 2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}$,

$$|R_{m_j}(\tilde{A}_j;x,y)|\le C|x-y|^{m_j}\sum_{|\alpha|=m_j}(\|D^\alpha A_j\|_{\mathrm{BMO}}+|(D^\alpha A_j)_{\tilde{Q}(x,y)}-(D^\alpha A_j)_{\tilde{Q}}|)$$

$$\le Ck|x-y|^{m_j}\sum_{|\alpha|=m_j}\|D^\alpha A_j\|_{\mathrm{BMO}}.$$

Note that $|x-y|\sim|x_0-y|$ for $x\in Q$ and $y\in R^n\setminus\tilde{Q}$, and we obtain, by the conditions on K,

$$|\mathrm{I}_5^{(1)}|\le C\int_{R^n}\left(\frac{|x-x_0|}{|x_0-y|^{m+n+1-\delta}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{m+n+\varepsilon-\delta}}\right)\prod_{j=1}^2|R_{m_j}(\tilde{A}_j;x,y)||h_i(y)|\mathrm{d}y$$

$$\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)$$

$$\times\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k^2\left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}}\right)|f_i(y)|\mathrm{d}y.$$

Thus, by the Minkowski's inequality,

$$\left(\sum_{i=1}^\infty|\mathrm{I}_5^{(1)}|^r\right)^{1/r}\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)$$

$$\times\sum_{k=0}^\infty\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k^2\left(\frac{|x-x_0|}{|x_0-y|^{n+1-\delta}}+\frac{|x-x_0|^\varepsilon}{|x_0-y|^{n+\varepsilon-\delta}}\right)|f(y)|_r\mathrm{d}y$$

$$\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\sum_{k=1}^\infty k^2(2^{-k}+2^{-\varepsilon k})\||f|_r\|_{L^{n/\delta}}$$

$$\le C\prod_{j=1}^2\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{L^{n/\delta}}.$$

For $I_5^{(2)}$, by the formula [6]:

$$R_{m_j}(\tilde{A}_j; x, y) - R_{m_j}(\tilde{A}_j; x_0, y) = \sum_{|\beta|<m} \frac{1}{\beta!} R_{m-|\beta|}(D^\beta \tilde{A}_j; x, x_0)(x-y)^\beta$$

and Lemma 1, we have

$$|R_{m_j}(\tilde{A}_j; x, y) - R_{m_j}(\tilde{A}_j; x_0, y)| \\ \le C \sum_{|\beta|<m_j} \sum_{|\alpha|=m_j} |x-x_0|^{m_j-|\beta|}|x-y|^{|\beta|}\|D^\alpha A_j\|_{\mathrm{BMO}}.$$

Thus

$$\left(\sum_{i=1}^\infty |I_5^{(2)}|^r\right)^{1/r} \le C \prod_{j=1}^2 \left(\sum_{|\alpha|=m_j} \|D^\alpha A_j\|_{\mathrm{BMO}}\right) \\ \times \sum_{k=0}^\infty \int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k \frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} |f(y)|_r \mathrm{d}y \\ \le C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}}\right) \||f|_r\|_{L^{n/\delta}}.$$

Similarly,

$$\left(\sum_{i=1}^\infty |I_5^{(3)}|^r\right)^{1/r} \le C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}}\right) \||f|_r\|_{L^{n/\delta}}.$$

For $I_5^{(4)}$, taking $t>1$ such that $1/t+\delta/n=1$, then

$$\left(\sum_{i=1}^\infty |I_5^{(4)}|^r\right)^{1/r} \le C \sum_{|\alpha_1|=m_1} \int_{R^n} \left| \frac{(x-y)^{\alpha_1} K(x,y)}{|x-y|^m} - \frac{(x_0-y)^{\alpha_1} K(x_0,y)}{|x_0-y|^m} \right| \\ \times |R_{m_2}(\tilde{A}_2; x, y)||D^{\alpha_1}\tilde{A}_1(y)||h(y)|_r \mathrm{d}y \\ + C \sum_{|\alpha_1|=m_1} \int_{R^n} |R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; x_0, y)| \\ \times \frac{|(x_0-y)^{\alpha_1} K(x_0,y)|}{|x_0-y|^m} |D^{\alpha_1}\tilde{A}_1(y)||h(y)|_r \mathrm{d}y \\ \le C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{\mathrm{BMO}} \sum_{|\alpha_1|=m_1} \sum_{k=1}^\infty k(2^{-k}+2^{-\varepsilon k}) \\ \times \left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}} |D^{\alpha_1}\tilde{A}_1(y)|^t \mathrm{d}y\right)^{1/t} \||f|_r\|_{L^{n/\delta}} \\ \le C \prod_{j=1}^2 \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}}\right) \||f|_r\|_{L^{n/\delta}}.$$

Similarly,

$$\left(\sum_{i=1}^{\infty}|\mathrm{I}_5^{(5)}|^r\right)^{1/r} \le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{L^{n/\delta}}.$$

For $\mathrm{I}_5^{(6)}$, taking $t_1, t_2 > 1$ such that $\delta/n + 1/t_1 + 1/t_2 = 1$, then

$$\begin{aligned}\left(\sum_{i=1}^{\infty}|\mathrm{I}_5^{(6)}|^r\right)^{1/r} &\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n}\left|\frac{(x-y)^{\alpha_1+\alpha_2}K(x,y)}{|x-y|^m}\right.\\ &\quad\left.-\frac{(x_0-y)^{\alpha_1+\alpha_2}K(x_0,y)}{|x_0-y|^m}\right||D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||h(y)|_r\mathrm{d}y\\ &\le C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k})\||f|_r\|_{L^{n/\delta}}\\ &\quad\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)|^{t_1}\mathrm{d}y\right)^{1/t_1}\\ &\quad\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_2}\tilde{A}_2(y)|^{t_2}\mathrm{d}y\right)^{1/t_2}\\ &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{L^{n/\delta}}.\end{aligned}$$

Thus

$$|\mathrm{I}_5| \le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{L^{n/\delta}}.$$

Proof of Theorem 1(*b*). It suffices to prove that there exists a constant C_Q such that

$$\frac{1}{|Q|}\int_Q ||T_\delta^A(f)(x)|_r - C_Q|\mathrm{d}x \le C\||f|_r\|_{B_p^\delta}$$

holds for any cube $Q = Q(0,d)$ with $d > 1$. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(0,d)$ with $d > 1$. Let $\tilde{Q}$ and $\tilde{A}_j(x)$ be the same as the proof of (a). Write, for $f = g + h = \{g_i\} + \{h_i\}$ with $g_i = f_i\chi_{\tilde{Q}}$ and $h_i = f_i\chi_{R^n\setminus\tilde{Q}}$,

$$\begin{aligned}&\frac{1}{|Q|}\int_Q ||T_\delta^A(f)(x)|_r - |T_\delta^{\tilde{A}}(h)(0)|_r|\mathrm{d}x\\ &\quad\le \frac{1}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}|T_A(f_i)(x) - T_{\tilde{A}}(h_i)(0)|^r\right)^{1/r}\mathrm{d}x\\ &\quad\le \frac{1}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\int_{R^n}\frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j;x,y)}{|x-y|^m}K(x,y)g_i(y)\mathrm{d}y\right|^r\right)^{1/r}\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\right.\right.\\
&\times\left.\left.\int_{R^n}\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}D^{\alpha_1}\tilde{A}_1(y)K(x,y)g_i(y)\mathrm{d}y\right|^r\right)^{1/r}\mathrm{d}x\\
&+\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\right.\right.\\
&\times\left.\left.\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)K(x,y)g_i(y)\mathrm{d}y\right|^r\right)^{1/r}\mathrm{d}x\\
&+\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\right.\right.\\
&\times\left.\left.\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}K(x,y)g_i(y)\mathrm{d}y\right|^r\right)^{1/r}\mathrm{d}x\\
&+\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|T_\delta^{\tilde{A}}(h_i)(x)-T_\delta^{\tilde{A}}(h_i)(0)\right|^r\right)^{1/r}\mathrm{d}x\\
&:=\mathrm{J}_1+\mathrm{J}_2+\mathrm{J}_3+\mathrm{J}_4+\mathrm{J}_5.
\end{aligned}
$$

Similar to the proof of (a), we get, for $1/t = 1/s - \delta/n$, $1 < s < p$, $1 < t_1, t_2 < \infty$ and $1/t_1 + 1/t_2 + s/p = 1$,

$$
\begin{aligned}
\mathrm{J}_1 &\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\frac{1}{|Q|}\int_Q|T_\delta(g)(x)|_r\mathrm{d}x\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\left(\frac{1}{|Q|}\int_Q|T_\delta(g)(x)|_r^q\mathrm{d}x\right)^{1/q}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)d^{-n(1/p-\delta/n)}\||f|_r\chi_{\tilde{Q}}\|_{L^p}\\
&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{B_p^\delta},\\
\mathrm{J}_2 &\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{\mathrm{BMO}}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|}\int_Q|T_\delta(D^{\alpha_1}\tilde{A}_1g)(x)|_r\mathrm{d}x\\
&\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{\mathrm{BMO}}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|T_\delta(D^{\alpha_1}\tilde{A}_1g)(x)|_r^t\mathrm{d}x\right)^{1/t}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{\mathrm{BMO}} \sum_{|\alpha_1|=m_1} |Q|^{-1/t} \||D^{\alpha_1} \tilde{A}_1 g|_r\|_{L^s} \\
&\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{\mathrm{BMO}} \\
&\quad \times \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{\tilde{Q}} |D^{\alpha_1} \tilde{A}_1(y)|^{ps/(p-s)} \mathrm{d}y \right)^{(p-s)/(ps)} \\
&\quad \times |Q|^{\delta/n-1/p} \||f|_r \chi_{\tilde{Q}}\|_{L^p} \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) \||f|_r\|_{B_p^\delta},
\end{aligned}
$$

$$
\mathrm{J}_3 \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) \||f|_r\|_{B_p^\delta},
$$

$$
\begin{aligned}
\mathrm{J}_4 &\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{|Q|} \int_Q |T_\delta(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 g)(x)|_r \mathrm{d}x \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{R^n} |T_\delta(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 f_1)(x)|_r^q \mathrm{d}x \right)^{1/q} \\
&\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} |Q|^{-1/s} \left(\int_{R^n} |D^{\alpha_1} \tilde{A}_1(x) D^{\alpha_2} \tilde{A}_2(x) g(x)|_r^s \mathrm{d}x \right)^{1/s} \\
&\leq C \sum_{|\alpha_1|=m_1} \left(\frac{1}{|Q|} \int_{\tilde{Q}} |D^{\alpha_1} \tilde{A}_1(x)|^{st_1} \mathrm{d}x \right)^{1/st_1} \\
&\quad \times \sum_{|\alpha_2|=m_2} \left(\frac{1}{|Q|} \int_{\tilde{Q}} |D^{\alpha_2} \tilde{A}_2(x)|^{st_2} \mathrm{d}x \right)^{1/st_2} |Q|^{\delta/n-1/p} \||f|_r \chi_{\tilde{Q}}\|_{L^p} \\
&\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) \||f|_r\|_{B_p^\delta}.
\end{aligned}
$$

For J_5, we write, for $x \in Q$,

$$
\begin{aligned}
&T_\delta^{\tilde{A}}(h_i)(x) - T_\delta^{\tilde{A}}(h_i)(0) \\
&\quad = \int_{R^n} \left(\frac{K(x, y)}{|x-y|^m} - \frac{K(0, y)}{|y|^m} \right) \prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, y) h_i(y) \mathrm{d}y \\
&\qquad + \int_{R^n} (R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; 0, y)) \frac{R_{m_2}(\tilde{A}_2; x, y)}{|y|^m} K(0, y) h_i(y) \mathrm{d}y
\end{aligned}
$$

$$+\int_{R^n}(R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;0,y))\frac{R_{m_1}(\tilde{A}_1;0,y)}{|y|^m}K(0,y)h_i(y)\mathrm{d}y$$

$$-\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\int_{R^n}\left[\frac{R_{m_2}(\tilde{A}_2;x,y)(x-y)^{\alpha_1}}{|x-y|^m}K(x,y)\right.$$

$$\left.-\frac{R_{m_2}(\tilde{A}_2;0,y)(-y)^{\alpha_1}}{|y|^m}K(0,y)\right]D^{\alpha_1}\tilde{A}_1(y)h_i(y)\mathrm{d}y$$

$$-\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\int_{R^n}\left[\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}K(x,y)\right.$$

$$\left.-\frac{R_{m_1}(\tilde{A}_1;0,y)(-y)^{\alpha_2}}{|y|^m}K(0,y)\right]D^{\alpha_2}\tilde{A}_2(y)h_i(y)\mathrm{d}y$$

$$+\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\int_{R^n}\left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m}K(x,y)\right.$$

$$\left.-\frac{(-y)^{\alpha_1+\alpha_2}}{|y|^m}K(0,y)\right]D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)h_i(y)\mathrm{d}y$$

$$=\mathrm{J}_5^{(1)}+\mathrm{J}_5^{(2)}+\mathrm{J}_5^{(3)}+\mathrm{J}_5^{(4)}+\mathrm{J}_5^{(5)}+\mathrm{J}_5^{(6)}.$$

Similar to the proof of (a), we get, for $1<t_1,t_2<\infty$ and $1/t_1+1/t_2+1/p=1$,

$$\left(\sum_{i=1}^{\infty}|\mathrm{J}_5^{(1)}|^r\right)^{1/r}\le C\int_{R^n}\left(\frac{|x|}{|y|^{m+n+1-\delta}}+\frac{|x|^{\varepsilon}}{|y|^{m+n+\varepsilon-\delta}}\right)\prod_{j=1}^{2}|R_{m_j}(\tilde{A}_j;x,y)||h(y)|_r\mathrm{d}y$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)$$

$$\times\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k^2\left(\frac{|x|}{|y|^{n+1-\delta}}+\frac{|x|^{\varepsilon}}{|y|^{n+\varepsilon-\delta}}\right)|f(y)|_r\mathrm{d}y$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)$$

$$\times\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-\varepsilon k})(2^kd)^{-n(1/p-\delta/n)}\||f|_r\chi_{2^k\tilde{Q}}\|_{L^p}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{B_p^{\delta}},$$

$$\left(\sum_{i=1}^{\infty}|J_5^{(2)}|^r\right)^{1/r} \le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}} k\frac{|x|}{|y|^{n+1-\delta}}|f(y)|_r\mathrm{d}y$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{B_p^\delta},$$

$$\left(\sum_{i=1}^{\infty}|J_5^{(3)}|^r\right)^{1/r} \le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{B_p^\delta},$$

$$\left(\sum_{i=1}^{\infty}|J_5^{(4)}|^r\right)^{1/r} \le C\sum_{|\alpha_1|=m_1}\int_{R^n}\left|\frac{(x-y)^{\alpha_1}K(x,y)}{|x-y|^m}-\frac{(-y)^{\alpha_1}K(0,y)}{|y|^m}\right|$$

$$\times|R_{m_2}(\tilde{A}_2;x,y)||D^{\alpha_1}\tilde{A}_1(y)||h(y)|_r\mathrm{d}y$$

$$+C\sum_{|\alpha_1|=m_1}\int_{R^n}|R_{m_2}(\tilde{A}_2;x,y)-R_{m_2}(\tilde{A}_2;0,y)|$$

$$\times\frac{|(-y)^{\alpha_1}K(0,y)|}{|y|^m}|D^{\alpha_1}\tilde{A}_1(y)||h(y)|_r\mathrm{d}y$$

$$\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{\mathrm{BMO}}$$

$$\times\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})(2^k d)^{-n(1/p-\delta/n)}\||f|_r\chi_{2^k\tilde{Q}}\|_{L^p}$$

$$\times\sum_{|\alpha_1|=m_1}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)|^{p'}\mathrm{d}y\right)^{1/p'}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{B_p^\delta},$$

$$\left(\sum_{i=1}^{\infty}|J_5^{(5)}|^r\right)^{1/r} \le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{B_p^\delta},$$

$$\left(\sum_{i=1}^{\infty}|J_5^{(6)}|^r\right)^{1/r} \le C\sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k})(2^k d)^{-n(1/p-\delta/n)}\||f|_r\chi_{2^k\tilde{Q}}\|_{L^p}$$

$$\times\sum_{|\alpha_1|=m_1}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)|^{t_1}\mathrm{d}y\right)^{1/t_1}$$

$$\times \sum_{|\alpha_2|=m_2} \left(\frac{1}{|2^k \tilde{Q}|} \int_{2^k \tilde{Q}} |D^{\alpha_2} \tilde{A}_2(y)|^{t_2} \mathrm{d}y \right)^{1/t_2}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) \||f|_r\|_{B_p^\delta}.$$

Thus

$$\mathrm{J}_5 \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) \||f|_r\|_{B_p^\delta}.$$

This completes the proof of Theorem 1.

Proof of Theorem 2(i). It suffices to show the conclusion for the case $\mathrm{BMO}(w) = \mathrm{BMO}(R^n)$ by Lemma 2, that is, it is only to prove that there exists a constant C_Q such that

$$\frac{1}{|Q|} \int_Q ||T^A(f)(x)|_r - C_Q| \mathrm{d}x \leq C \||f|_r\|_{L^\infty(w)}$$

holds for any cube Q. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(x_0, d)$. Let $\tilde{Q}$ and $\tilde{A}_j(x)$ be the same as the proof of Theorem 1. Write, for $f = g + h = \{g_i\} + \{h_i\}$ with $g_i = f_i \chi_{\tilde{Q}}$ and $h_i = f_i \chi_{R^n \setminus \tilde{Q}}$,

$$\frac{1}{|Q|} \int_Q ||T^A(f)(x)|_r - |T^{\tilde{A}}(h)(x_0)|_r| \mathrm{d}x$$

$$\leq \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} |T_A(f_i)(x) - T_{\tilde{A}}(h_i)(x_0)|^r \right)^{1/r} \mathrm{d}x$$

$$\leq \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left| \int_{R^n} \frac{\prod_{j=1}^2 R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m} K(x,y) g_i(y) \mathrm{d}y \right|^r \right)^{1/r} \mathrm{d}x$$

$$+ \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left| \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \right.\right.$$

$$\left.\left. \times \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1} \tilde{A}_1(y) K(x,y) g_i(y) \mathrm{d}y \right|^r \right)^{1/r} \mathrm{d}x$$

$$+ \frac{1}{|Q|} \int_Q \left(\sum_{i=1}^{\infty} \left| \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \right.\right.$$

$$\left.\left. \times \int_{R^n} \frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} D^{\alpha_2} \tilde{A}_2(y) K(x,y) g_i(y) \mathrm{d}y \right|^r \right)^{1/r} \mathrm{d}x$$

$$+\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}\left|\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\right.\right.$$

$$\left.\left.\times\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}K(x,y)g_i(y)\mathrm{d}y\right|^r\right)^{1/r}\mathrm{d}x$$

$$+\frac{1}{|Q|}\int_Q\left(\sum_{i=1}^{\infty}|T_\delta^{\tilde{A}}(h_i)(x)-T_\delta^{\tilde{A}}(h_i)(x_0)|^r\right)^{1/r}\mathrm{d}x$$

$$:=\mathrm{L}_1+\mathrm{L}_2+\mathrm{L}_3+\mathrm{L}_4+\mathrm{L}_5.$$

By the L^s-boundedness of $|T|_r$ for $1<s\le\infty$ and using the same argument as in the proof of Theorem 1, we get, for $1/t_1+1/t_2=1$,

$$\mathrm{L}_1\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\frac{1}{|Q|}\int_Q|T(g)(x)|_r\mathrm{d}x$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||T(g)|_r\|_{L^\infty}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{L^\infty(w)},$$

$$\mathrm{L}_2\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{\mathrm{BMO}}\sum_{|\alpha_1|=m_1}\frac{1}{|Q|}\int_Q|T(D^{\alpha_1}\tilde{A}_1g)(x)|_r\mathrm{d}x$$

$$\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{\mathrm{BMO}}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|T(D^{\alpha_1}\tilde{A}_1g)(x)|_r^s\mathrm{d}x\right)^{1/s}$$

$$\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{\mathrm{BMO}}\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{R^n}|D^{\alpha_1}\tilde{A}_1(x)g(x)|_r^s\mathrm{d}x\right)^{1/s}$$

$$\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{\mathrm{BMO}}$$

$$\times\sum_{|\alpha_1|=m_1}\left(\frac{1}{|Q|}\int_{\tilde{Q}}|D^{\alpha_1}A_1(x)-(D^{\alpha_1}A_1)_{\tilde{Q}}|^s\mathrm{d}x\right)^{1/s}\||f|_r\|_{L^\infty}$$

$$\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{L^\infty(w)},$$

$$\mathrm{L}_3\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{L^\infty(w)},$$

$$\mathrm{L}_4 \le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \frac{1}{|Q|}\int_Q |T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 g)(x)|_r \mathrm{d}x$$

$$\le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \left(\frac{1}{|Q|}\int_{R^n} |T(D^{\alpha_1}\tilde{A}_1 D^{\alpha_2}\tilde{A}_2 g)(x)|_r^s \mathrm{d}x\right)^{1/s}$$

$$\le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} w(Q)^{-1/s}\left(\int_{R^n} |D^{\alpha_1}\tilde{A}_1(x) D^{\alpha_2}\tilde{A}_2(x) g(x)|_r^s \mathrm{d}x\right)^{1/s}$$

$$\le C \sum_{|\alpha_1|=m_1,|\alpha_2|=m_2} \left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\alpha_1}\tilde{A}_1(x)|^{st_1} \mathrm{d}x\right)^{1/st_1}$$

$$\times \left(\frac{1}{|Q|}\int_{\tilde{Q}} |D^{\alpha_2}\tilde{A}_2(x)|^{st_2} \mathrm{d}x\right)^{1/st_2} \||f|_r\|_{L^\infty}$$

$$\le C \prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{L^\infty(w)},$$

$$\mathrm{L}_5 \le C \prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{L^\infty(w)}.$$

Proof of Theorem 2(*ii*). It suffices to prove that there exists a constant C_Q such that

$$\frac{1}{w(Q)}\int_Q ||T^A(f)(x)|_r - C_Q| w(x)\mathrm{d}x \le C\||f|_r\|_{B_p(w)}$$

holds for any cube $Q = Q(0, d)$ with $d > 1$. Without loss of generality, we may assume $l = 2$. Fix a cube $Q = Q(0, d)$ with $d > 1$. Let $\tilde{Q}$ and $\tilde{A}_j(x)$ be the same as the proof of Theorem 1. Write, for $f = g + h = \{g_i\} + \{h_i\}$ with $g_i = f_i\chi_{\tilde{Q}}$ and $h_i = f_i\chi_{R^n\setminus\tilde{Q}}$,

$$\frac{1}{w(Q)}\int_Q ||T^A(f)(x)|_r - |T^{\tilde{A}}(h)(0)|_r| w(x)\mathrm{d}x$$

$$\le \frac{1}{w(Q)}\int_Q \left(\sum_{i=1}^{\infty} |T_A(f_i)(x) - T_{\tilde{A}}(h_i)(0)|^r\right)^{1/r} w(x)\mathrm{d}x$$

$$\le \frac{1}{w(Q)}\int_Q \left(\sum_{i=1}^{\infty}\left|\int_{R^n} \frac{\prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, y)}{|x-y|^m} K(x, y) g_i(y)\mathrm{d}y\right|^r\right)^{1/r} w(x)\mathrm{d}x$$

$$+ \frac{1}{w(Q)}\int_Q \left(\sum_{i=1}^{\infty}\left|\sum_{|\alpha_1|=m_1}\frac{1}{\alpha_1!}\right.\right.$$

$$\left.\left.\times \int_{R^n} \frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} D^{\alpha_1}\tilde{A}_1(y) K(x, y) g_i(y)\mathrm{d}y\right|^r\right)^{1/r} w(x)\mathrm{d}x$$

$$+\frac{1}{w(Q)}\int_Q\left(\sum_{i=1}^{\infty}\left|\sum_{|\alpha_2|=m_2}\frac{1}{\alpha_2!}\right.\right.$$

$$\left.\left.\times\int_{R^n}\frac{R_{m_1}(\tilde{A}_1;x,y)(x-y)^{\alpha_2}}{|x-y|^m}D^{\alpha_2}\tilde{A}_2(y)K(x,y)g_i(y)\mathrm{d}y\right|^r\right)^{1/r}w(x)\mathrm{d}x$$

$$+\frac{1}{w(Q)}\int_Q\left(\sum_{i=1}^{\infty}\left|\sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2}\frac{1}{\alpha_1!\alpha_2!}\right.\right.$$

$$\left.\left.\times\int_{R^n}\frac{(x-y)^{\alpha_1+\alpha_2}D^{\alpha_1}\tilde{A}_1(y)D^{\alpha_2}\tilde{A}_2(y)}{|x-y|^m}K(x,y)g_i(y)\mathrm{d}y\right|^r\right)^{1/r}w(x)\mathrm{d}x$$

$$+\frac{1}{w(Q)}\int_Q\left(\sum_{i=1}^{\infty}|T_\delta^{\tilde{A}}(h_i)(x)-T_\delta^{\tilde{A}}(h_i)(0)|^r\right)^{1/r}w(x)\mathrm{d}x$$

$$:=\mathrm{M}_1+\mathrm{M}_2+\mathrm{M}_3+\mathrm{M}_4+\mathrm{M}_5.$$

Similar to the proof of Theorem 1, we get

$$\begin{aligned}\mathrm{M}_1&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\frac{1}{w(Q)}\int_Q|T(g)(x)|_r w(x)\mathrm{d}x\\&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\left(\frac{1}{w(Q)}\int_Q|T(g)(x)|_r^p w(x)\mathrm{d}x\right)^{1/p}\\&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)w(\tilde{Q})^{-1/p}\||f|_r\chi_{\tilde{Q}}\|_{L^p(w)}\\&\le C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{B_p(w)}.\end{aligned}$$

For M_2, since $w\in A_1$, w satisfies the reverse of Hölder's inequality:

$$\left(\frac{1}{|Q|}\int_Q w(x)^q\mathrm{d}x\right)^{1/q}\le\frac{C}{|Q|}\int_Q w(x)\mathrm{d}x$$

for all cube Q and some $1<q<\infty$ [10,16]. Thus, taking $s,t>1$ such that $st<p$ and $q=(pt-st)/(p-st)$, then

$$\begin{aligned}\mathrm{M}_2&\le C\sum_{|\beta_2|=m_2}\|D^{\alpha_2}A_2\|_{\mathrm{BMO}}\sum_{|\alpha_1|=m_1}\frac{1}{w(Q)}\int_Q|T(D^{\alpha_1}\tilde{A}_1g)(x)|_r w(x)\mathrm{d}x\\&\le C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{\mathrm{BMO}}\end{aligned}$$

$$\times \sum_{|\alpha_1|=m_1} \left(\frac{1}{w(Q)} \int_{R^n} |T(D^{\alpha_1} \tilde{A}_1 g)(x)|_r^s w(x) \mathrm{d}x \right)^{1/s}$$

$$\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{\mathrm{BMO}} \sum_{|\alpha_1|=m_1} w(Q)^{-1/s} \|D^{\alpha_1} \tilde{A}_1 |g|_r\|_{L^s(w)}$$

$$\leq C \sum_{|\alpha_2|=m_2} \|D^{\alpha_2} A_2\|_{\mathrm{BMO}} w(Q)^{-1/s} \sum_{|\alpha_1|=m_1} \left(\int_{\tilde{Q}} |D^{\alpha} \tilde{A}_1(y)|^{st'} \mathrm{d}y \right)^{1/st'}$$

$$\times \left(\int_{\tilde{Q}} |f(x)|_r^{st} w(x)^t \mathrm{d}x \right)^{1/st}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) |Q|^{1/st'} w(Q)^{-1/s}$$

$$\times \left(\int_{\tilde{Q}} |f(x)|_r^p w(x) \mathrm{d}x \right)^{1/p} \left(\int_{\tilde{Q}} w(x)^q \mathrm{d}x \right)^{(p-s)/pqs}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) w(\tilde{Q})^{-1/p} \| |f|_r \chi_{\tilde{Q}} \|_{L^p(w)}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) \| |f|_r \|_{B_p(w)},$$

$$\mathrm{M}_3 \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) \| |f|_r \|_{B_p(w)}.$$

For M_4, taking $s, t_1, t_2, t_3 > 1$ such that $1/t_1 + 1/t_2 + 1/t_3 = 1$, $st_3 < p$ and $q = (pt_3 - st_3)/(p - st_3)$, then, by the reverse of Hölder's inequality,

$$\mathrm{M}_4 \leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \frac{1}{w(Q)} \int_Q |T(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 g)(x)|_r w(x) \mathrm{d}x$$

$$\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} \left(\frac{1}{w(Q)} \int_{R^n} |T(D^{\alpha_1} \tilde{A}_1 D^{\alpha_2} \tilde{A}_2 g)(x)|_r^s w(x) \mathrm{d}x \right)^{1/s}$$

$$\leq C \sum_{|\alpha_1|=m_1, |\alpha_2|=m_2} w(Q)^{-1/s} \left(\int_{R^n} |D^{\alpha_1} \tilde{A}_1(x) D^{\alpha_2} \tilde{A}_2(x) g(x)|_r^s w(x) \mathrm{d}x \right)^{1/s}$$

$$\leq C \sum_{|\alpha_1|=m_1} \left(\int_{\tilde{Q}} |D^{\alpha_1} \tilde{A}_1(x)|^{st_1} \mathrm{d}x \right)^{1/st_1}$$

$$\times \sum_{|\alpha_2|=m_2} \left(\int_{\tilde{Q}} |D^{\alpha_2} \tilde{A}_2(x)|^{st_2} \mathrm{d}x \right)^{1/st_2} w(Q)^{-1/s} \left(\int_{\tilde{Q}} |f(x)|_r^{st_3} w(x)^{t_3} \mathrm{d}x \right)^{1/st_3}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) w(\tilde{Q})^{-1/p} \| |f|_r \chi_{\tilde{Q}} \|_{L^p(w)}$$

$$\leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) \| |f|_r \|_{B_p(w)}.$$

For M_5, we write, for $x \in Q$,

$$\begin{aligned}
&T^{\tilde{A}}(h_i)(x) - T^{\tilde{A}}(h_i)(0) \\
&= \int_{R^n} \left(\frac{K(x,y)}{|x-y|^m} - \frac{K(0,y)}{|y|^m} \right) \prod_{j=1}^{2} R_{m_j}(\tilde{A}_j; x, y) h_i(y) \mathrm{d}y \\
&\quad + \int_{R^n} (R_{m_1}(\tilde{A}_1; x, y) - R_{m_1}(\tilde{A}_1; 0, y)) \frac{R_{m_2}(\tilde{A}_2; x, y)}{|y|^m} K(0,y) h_i(y) \mathrm{d}y \\
&\quad + \int_{R^n} (R_{m_2}(\tilde{A}_2; x, y) - R_{m_2}(\tilde{A}_2; 0, y)) \frac{R_{m_1}(\tilde{A}_1; 0, y)}{|y|^m} K(0,y) h_i(y) \mathrm{d}y \\
&\quad - \sum_{|\alpha_1|=m_1} \frac{1}{\alpha_1!} \int_{R^n} \left[\frac{R_{m_2}(\tilde{A}_2; x, y)(x-y)^{\alpha_1}}{|x-y|^m} K(x,y) \right. \\
&\qquad \left. - \frac{R_{m_2}(\tilde{A}_2; 0, y)(-y)^{\alpha_1}}{|y|^m} K(0,y) \right] D^{\alpha_1} \tilde{A}_1(y) h_i(y) \mathrm{d}y \\
&\quad - \sum_{|\alpha_2|=m_2} \frac{1}{\alpha_2!} \int_{R^n} \left[\frac{R_{m_1}(\tilde{A}_1; x, y)(x-y)^{\alpha_2}}{|x-y|^m} K(x,y) \right. \\
&\qquad \left. - \frac{R_{m_1}(\tilde{A}_1; 0, y)(-y)^{\alpha_2}}{|y|^m} K(0,y) \right] D^{\alpha_2} \tilde{A}_2(y) h_i(y) \mathrm{d}y \\
&\quad + \sum_{|\alpha_1|=m_1,\ |\alpha_2|=m_2} \frac{1}{\alpha_1!\alpha_2!} \int_{R^n} \left[\frac{(x-y)^{\alpha_1+\alpha_2}}{|x-y|^m} K(x,y) \right. \\
&\qquad \left. - \frac{(-y)^{\alpha_1+\alpha_2}}{|y|^m} K(0,y) \right] D^{\alpha_1} \tilde{A}_1(y) D^{\alpha_2} \tilde{A}_2(y) h_i(y) \mathrm{d}y \\
&= \mathrm{M}_5^{(1)} + \mathrm{M}_5^{(2)} + \mathrm{M}_5^{(3)} + \mathrm{M}_5^{(4)} + \mathrm{M}_5^{(5)} + \mathrm{M}_5^{(6)}.
\end{aligned}$$

Similar to the proof of Theorem 1 and notice that $w \in A_1 \subset A_p$, we get

$$\left(\sum_{i=1}^{\infty} |\mathrm{M}_5^{(1)}|^r \right)^{1/r} \leq C \int_{R^n} \left(\frac{|x|}{|y|^{m+n+1}} + \frac{|x|^{\varepsilon}}{|y|^{m+n+\varepsilon}} \right) \prod_{j=1}^{2} |R_{m_j}(\tilde{A}_j; x, y)| |h(y)|_r \mathrm{d}y$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)$$

$$\times\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k^2\left(\frac{|x|}{|y|^{n+1}}+\frac{|x|^{\varepsilon}}{|y|^{n+\varepsilon}}\right)|f(y)|_r\mathrm{d}y$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)$$

$$\times\sum_{k=1}^{\infty}k^2(2^{-k}+2^{-\varepsilon k})w(2^k\tilde{Q})^{-1/p}\left(\int_{2^k\tilde{Q}}|f(y)|_r^p w(y)\mathrm{d}y\right)^{1/p}$$

$$\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)\mathrm{d}y\right)^{1/p}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)^{-1/(p-1)}\mathrm{d}y\right)^{(p-1)/p}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{B_p(w)},$$

$$\left(\sum_{i=1}^{\infty}|\mathrm{M}_5^{(2)}|^r\right)^{1/r}\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\frac{|x|}{|y|^{n+1}}|f(y)|_r\mathrm{d}y$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{B_p(w)},$$

$$\left(\sum_{i=1}^{\infty}|\mathrm{M}_5^{(3)}|^r\right)^{1/r}\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{B_p(w)}.$$

For $\mathrm{M}_5^{(4)}$, choose $1 < s < p$, notice that $w \in A_1 \subset A_{p/s}$, we get

$$\left(\sum_{i=1}^{\infty}|\mathrm{M}_5^{(4)}|^r\right)^{1/r}\leq C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{\mathrm{BMO}}$$

$$\times\sum_{k=0}^{\infty}\int_{2^{k+1}\tilde{Q}\setminus 2^k\tilde{Q}}k\left(\frac{|x|}{|y|^{n+1}}+\frac{|x|^{\varepsilon}}{|y|^{n+\varepsilon}}\right)|D^{\alpha_1}\tilde{A}_1(y)||f(y)|_r\mathrm{d}y$$

$$\leq C\sum_{|\alpha_2|=m_2}\|D^{\alpha_2}A_2\|_{\mathrm{BMO}}\sum_{k=0}^{\infty}\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^{\varepsilon}}{(2^kd)^{n+\varepsilon}}\right)$$

$$\times\left(\int_{2^{k+1}\tilde{Q}}|f(y)|_r^s\mathrm{d}y\right)^{1/s}\mathrm{d}y\left(\int_{2^{k+1}\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)|^{s'}\mathrm{d}y\right)^{1/s'}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\sum_{k=1}^{\infty}k(2^{-k}+2^{-\varepsilon k})w(2^k\tilde{Q})^{-1/p}$$

$$\times\left(\int_{2^k\tilde{Q}}|f(y)|_r^p w(y)\mathrm{d}y\right)^{1/p}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)\mathrm{d}y\right)^{1/p}$$

$$\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)^{-s/(p-s)}\mathrm{d}y\right)^{(p-s)/ps}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{B_p(w)},$$

$$\left(\sum_{i=1}^{\infty}|\mathrm{M}_5^{(5)}|^r\right)^{1/r}\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{B_p(w)}.$$

For $\mathrm{L}_5^{(6)}$, choose $1<t_1,t_2,t_3<\infty$ such that $t_3<p$ and $1/t_1+1/t_2+1/t_3=1$. Notice that $w\in A_1\subset A_{p/t_3}$, we get

$$\left(\sum_{i=1}^{\infty}|\mathrm{M}_5^{(6)}|^r\right)^{1/r}\leq C\sum_{|\alpha_1|=m_1,|\alpha_2|=m_2}\int_{R^n}\left|\frac{(x-y)^{\alpha_1+\alpha_2}K(x,y)}{|x-y|^m}-\frac{(-y)^{\alpha_1+\alpha_2}K(0,y)}{|y|^m}\right|$$

$$\times|D^{\alpha_1}\tilde{A}_1(y)||D^{\alpha_2}\tilde{A}_2(y)||h(y)|_r\mathrm{d}y$$

$$\leq C\sum_{k=0}^{\infty}\left(\frac{d}{(2^kd)^{n+1}}+\frac{d^{\varepsilon}}{(2^kd)^{n+\varepsilon}}\right)\left(\int_{2^{k+1}\tilde{Q}}|f(y)|_r^{t_3}\mathrm{d}y\right)^{1/t_3}\mathrm{d}y$$

$$\times\sum_{|\alpha_1|=m_1}\left(\int_{2^{k+1}\tilde{Q}}|D^{\alpha_1}\tilde{A}_1(y)|^{t_1}\mathrm{d}y\right)^{1/t_1}$$

$$\times\sum_{|\alpha_2|=m_2}\left(\int_{2^{k+1}\tilde{Q}}|D^{\alpha_2}\tilde{A}_2(y)|^{t_2}\mathrm{d}y\right)^{1/t_2}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\sum_{k=1}^{\infty}(2^{-k}+2^{-\varepsilon k})w(2^k\tilde{Q})^{-1/p}$$

$$\times\left(\int_{2^k\tilde{Q}}|f(y)|_r^p w(y)\mathrm{d}y\right)^{1/p}\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)\mathrm{d}y\right)^{1/p}$$

$$\times\left(\frac{1}{|2^k\tilde{Q}|}\int_{2^k\tilde{Q}}w(y)^{-t_3/(p-t_3)}\mathrm{d}y\right)^{(p-t_3)/pt_3}$$

$$\leq C\prod_{j=1}^{2}\left(\sum_{|\alpha_j|=m_j}\|D^{\alpha_j}A_j\|_{\mathrm{BMO}}\right)\||f|_r\|_{B_p(w)}.$$

Thus

$$\mathrm{M}_5 \leq C \prod_{j=1}^{2} \left(\sum_{|\alpha_j|=m_j} \|D^{\alpha_j} A_j\|_{\mathrm{BMO}} \right) \| |f|_r \|_{B_p(w)}.$$

This completes the proof of Theorem 2.

4. Applications

Now we shall apply the theorems of the paper to some particular operators such as the Calderón–Zygmund singular integral operator and fractional integral operator.

Application 1. Calderón–Zygmund singular integral operator.

Let T be the Calderón–Zygmund operator [7,10,16]. Then it is easy to see that T satisfies the conditions in Theorem 2. Thus the conclusions of Theorem 2 hold for T^A.

Application 2. Fractional integral operator with rough kernel.

For $0 < \delta < n$, let T_δ be the fractional integral operator with rough kernel defined by [1,9,11]

$$T_\delta f(x) = \int_{R^n} \frac{\Omega(x-y)}{|x-y|^{n-\delta}} f(y) \mathrm{d}y,$$

the vector-valued multilinear operator related to T_δ is defined by

$$|T_\delta^A(f)(x)|_r = \left(\sum_{i=1}^{\infty} |T_\delta^A(f_i)(x)|^r \right)^{1/r},$$

where

$$T_\delta^A(f)(x) = \int_{R^n} \frac{\prod_{j=1}^{l} R_{m_j+1}(A_j; x, y)}{|x-y|^{m+n-\delta}} \Omega(x-y) f(y) \mathrm{d}y$$

and Ω is homogeneous of degree zero on R^n, $\int_{S^{n-1}} \Omega(x')\mathrm{d}\sigma(x') = 0$ and $\Omega \in \mathrm{Lip}_\varepsilon(S^{n-1})$ for some $0 < \varepsilon \leq 1$, that is, there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leq M|x-y|^\varepsilon$. Then T_δ satisfies the conditions in Theorem 1. Thus the conclusions of Theorem 1 hold for T_δ^A.

Acknowledgements

The author would like to express his deep gratitude to the referee for his valuable comments and suggestions. This work is supported by the NNSF (Grant: 10271071).

References

[1] Chanillo S, A note on commutators, *Indiana Univ. Math. J.* **31** (1982) 7–16
[2] Chanillo S, Remarks on commutators of pseudo-differential operators, *Contemporary Math. of the Am. Math. Soc.* **205** (1997) 33–37

[3] Chen W and Hu G, Weak type (H^1, L^1) estimate for multilinear singular integral operator, *Adv. Math. (China)* **30** (2001) 63–69

[4] Cohen J, A sharp estimate for a multilinear singular integral on R^n, *Indiana Univ. Math. J.* **30** (1981) 693–702

[5] Cohen J and Gosselin J, On multilinear singular integral operators on R^n, *Studia Math.* **72** (1982) 199–223

[6] Cohen J and Gosselin J, A BMO estimate for multilinear singular integral operators, *Illinois J. Math.* **30** (1986) 445–465

[7] Coifman R and Meyer Y, Wavelets, Calderón–Zygmund and multilinear operators, *Cambridge Studies in Advanced Math.* (Cambridge: Cambridge University Press) (1997) vol. 48

[8] Coifman R, Rochberg R and Weiss G, Factorization theorems for Hardy spaces in several variables, *Ann. Math.* **103** (1976) 611–635

[9] Ding Y and Lu S Z, Weighted boundedness for a class of rough multilinear operators, *Acta Math. Sinica* **17** (2001) 517–526

[10] Garcia-Cuerva J and Rubio de Francia J L, Weighted norm inequalities and related topics, (Amsterdam: North-Holland Math) (1985) vol. 16

[11] Harboure E, Segovia C and Torrea J L, Boundedness of commutators of fractional and singular integrals for the extreme values of p, *Illinois J. Math.* **41** (1997) 676–700

[12] Muckenhoupt B and Wheeden R, Weighted BMO and the Hilbert transform, *Studia Math.* **54** (1976) 221–237

[13] Perez C and Pradolini G, Sharp weighted endpoint estimates for commutators of singular integral operators, *Michigan Math. J.* **49** (2001) 23–37

[14] Perez C and Trujillo-Gonzalez R, Sharp weighted estimates for multilinear commutators, *J. London Math. Soc.* **65** (2002) 672–692

[15] Perez C and Trujillo-Gonzalez R, Sharp weighted estimates for vector-valued singular integral operators and commutators, *Tohoku Math. J.* **55** (2003) 109–129

[16] Stein E M, Harmonic analysis: Real variable methods, orthogonality and oscillatory integrals (Princeton, NJ: Princeton Univ. Press) (1993)