Proc. Indian Acad. Sci. (Math. Sci.) Vol. 115, No. 2, May 2005, pp. 147–165.

Twisted vector bundles on pointed nodal curves

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MS received 19 January 2005

Abstract. Motivated by the quest for a good compactification of the moduli space of G-bundles on a nodal curve we establish a striking relationship between Abramovich's and Vistoli's twisted bundles and Gieseker vector bundles.

Keywords. Twisted bundles; nodal curves.

1. Introduction

This paper is the result of an attempt to understand a recent draft of Seshadri [8] and is meant as a contribution in the quest for a good compactification of the moduli space (or stack) of G-bundles on a nodal curve.

We are led by the idea that such a compactification should behave well in families and also under partial normalization of nodal curves. This statement may be reformulated by saying that we are looking for an object which has the right to be called the moduli stack of stable maps into the classifying stack BG of a reductive group G.

For finite groups G the stack of stable maps into BG has been recently constructed by Abramovich, Corti and Vistoli [1,2] by means of the so-called twisted bundles. On the other hand, as shown in [6], the notion of Gieseker vector bundles leads to the construction of the stack of stable maps into $B\mathrm{GL}_r$.

In this note we establish a connection between the straightforward generalization of the notion of twisted bundles to the case of the non-finite reductive group GL_r and Gieseker vector bundles. My hope is that this relationship – whose observation is entirely due to Seshadri, and which in my mind is really striking – may help to find the right notion for a more general reductive groups G.

2. Twisted G-bundles

Throughout this section k denotes an algebraically closed field and G a reductive group over k.

A twisted G-bundle is a twisted object in the sense of §3 of [1] where the target stack $\mathcal{M}$ is taken to be the classifying stack BG. For convenience, we recall the necessary definitions from *loc. cit.*

DEFINITION 2.1

1. An *n-marked curve* consists of data $(U \to S, \Sigma_i)$ where $\pi\colon U \to S$ is a nodal curve and $\Sigma_1, \ldots, \Sigma_n \subset U$ are pairwise disjoint closed subschemes whose support do not

intersect the singular locus U_{sing} of π and are such that the projections $\Sigma_i \to S$ are étale.

2. A *morphism between two n-marked curves* $(U \to S, \Sigma_i^U)$ *and* $(V \to S, \Sigma_i^V)$ is an S-morphism $f: U \to V$ such that $f(\Sigma_i^U) \subseteq \Sigma_i^V$ for each i. Such a morphism is called *strict*, if for each i the support of $f^{-1}(\Sigma_i^V)$ coincides with the support of Σ_i^U and if furthermore the support of $f^{-1}(V_{\text{sing}})$ coincides with one of the U_{sing}.
3. The *pull back* of an n-marked curve $(U \to S, \Sigma_i)$ by a morphism $S' \to S$ is the n-marked curve $(U \times_S S', \Sigma_i \times_S S')$.
4. An *n-pointed nodal curve* is an n-marked curve where the projections $\Sigma_i \to S$ are isomorphisms.
5. Let $(U \to S, \Sigma_i)$ be an n-marked curve. The complement (inside U) of the union of the singular locus U_{sing} and the markings Σ_i is called the *generic locus* of U and is denoted by U_{gen}.

DEFINITION 2.2

1. An *action of a finite group* Γ on an n-marked nodal curve $(U \to S, \Sigma_i)$ is an action of Γ on U as an S-scheme which leaves the Σ_i invariant. Such an action is called *tame*, if for each geometric point u of U the stabilizer $\Gamma_u \subseteq \Gamma$ of u has order prime to the characteristic of u.
2. Let S be a k-scheme. Let $(U \to S, \Sigma_i)$ be an n-marked nodal curve and let η be a principal G-bundle on U. An *essential action of a finite group* Γ *on* (η, U) is a pair of actions of Γ on η and on $(U \to S, \Sigma_i)$ such that

   (i) the actions of Γ on η and on U are compatible, i.e. if $\pi: \eta \to U$ denotes the projection, then $\pi \circ \gamma = \gamma \circ \pi$ for each $\gamma \in \Gamma$,
   (ii) if $\gamma \in \Gamma$ is an element different from the identity and u is a geometric point of U fixed by γ, then the automorphism of the fiber η_u induced by γ is not trivial.

3. An essential action of a finite group Γ on (η, U) is called *tame*, if the action of Γ on $(U \to S, \Sigma_i)$ is tame.

DEFINITION 2.3

Let S be a k-scheme. Let $C \to S$ be an n-pointed nodal curve and let ξ be a principal G-bundle over C_{gen}. A *chart* (U, η, Γ) *for* ξ consists of the following data:

(1) An n-marked curve $U \to S$ and a strict morphism $\phi: U \to C$.
(2) A principal G-bundle η on U.
(3) An isomorphism $\eta \times_U U_{\text{gen}} \tilde{\to} \xi \times_C U_{\text{gen}}$ of G-bundles on U_{gen}.
(4) A finite group Γ.
(5) A tame, essential action of Γ on (η, U).

These data are required to satisfy the following conditions:

(i) The action of Γ leaves the morphisms $U \to C$ and $\eta \times_U U_{\text{gen}} \tilde{\to} \xi \times_C U_{\text{gen}}$ invariant.
(ii) The induced morphism $U/\Gamma \to C$ is étale.

PROPOSITION 2.4 (see [1], Proposition 3.2.3)

Let $C \to S$ be an n-pointed nodal curve over a k-scheme S and let ξ be a principal G-bundle on C_{gen}. Let (U, η, Γ) be a chart for ξ. Then the following holds:

(1) *The action of* Γ *on* U_{gen} *is* free.
Let s be a geometric point of S and u be a closed point of the curve U_s. Let $\Gamma_u \subseteq \Gamma$ be the stabilizer of u. Then Γ_u is a cyclic group. Let e be its order and let γ_u be a generator of Γ_u. Then
(2) *if u is a regular point, the action of γ_u on the tangent space of U_s at u is via multiplication by a primitive e-th root of unity*,
(3) *if u is a singular point, Γ_u leaves each of the two branches of U_s at u invariant. The action of γ_u on the tangent space of each of the branches is via multiplication with a primitive e-th root of unity.*

DEFINITION 2.5

Let $C \to S$ be an n-pointed nodal curve over a k-scheme S and let ξ be a principal G-bundle on C_{gen}. A chart (U, η, Γ) for ξ is called *balanced*, if for each geometric fiber of $U \to S$ and each singular point u on it the action of γ_u on the tangent spaces of the two branches is via multiplication with primitive roots of unity which are inverse to each other.

DEFINITION 2.6

Let $C \to S$ be an n-pointed nodal curve over a k-scheme S and let ξ be a principal G-bundle on C_{gen}. Two charts (U_1, η_1, Γ_1) and (U_2, η_2, Γ_2) of ξ are called *compatible*, if for each pair of u_1, u_2 of geometric points U_1, U_2 lying above the same geometric point u of C the following holds:

Let C^{sh} denote the strict henselization of C at u. For $j = 1, 2$ let $\Gamma'_j \subseteq \Gamma_j$ denote the stabilizer subgroup of the point u_j. Let U_j^{sh} denote the strict henselization of U_j at u_j, and let $\eta_j^{\text{sh}} := \eta_j \times_{U_j} U_j^{\text{sh}}$. Then there exists an isomorphism $\theta\colon \Gamma'_1 \to \Gamma'_2$, a θ-equivariant isomorphism $\phi\colon U_1^{\text{sh}} \xrightarrow{\sim} U_2^{\text{sh}}$ of C^{sh}-schemes, and a θ-equivariant isomorphism $\eta_1^{\text{sh}} \xrightarrow{\sim} \phi^* \eta_2^{\text{sh}}$ of G-bundles.

DEFINITION 2.7

Let g and n be two non-negative integers. An *n-pointed twisted G-bundle of genus g* is a triple $(\xi, C \to S, \mathcal{A})$ where

(1) S is a k-scheme,
(2) $C \to S$ is a proper n-pointed nodal curve of finite presentation with geometrically connected fibers of genus g,
(3) ξ is a principal G-bundle on C_{gen},
(4) $\mathcal{A} = \{(U_\alpha, \eta_\alpha, \Gamma_\alpha)\}$ is a balanced atlas, i.e. a collection of mutually compatible balanced charts for ξ, such that the images of the U_α cover C.

DEFINITION 2.8

Let $(\xi, C \to S, \mathcal{A})$ be an n-pointed twisted G-bundle of genus g. A morphism of k-schemes $S' \to S$ induces a triple $(\xi', C' \to S', \mathcal{A}')$ as follows:

- The n-pointed nodal curve $C' \to S'$ is the pull back of $C \to S$ by $S' \to S$.
- Thus we have a morphism $C'_{\text{gen}} \to C_{\text{gen}}$, and the G-bundle ξ' is the pull back of ξ by this morphism.
- Let $\{U_\alpha, \eta_\alpha, \Gamma_\alpha\}$ be the set of charts which make up the atlas $\mathcal{A}$. Then $\mathcal{A}' = \{U'_\alpha, \eta'_\alpha, \Gamma_\alpha\}$, where $U'_\alpha \to S'$ is the pull back of the n-marked curve $U_\alpha \to S$, and η'_α is the pull back of η_α by the morphism $U'_\alpha \to U_\alpha$. Since the $(U'_\alpha, \eta'_\alpha, \Gamma_\alpha)$ are charts for ξ' which are balanced and mutually compatible (see [1], Proposition 3.4.3), $\mathcal{A}'$ is a balanced atlas.

Thus the triple $(\xi', C' \to S', \mathcal{A}')$ is an n-pointed twisted G-bundle of genus g. It is called the *pull back* of $(\xi, C \to S, \mathcal{A})$ by the morphism $S' \to S$.

DEFINITION 2.9

A *morphism* between two n-pointed twisted G-bundles $(\xi', C' \to S', \mathcal{A}')$ and $(\xi, C \to S, \mathcal{A})$ consists of a Cartesian diagram

$$\begin{array}{ccc} C' & \longrightarrow & C \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

and an isomorphism $\xi' \xrightarrow{\sim} \xi \times_{C_{\text{gen}}} C'_{\text{gen}}$ such that the pull-back of the charts in $\mathcal{A}$ (considered as charts for ξ') are compatible with all the charts in $\mathcal{A}'$.

3. Review of Gieseker vector bundles

In this section I will recall some definitions from my earlier papers [4] and [5].

Let k be an algebraically closed field. Let $n \geq 1$ be an integer and let $R_1, \dots, R_n$ be n copies of the projective line $\mathbb{P}^1$. On each R_i we choose two distinct points x_i and y_i. Let R be the nodal curve over k constructed from $R_1, \dots, R_n$ by identifying y_i with x_{i+1} for $i = 1, \dots, n-1$. We call such a curve R a *chain of projective lines* of length n with components $R_1, \dots, R_n$. On the extremal components R_1 and R_n we have the two points x_1 and y_n respectively, which are smooth points of R.

DEFINITION 3.1

A vector bundle E of rank r on R is called *admissible*, if

(1) for each $i \in [1, n]$ the restriction of E on the component R_i is of the form

$$d_i \mathcal{O}_{R_i}(1) \oplus (r - d_i)\mathcal{O}_{R_i}$$

for some integer $d_i \geq 1$ and

(2) there exists no nonvanishing global section of E over R which vanishes in the two points x_1 and y_n.

Let C be an irreducible curve with exactly one double point p. Let $\widetilde{C} \to C$ be the normalization of C and let $p_1, p_2 \in \widetilde{C}$ be the two points lying above p. Let $C_0 := C$. For $n \geq 1$ we let C_n denote the reducible nodal curve which is constructed from $\widetilde{C}$ and a chain $R = R_1 \cup \dots \cup R_n$ of projective lines by identifying the points p_1, x_1 and p_2, y_n respectively.

DEFINITION 3.2

A *Gieseker vector bundle* on C is a pair $(C' \to C, \mathcal{F})$ where $C' = C_n$ for some $n \geq 0$, the morphism $C' \to C$ is the one which contracts the chain of projective lines to the point p and $\mathcal{F}$ is a vector bundle on C' whose restriction to the chain of projective lines is admissible in the sense of 3.1.

DEFINITION 3.3

A *Gieseker vector bundle datum* on the two-pointed curve $(\widetilde{C}, p_1, p_2)$ is a triple $(C' \to C, F, p')$, where $(C' \to C, \mathcal{F})$ is a Gieseker vector bundle on C and p' is a singular point on C'.

Let V and W be two r-dimensional k-vector spaces. In [4] I have constructed a certain compactification $\mathrm{KGL}(V, W)$ of the space $\mathrm{Isom}(V, W)$ of linear isomorphisms from V to W which has properties similar to De Concini's and Procesi's so-called wonderful compactification of adjoint linear groups. We need the following fact about $\mathrm{KGL}(V, W)$ whose proof can be found in §9 of [4].

The variety $\mathrm{KGL}(V, W)$ is the disjoint union of strata $\mathbf{O}_{I,J} \subset \mathrm{KGL}(V, W)$ indexed by pairs of subsets $I, J \in [0, r-1]$ such that $\min(I) + \min(J) \geq r$. Let I, J be such a pair. Let us write $I = \{i_1, \ldots, i_{n_1}\}$ and $J = \{j_1, \ldots, j_{n_2}\}$ where $i_1 < \cdots < i_{n_1} < i_{n_1+1} := r$ and $j_1 < \cdots < j_{n_2} < j_{n_2+1} := r$. A ($k$-valued) point in $\mathbf{O}_{I,J}$ is given by the data

$$\Phi = (F_\bullet(V), F_\bullet(W), \overline{\varphi}_1, \ldots, \overline{\varphi}_{n_1}, \overline{\psi}_1, \ldots, \overline{\psi}_{n_2}, \Phi'),$$

where

(1) $F_\bullet(V)$ denotes a flag

$$0 = F_0(V) \subsetneq F_1(V) \subsetneq \cdots \subsetneq F_{n_2}(V) \subseteq F_{n_2+1}(V) \subsetneq \cdots \subsetneq F_{n_1+n_2+1}(V) = V$$

with $\dim F_\nu(V) = r - j_{n_2+1-\nu}$ for $\nu \in [0, n_2]$ and $\dim F_\nu(V) = i_{\nu-n_2}$ for $\nu \in [n_2+1, n_1+n_2+1]$.

(2) $F_\bullet(W)$ denotes a flag

$$\begin{aligned} 0 = F_0(W) \subsetneq F_1(W) \subsetneq \cdots \subsetneq F_{n_1}(W) \\ \subseteq F_{n_1+1}(W) \subsetneq \cdots \subsetneq F_{n_1+n_2+1}(W) = W, \end{aligned}$$

where $\dim F_\nu(W) = r - i_{n_1+1-\nu}$ for $\nu \in [0, n_1]$ and $\dim F_\nu(W) = i_{\nu-l}$ for $\nu \in [n_1+1, n_1+n_2+1]$.

(3) The symbol $\overline{\varphi}_\nu$ denotes the homothety class of an isomorphism from the subquotient $F_{n_1-\nu+1}(W)/F_{n_1-\nu}(W)$ of W to the subquotient $F_{n_2+\nu+1}(V)/F_{n_2+\nu}(V)$ of V.

(4) The symbol $\overline{\psi}_\nu$ denotes the homothety class of an isomorphism from the subquotient $F_{n_2-\nu+1}(V)/F_{n_2-\nu}(V)$ of V to the subquotient $F_{n_1+\nu+1}(W)/F_{n_1+\nu}(W)$ of W.

(5) The symbol Φ' denotes an isomorphism from the subquotient $F_{n_2+1}(V)/F_{n_2}(V)$ of V to the subquotient $F_{n_1+1}(W)/F_{n_1}(W)$ of W.

The relationship between Gieseker vector bundles and the compactification $\mathrm{KGL}(V, W)$ is given by the following theorem.

Theorem 3.4 ([5], Theorem 9.5). *There exists a natural bijection from the set of all Gieseker vector bundle data on* $(\widetilde{C}, p_1, p_2)$ *to the set of all pairs* $(\mathcal{E}, \Phi)$, *where* $\mathcal{E}$ *is a vector bundle on* $\widetilde{C}$ *and* Φ *is a* k*-valued point in* $\mathrm{KGL}(\mathcal{E}[p_1], \mathcal{E}[p_2])$.

More precisely, let $(C' \to C, \mathcal{F}, p')$ *be a Gieseker vector bundle datum on* $(\widetilde{C}, p_1, p_2)$. *Let* $R = R_1 \cup \cdots \cup R_n$ *be the chain of projective lines in* C'. *Let* $y_0 := p_1$ *and* $x_{n+1} := p_2$. *Let* $n_1 + n_2 = n$ *be such that the singular point* $p' \in C'$ *comes from identifying the points* y_{n_1} *and* x_{n_1+1}. *Let* d_i *be the degree of* $\mathcal{F}$ *restricted to* R_i. *Let* $(\mathcal{E}, \Phi)$ *be the pair associated to the given Gieseker vector bundle datum* $(C' \to C, \mathcal{F}, p')$. *Then* Φ *is in fact a point in the stratum* $\mathbf{O}_{I,J}$, *where* $I = \{i_1, \ldots, i_{n_1}\}$, $J = \{j_1, \ldots, j_{n_2}\}$ *and the* i_ν, j_ν *are defined by*

$$i_\nu = r - \sum_{i=\nu}^{n_1} d_i, \quad j_\nu = r - \sum_{i=n_1+1}^{n-\nu+1} d_i.$$

The special case $n = 0$ *is included here in the sense that* $I = J = \emptyset$ *and* $\Phi \in \mathbf{O}_{\emptyset,\emptyset} = \mathrm{Isom}(\mathcal{E}[p_1], \mathcal{E}[p_2])$.

4. Twisted GL_r-bundles on a fixed curve

Throughout this section k denotes an algebraically closed field and r a positive integer.

Let (C, p_i) be an n-pointed nodal curve over k. Let $\mathrm{TVB}_r(C, p_i)$ be the set of isomorphism classes of n-pointed twisted GL_r-bundles of the form

$$(\xi, C \to \mathrm{Spec}(k), \mathcal{A}).$$

The case of a one-pointed smooth curve

Assume that C is smooth and that $n = 1$, i.e. $(C, p_i) = (C, p)$ is a one-pointed smooth curve. Let $\mathrm{PB}_r(C, p)$ be the set of isomorphism classes of vector bundles E of rank r on C together with a flag in the fiber at p.

Theorem 4.1. *There is a natural surjection*

$$\mathrm{TVB}_r(C, p) \to \mathrm{PB}_r(C, p).$$

We skip the proof of Theorem 4.1, since on the one hand the result is well-known [3,7] and on the other hand there is a proof analogous to (and easier than) the proof of Theorem 4.2 which we give in detail below.

The case of a nodal curve with one singularity

Assume now that $n = 0$ and C has exactly one double point. Let $\mathrm{GVB}_r(C)$ be the set of isomorphism classes of Gieseker vector bundles of rank r on C.

Theorem 4.2. *There is a natural surjection*

$$\mathrm{TVB}_r(C) \to \mathrm{GVB}_r(C).$$

The rest of the paper is concerned with the proof of Theorem 4.2.

5. Construction

Let C be a nodal curve over $\mathrm{Spec}(k)$ with one singular point p. Let $(\xi, C \to \mathrm{Spec}(k), \mathcal{A})$ be an object of $\mathrm{TVB}_r(C)$. Let (U, η, Γ) be a chart belonging to $\mathcal{A}$ such that there is a point $q \in U$ which is mapped to p.

We denote by $\widehat{\mathcal{O}}_p$ and $\widehat{\mathcal{O}}_q$ the completion of the local rings $\mathcal{O}_{C,p}$ and $\mathcal{O}_{U,q}$ respectively. Let $\Gamma_q \subseteq \Gamma$ be the subgroup consisting of those elements, which leave q invariant. Γ_q acts on $\widehat{\mathcal{O}}_q$, and $\widehat{\mathcal{O}}_p$ may be identified with the set of invariants under that action. By Proposition 2.4, the group Γ_q is cyclic of some order e (which is prime to $\mathrm{char}(k)$ by the tameness assumption). Let γ be a generator of Γ_q.

We choose an isomorphism

$$\widehat{\mathcal{O}}_p \xrightarrow{\sim} k[[s,t]]/(s \cdot t). \tag{1}$$

It follows from Proposition 2.4(3) that there exists an isomorphism

$$\widehat{\mathcal{O}}_q \xrightarrow{\sim} k[[u,v]]/(u \cdot v) \tag{2}$$

and a primitive e-th root of unity ζ such that the diagrams

$$\begin{array}{ccc} \widehat{\mathcal{O}}_q & \xrightarrow{\cong} & k[[u,v]]/(u \cdot v) \\ \big\downarrow{\scriptstyle\gamma} & & \big\downarrow \\ \widehat{\mathcal{O}}_q & \xrightarrow{\cong} & k[[u,v]]/(u \cdot v) \end{array} \qquad \begin{array}{c} u \\ \big\downarrow \\ \zeta u \end{array} \qquad \begin{array}{c} v \\ \big\downarrow \\ \zeta^{-1} v \end{array}$$

and

$$\begin{array}{ccc} \widehat{\mathcal{O}}_q & \xrightarrow{\cong} & k[[u,v]]/(u \cdot v) \\ \big\uparrow & & \big\uparrow \\ \widehat{\mathcal{O}}_p & \xrightarrow{\cong} & k[[s,t]]/(s \cdot t) \end{array} \qquad \begin{array}{c} u^e \\ \big\uparrow \\ s \end{array} \qquad \begin{array}{c} v^e \\ \big\uparrow \\ t \end{array}$$

are commutative.

Let $\widehat{K}_p$ be the total quotient ring of $\widehat{\mathcal{O}}_p$. Then we have $\mathrm{Spec}(\widehat{K}_p) = \mathrm{Spec}(\widehat{\mathcal{O}}_p) \times_C C_{\mathrm{gen}}$ and the isomorphism (1) induces an isomorphism $\widehat{K}_p \xrightarrow{\sim} k((s)) \times k((t))$. We choose an isomorphism

$$\xi \times_{C_{\mathrm{gen}}} \mathrm{Spec}(\widehat{K}_p) \xrightarrow{\sim} \mathrm{GL}_r \times \mathrm{Spec}(\widehat{K}_p). \tag{3}$$

The group Γ_q acts on $\eta \times_U \mathrm{Spec}(\widehat{\mathcal{O}}_q)$ (since it acts compatibly on η, U, $\mathrm{Spec}(\widehat{\mathcal{O}}_q)$). To analyse this action we need the following lemma.

Lemma 5.1. *Let k be an algebraically closed field. Let $(R, \mathfrak{m})$ be a local k-algebra with residue field $R/\mathfrak{m} = k$. Let Γ be a cyclic group of order e prime to the characteristic of k and let $\gamma \in \Gamma$ be a generator. Assume that Γ acts on R such that the induced action on k is trivial. Let M be a trivial R-module of rank r on which Γ acts such that $\gamma(ax) = \gamma(a)\gamma(x)$ for all $a \in R$, $x \in M$. Then there is a basis $x_1, \ldots, x_r$ of M such that $\gamma(x_i) = \zeta_i x_i$ for some e-th roots of unity ζ_i.*

Proof. Let $e_1, \ldots, e_r$ be an arbitrary basis of M. Let $a = (a_{i,j}) \in \mathrm{GL}_r(R)$ be defined by $\gamma(e_j) = \sum_{i=1}^{r} a_{i,j} e_i$. Since γ is of order e, it follows that

$$\prod_{j=0}^{e-1} \gamma^j(a) = 1.$$

We have to show that there is a matrix $b \in \mathrm{GL}_r(R)$ such that

$$a \cdot \gamma(b) = b \cdot z$$

for some diagonal matrix $z \in \mathrm{GL}_r(k)$ with $z^e = 1$.

Representation theory of finite groups tells us that there is a matrix $c \in \mathrm{GL}_r(k)$ and a diagonal matrix $z \in \mathrm{GL}_r(k)$ with $z^e = 1$ such that $a \cdot c \equiv c \cdot z$ modulo $\mathfrak{m}$. Let $a' := c^{-1} \cdot a \cdot c$ and let b' be the matrix

$$b' := \sum_{i=0}^{e-1} \left(\prod_{j=0}^{i-1} \gamma^j(a') \right) z^{-i}.$$

Since $b \equiv e \cdot 1$ modulo $\mathfrak{m}$, it follows that $b' \in \mathrm{GL}_r(R)$. Using the fact that $\prod_{i=0}^{e-1} \gamma^i(a') = 1$ a simple calculation shows that

$$\gamma(b') = (a')^{-1} \cdot b' \cdot z.$$

Therefore, if we set $b := c \cdot b'$, we get the desired equality. □

COROLLARY 5.2

There exists an isomorphism

$$\eta \times_U \mathrm{Spec}(\widehat{\mathcal{O}}_q) \xrightarrow{\sim} \mathrm{GL}_r \times \mathrm{Spec}(\widehat{\mathcal{O}}_q) \tag{4}$$

of principal GL_r*-bundles on* $\mathrm{Spec}(\widehat{\mathcal{O}}_q)$, *and elements* $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}/e\mathbb{Z}$ *such that the following diagram commutes*:

$$\begin{array}{ccc} \eta \times_U \mathrm{Spec}(\widehat{\mathcal{O}}_q) & \xrightarrow{\cong} & \mathrm{GL}_r \times \mathrm{Spec}(\widehat{\mathcal{O}}_q) \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \mathrm{diag}(\zeta^{\alpha_1}, \ldots, \zeta^{\alpha_r}) \times \gamma} \\ \eta \times_U \mathrm{Spec}(\widehat{\mathcal{O}}_q) & \xrightarrow{\cong} & \mathrm{GL}_r \times \mathrm{Spec}(\widehat{\mathcal{O}}_q) \end{array}$$

where the morphism $\mathrm{diag}(\zeta^{\alpha_1}, \ldots, \zeta^{\alpha_r})\colon \mathrm{GL}_r \to \mathrm{GL}_r$ *is a multiplication from the left with the matrix whose only non-zero entries are the values* $\zeta^{\alpha_1}, \ldots, \zeta^{\alpha_r}$ *on the diagonal.*

Proof. This is immediate from Lemma 5.1. □

Let $\widehat{K}_q$ be the total quotient ring of $\widehat{\mathcal{O}}_q$. The Γ-equivariant isomorphism $\eta \times_U U_{\mathrm{gen}} \xrightarrow{\sim} \xi \times_{C_{\mathrm{gen}}} U_{\mathrm{gen}}$, which is part of the data of the chart (U, η, Γ), induces a Γ_q-equivariant isomorphism

$$\eta \times_U \mathrm{Spec}(\widehat{K}_q) \xrightarrow{\sim} \xi \times_{C_{\mathrm{gen}}} \mathrm{Spec}(\widehat{K}_q) \tag{5}$$

of principal GL_r-bundles over $\mathrm{Spec}(\widehat{K}_q)$.

Via the isomorphisms (3) and (4) such an isomorphism is given by a matrix $F \in \mathrm{GL}_r(\widehat{K}_q)$ such that

$$\gamma(F) = F \cdot \mathrm{diag}(\zeta^{\alpha_1}, \ldots, \zeta^{\alpha_r}).$$

The isomorphism (2) induces an isomorphism $\mathrm{GL}_r(\widehat{K}_q) \xrightarrow{\sim} \mathrm{GL}_r(k((u))) \times \mathrm{GL}_r(k((v)))$ and we denote by $(F^1(u), F^2(v))$ the image of F under this isomorphism. The above condition on F translates into the conditions

$$F^1_{i,j}(\zeta u) = \zeta^{\alpha_j} F^1_{i,j}(u), \tag{6}$$

$$F^2_{i,j}(\zeta^{-1} v) = \zeta^{\alpha_j} F^2_{i,j}(v), \tag{7}$$

for the entries $F^1_{i,j}(u) \in k((u))$ and $F^2_{i,j}(v) \in k((v))$ of the matrices $F^1(u)$ and $F^2(v)$.

After possibly changing the isomorphism (4) by a permutation matrix, we can choose integers $a_1, \ldots, a_r$ with

$$0 \leq a_1 \leq a_2 \leq \cdots \leq a_r < e \quad \text{and} \quad a_i \equiv \alpha_i \bmod e\mathbb{Z}. \tag{8}$$

Conditions (6), (7) imply that there are matrices $H^1(s)$ and $H^2(t)$ with entries $H^1_{i,j}(s) \in k((s))$ and $H^2_{i,j}(t) \in k((t))$ such that

$$F^1_{i,j}(u) = u^{a_j} H^1_{i,j}(u^e), \tag{9}$$

$$F^2_{i,j}(v) = v^{-a_j} H^2_{i,j}(v^e). \tag{10}$$

We will now use the GL_r-bundle ξ over C_{gen}, the isomorphisms (1) and (3), the numbers $a_1, \ldots, a_r$ and the matrices $H^1(s)$ and $H^2(t)$, to construct a Gieseker vector bundle of rank r on the curve C.

Let p_1 and p_2 denote the closed point of $\mathrm{Spec}(k[[s]])$ and $\mathrm{Spec}(k[[t]])$ respectively. Let $\mathcal{V}$ be the trivial vector bundle $\mathcal{O}^{[1,r]}$ on the disjoint union $\mathrm{Spec}(k[[s]]) \sqcup \mathrm{Spec}(k[[t]])$ (the normalization of $\mathrm{Spec}(k[[s,t]]/(s \cdot t))$), and let V and W be its fiber at p_1 and p_2 respectively. Of course, both V and W are naturally identified with $k^{[1,r]}$.

The numbers $a_1, \ldots, a_r$ define a partition

$$[1, r] = D_1 \sqcup D_2 \sqcup \cdots \sqcup D_m$$

characterized by the following properties:

(1) D_1 is the (possibly empty) set of all indices i such that $a_i = 0$.
(2) For $\nu \geq 2$ the set D_ν is non-empty.
(3) If $1 \leq \nu < \nu' \leq m$, $i \in D_\nu$ and $j \in D_{\nu'}$ then $a_i < a_j$.
(4) For all $\nu \in [1, m]$ and $i, j \in D_\nu$ we have $a_i = a_j$.

We define filtrations

$$0 = F_0(V) \subseteq F_1(V) \subsetneq F_2(V) \subsetneq \cdots \subsetneq F_{m-1}(V) \subsetneq F_m(V) = V,$$

$$0 = F_0(W) \subsetneq F_1(W) \subsetneq F_2(W) \subsetneq \cdots \subsetneq F_{m-1}(W) \subseteq F_m(W) = W,$$

by setting

$$F_i(V) := k^{D_1 \sqcup \cdots \sqcup D_i} \quad \text{and} \quad F_i(W) := k^{D_{m-i+1} \sqcup \cdots \sqcup D_m}$$

for $i = 0, \dots, m$. For $i = 1, \dots, m-1$, let

$$\varphi_i \colon F_{m-i}(W)/F_{m-i-1}(W) = k^{D_{i+1}} \xrightarrow{\sim} k^{D_{i+1}} = F_{i+1}(V)/F_i(V)$$

be the identity morphism on $k^{D_{i+1}}$ and let $\overline{\varphi}_i$ be the homothety class of φ_i. Finally let

$$\Phi' \colon F_1(V)/F_0(V) = k^{D_1} \xrightarrow{\sim} k^{D_1} = F_m(W)/F_{m-1}(W)$$

be the identity morphism on k^{D_1}.

By [4], §9.3 the data

$$((F_\bullet(V), F_\bullet(W)), \overline{\varphi}_1, \dots, \overline{\varphi}_{m-1}, \Phi')$$

defines a k-valued point of $\mathrm{KGL}(V, W)$, i.e. a generalized isomorphism Φ from V to W.

Let $\widetilde{C} \to C$ be the normalization of the curve C. By a slight abuse of notation we denote also by p_1, p_2 the two points of $\widetilde{C}$ which lie above the singular point p of C. Let $\mathcal{E}_\xi$ be the rank r vector bundle on $C_{\mathrm{gen}} = \widetilde{C} \setminus \{p_1, p_2\}$ associated to the principal GL_r-bundle ξ.

We use the isomorphism

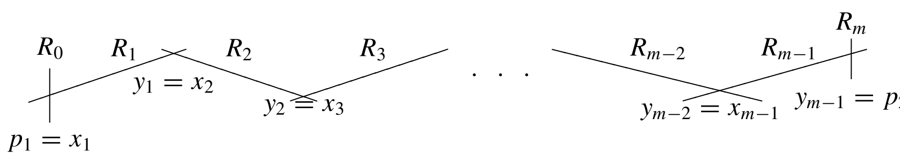
$$\begin{array}{ccc} (k((s)) \times k((t)))^{[1,r]} & \xrightarrow{(H^1, H^2)} & (k((s)) \times k((t)))^{[1,r]} \\ \Big\| & & \cong \Big\uparrow (1), (3) \\ \mathcal{V} \otimes_{k[[s]] \times k[[t]]} (k((s)) \times k((t))) & & \mathcal{E}_\xi \otimes_{\mathcal{O}_{\widetilde{C}}} \widehat{K}_p \end{array}$$

as a glueing datum to define a vector bundle $\mathcal{E}$ on $\widetilde{C}$, whose fibers at the points p_1 and p_2 are naturally identified with V and W respectively. By Theorem 3.4 the pair $(\mathcal{E}, \Phi)$ induces a Gieseker vector bundle datum $(C' \to C, \mathcal{F}, p')$ on $(\widetilde{C}, p_1, p_2)$ which in turn induces a Gieseker vector bundle $(C' \to C, \mathcal{F})$ on C.

For the convenience of the reader I will now describe the Gieseker vector bundle $(C' \to C, \mathcal{F})$ explicitly. Let $R_0 := \mathrm{Spec}(k[[s]])$, $R_m := \mathrm{Spec}(k[[t]])$. If $m = 1$, we set $R = \mathrm{Spec}(k[[s, t]]/(s \cdot t))$, which is nothing else but the nodal curve which arises from $R_0 \sqcup R_m$ by identifying the points p_1 and p_2. If $m \geq 2$, let $R_1, \dots, R_{m-1}$ be $m-1$ copies of the projective line $\mathbb{P}^1$ and let x_i, y_i be two distinct points in R_i. Let R be the nodal curve which arises from the union

$$R_0 \sqcup R_1 \sqcup \cdots \sqcup R_{m-1} \sqcup R_m$$

by identifying $p_1 \in R_0$ and $p_2 \in R_m$ with $x_1 \in R_1$ and $y_{m-1} \in R_{m-1}$ respectively and by identifying $y_i \in R_i$ with $x_{i+1} \in R_{i+1}$ for $i \in [1, m-2]$:



.

Let $\mathcal{O}_{R_i}(1)$ be the defining bundle on $R_i = \mathbb{P}^1$ together with isomorphisms

$$\mathcal{O}_{R_i}(1)[x_i] \xrightarrow{\sim} k \quad \text{and} \quad \mathcal{O}_{R_i}(1)[y_i] \xrightarrow{\sim} k. \tag{11}$$

We define the rank r vector bundles

$$E_i := \mathcal{O}_{R_i}^{D_1 \sqcup \cdots \sqcup D_i} \oplus \mathcal{O}_{R_i}(1)^{D_{i+1}} \oplus \mathcal{O}_{R_i}^{D_{i+2} \sqcup \cdots \sqcup D_m}$$

on R_i together with the isomorphisms

$$E_i[x_i] \xrightarrow{\sim} k^{[1,r]} \quad \text{and} \quad E_i[y_i] \xrightarrow{\sim} k^{[1,r]} \tag{12}$$

induced by (11).

The maximal ideal $sk[[s]]$ of $k[[s]]$ is a free module of rank one and as such defines a line bundle $\mathcal{O}_{R_0}(-1)$ on $R_0 = \mathrm{Spec}(k[[s]])$. We consider this line bundle together with the isomorphism

$$\mathcal{O}_{R_0}(-1)[p_2] \xrightarrow{\sim} k \tag{13}$$

given by $sk[[s]]/s^2k[[s]] \to k$, $s \mapsto 1$. The generic fiber of $\mathcal{O}_{R_0}(-1)$ is identified with $k((s))$ via the inclusion $sk[[s]] \hookrightarrow k[[s]]$. Then we have the rank r vector bundles

$$E_0 := \mathcal{O}_{R_0}^{D_1} \oplus \mathcal{O}_{R_0}(-1)^{D_2 \sqcup \cdots \sqcup D_m} \quad \text{and} \quad E_m := \mathcal{O}_{R_m}^{[1,r]}$$

on R_0 and R_m together with isomorphisms

$$E_0[p_1] \xrightarrow{\sim} k^{[1,r]} \quad \text{and} \quad E_m[p_2] \xrightarrow{\sim} k^{[1,r]} \tag{14}$$

(the first one being induced by (13)) and isomorphisms

$$E_0 \otimes_{\mathcal{O}_{R_0}} k((s)) \xrightarrow{\sim} k((s))^{[1,r]} \quad \text{and} \quad E_m \otimes_{\mathcal{O}_{R_m}} k((t)) \xrightarrow{\sim} k((t))^{[1,r]}. \tag{15}$$

The vector bundles $E_0, \ldots, E_m$ glue together via the isomorphisms (12) and (14) to form a rank r vector bundle E on R.

Let $C' \to C$ be the modification of C obtained by glueing together R and C_{gen} along the isomorphism

$$\begin{array}{ccc} \mathrm{Spec}(k((s))) \sqcup \mathrm{Spec}(k((t))) & \xrightarrow{(1)} & \mathrm{Spec}(\widehat{K}_p) \\ \downarrow & & \downarrow \\ R & & C_{\mathrm{gen}} \end{array}$$

and let $\mathcal{F}$ be the rank r vector bundle on C' obtained by glueing together E and $\mathcal{E}_{\mathrm{gen}}$ via the isomorphism

$$\begin{array}{ccc} (k((s)) \times k((t)))^{[1,r]} & \xrightarrow{(H^1, H^2)} & (k((s)) \times k((t)))^{[1,r]} \\ \cong \uparrow (15) & & \cong \uparrow (1), (3) \\ E \otimes_{\mathcal{O}_R} (k((s)) \times k((t))) & & \mathcal{E}_\xi \otimes_{\mathcal{O}_{\widetilde{C}}} \widehat{K}_p \end{array}.$$

It is easy to check that $(C' \to C, \mathcal{F})$ is indeed a Gieseker vector bundle on C.

It remains to be shown that the association

$$(\xi, C \to \mathrm{Spec}(k), \mathcal{A}) \;\mapsto\; (C' \to C, \mathcal{F})$$

is surjective and does not depend on the choices (1), (2), (3) and (4) which we made during the construction. This will be done in the following sections.

6. Independence of the isomorphisms (1) and (2)

Let

$$\widehat{\mathcal{O}}_p \xrightarrow{\sim} k[[s,t]]/(s\cdot t) \tag{$1'$}$$

be another isomorphism and let

$$\widehat{\mathcal{O}}_q \xrightarrow{\sim} k[[u,v]]/(u\cdot v) \tag{$2'$}$$

be an isomorphism with the required property with respect to ($1'$). For the moment we make the following assumption:

The images of the two minimal ideals of $\widehat{\mathcal{O}}_p$ under (1) and ($1'$) are the same. (*)

Then there are units $\sigma(s), \pi(s) \in k[[s]]^\times$ and $\tau(t), \omega(t) \in k[[t]]^\times$ such that the following diagrams commute:

$$\begin{array}{ccc} k[[s,t]]/(s\cdot t) & \xrightarrow[t\mapsto t\tau(t)]{s\mapsto s\sigma(s)} & k[[s,t]]/(s\cdot t) \\ & {\scriptstyle (1)}\nwarrow \quad \nearrow{\scriptstyle (1')} & \\ & \widehat{\mathcal{O}}_p & \end{array} ,$$

$$\begin{array}{ccc} k[[u,v]]/(u\cdot v) & \xrightarrow[v\mapsto v\omega(v^e)]{u\mapsto u\pi(u^e)} & k[[u,v]]/(u\cdot v) \\ & {\scriptstyle (2)}\nwarrow \quad \nearrow{\scriptstyle (2')} & \\ & \widehat{\mathcal{O}}_p & \end{array} .$$

Furthermore we have $\pi^e = \sigma$ and $\omega^e = \tau$.

It should be noticed that the e-th root of unity ζ is independent of whether we choose (1) or ($1'$), since it is the eigenvalue of γ operating on the tangent space of one of the branches of $\mathrm{Spec}(\widehat{\mathcal{O}}_q)$ and by assumption ($*$) both the isomorphisms (1) and ($1'$) map that branch $\mathrm{Spec}(\widehat{\mathcal{O}}_q)$ to the same branch $\{v=0\}$ of $\mathrm{Spec}(k[[u,v]]/(u\cdot v))$. Therefore the elements $\alpha_1,\ldots,\alpha_r \in \mathbb{Z}/e\mathbb{Z}$ and the numbers $a_1,\ldots,a_r$ are independent of whether we choose (1) or ($1'$).

Let $(\tilde{F}_1(u), \tilde{F}_2(v))$ be the image of F under the isomorphism $\mathrm{GL}_r(\widehat{K}_q) \xrightarrow{\sim} \mathrm{GL}_r(k[[u]]) \times \mathrm{GL}_r(k[[v]])$ induced by ($2'$). Then we have $\tilde{F}^1(u) = F^1(u\pi(u^e))$ and $\tilde{F}^2(v) = F^2(v\omega(v^e))$ and it follows that

$$\tilde{F}^1_{i,j}(u) = u^{a_j}\cdot \tilde{H}^1_{i,j}(u^e),$$

$$\tilde{F}^2_{i,j}(v) = v^{-a_j}\cdot \tilde{H}^2_{i,j}(v^e),$$

where

$$\tilde{H}^1_{i,j}(s) = \pi^{a_j} H^1_{i,j}(s\sigma),$$

$$\tilde{H}^2_{i,j}(t) = \omega^{-a_j} H^2_{i,j}(t\tau).$$

Therefore the following diagram commutes:

$$
\begin{array}{ccccc}
(k((s)) \times k((t)))^{[1,r]} & & \xrightarrow{(H^1, H^2)} & & (k((s)) \times k((t)))^{[1,r]} \\
\Big\downarrow & & \overset{\cong}{\nearrow}{\scriptstyle (1),(3)} & & \Big\downarrow {\scriptstyle s \mapsto s\sigma \;|\; t \mapsto t\tau} \\
 & & \mathcal{E}_\xi \otimes_{\mathcal{O}_{\widetilde{C}}} \widehat{K}_p & & \\
 & & \underset{\cong}{\searrow}{\scriptstyle (1'),(3)} & & \\
(k((s)) \times k((t)))^{[1,r]} & & \xrightarrow{(\tilde{H}^1, \tilde{H}^2)} & & (k((s)) \times k((t)))^{[1,r]}
\end{array}
$$

where the left vertical arrow maps an element $(x(s), y(t))$ to the element

$$(\mathrm{diag}(\pi^{a_1}, \dots, \pi^{a_r})x(s\sigma), \mathrm{diag}(\omega^{-a_1}, \dots, \omega^{-a_r})y(t\tau)).$$

Let $\tilde{\mathcal{E}}$ be the vector bundle on $\widetilde{C}$ obtained by glueing datum $(\tilde{H}^1, \tilde{H}^2)$. Then the above diagram shows that there is an isomorphism $\mathcal{E} \xrightarrow{\sim} \tilde{\mathcal{E}}$ which induces the isomorphisms

$$\mathcal{E}[p_1] = k^{[1,r]} \xrightarrow{\mathrm{diag}(\pi(0)^{-a_1}, \dots, \pi(0)^{-a_r})} k^{[1,r]} = \tilde{\mathcal{E}}[p_1],$$
$$\mathcal{E}[p_2] = k^{[1,r]} \xrightarrow{\mathrm{diag}(\omega(0)^{a_1}, \dots, \omega(0)^{a_r})} k^{[1,r]} = \tilde{\mathcal{E}}[p_2],$$

between the fibers at p_1 and p_2 respectively. Thus it maps the generalized isomorphism $\Phi \in \mathrm{KGL}(k^{[1,r]}, k^{[1,r]}) = \mathrm{KGL}(\mathcal{E}[p_1], \mathcal{E}[p_2])$ to the generalized isomorphism $\Phi \in \mathrm{KGL}(k^{[1,r]}, k^{[1,r]}) = \mathrm{KGL}(\tilde{\mathcal{E}}[p_1], \tilde{\mathcal{E}}[p_2])$. This shows that the pairs $(\mathcal{E}, \Phi)$ and $(\tilde{\mathcal{E}}, \Phi)$ are isomorphic. Consequently this is also true for the associated Gieseker vector bundles.

To get rid of the assumption $(*)$ we now investigate what happens if we change the isomorphisms (1), (2) by composing them with the automorphisms

$$k[[s,t]]/(s \cdot t) \xrightarrow[t \mapsto s]{s \mapsto t} k[[s,t]]/(s \cdot t),$$
$$k[[u,v]]/(u \cdot v) \xrightarrow[v \mapsto u]{u \mapsto v} k[[u,v]]/(u \cdot v),$$

respectively.

This means that ζ^{-1} takes the role of ζ and consequently the set $\{\alpha_1, \dots, \alpha_r\} \subseteq \mathbb{Z}/e\mathbb{Z}$ from 5.2 is replaced by the set $\{-\alpha_1, \dots, -\alpha_r\}$. It follows that in (8) we would choose integers $\tilde{a}_1, \dots, \tilde{a}_r$ instead of $a_1, \dots, a_r$, where

$$\tilde{a}_i = \begin{cases} a_i, & \text{for } i \in [1, i_1] = D_1, \\ e - a_{r+i_1+1-i}, & \text{for } i \in [i_1 + 1, r]. \end{cases}$$

Then the matrix F is replaced by the matrix $\tilde{F} = F \cdot \Lambda$, where

$$\Lambda = \left[\begin{array}{c|c} \mathbb{I}_{i_1} & 0 \\ \hline 0 & \begin{matrix} & & 1 \\ & \iddots & \\ 1 & & \end{matrix} \end{array}\right]$$

is the permutation matrix belonging to the permutation $\lambda \in S_r$, where

$$\lambda(i) = \begin{cases} i & \text{for } i \in [1, i_1] \\ r + i_1 + 1 - i, & \text{for } i \in [i_1 + 1, r] \end{cases}$$

and the matrices $H^1(s)$ and $H^2(t)$ are replaced by the matrices $\tilde{H}^1(s)$ and $\tilde{H}^2(t)$ respectively, where

$$\tilde{H}^1(s) = H^2(s) \cdot \Lambda \cdot \begin{bmatrix} \mathbb{I}_{i_1} & 0 \\ 0 & s^{-1}\mathbb{I}_{r-i_1} \end{bmatrix}$$

and

$$\tilde{H}^2(t) = H^1(t) \cdot \Lambda \cdot \begin{bmatrix} \mathbb{I}_{i_1} & 0 \\ 0 & t\mathbb{I}_{r-i_1} \end{bmatrix}.$$

The numbers $\tilde{a}_1, \dots, \tilde{a}_r$ define the partition

$$[1, r] = \tilde{D}_1 \sqcup \tilde{D}_2 \sqcup \dots \sqcup \tilde{D}_m,$$

where $\tilde{D}_1 = D_1$ and $\tilde{D}_i = \lambda(D_{m+2-i})$ for $i \in [2, m]$. Let $(\tilde{R} = \tilde{R}_0 \cup \dots \cup \tilde{R}_m, \tilde{E})$ be the nodal curve associated to this partition, together with isomorphisms $\tilde{E} \otimes_{\mathcal{O}_{\tilde{R}}} k((s)) \xrightarrow{\sim} k((s))^{[1,r]}$ and $\tilde{E} \otimes_{\mathcal{O}_{\tilde{R}}} k((t)) \xrightarrow{\sim} k((t))^{[1,r]}$ as in (15).

Now one checks easily that there is an isomorphism

$$\rho\colon (R, E) \xrightarrow{\sim} (\tilde{R}, \tilde{E})$$

which sends the component R_i to $\tilde{R}_{m-i}$ $(i = 0, \dots, m)$, such that the following diagram commutes:

$$\begin{array}{ccc} E \otimes_{\mathcal{O}_R} (k((s)) \times k((t))) & \xrightarrow{\cong} & (k((s)) \times k((t)))^{[1,r]} \\ \downarrow \rho & & \rho' \downarrow \\ \tilde{E} \otimes_{\mathcal{O}_{\tilde{R}}} (k((s)) \times k((t))) & \xrightarrow{\cong} & (k((s)) \times k((t)))^{[1,r]} \end{array},$$

where the morphism ρ' is given by

$$(x(s), y(t)) \mapsto \left(\Lambda \cdot \begin{bmatrix} \mathbb{I}_{i_1} & 0 \\ 0 & s^{-1}\mathbb{I}_{r-i_1} \end{bmatrix} \cdot y(s)\ ,\ \Lambda \cdot \begin{bmatrix} \mathbb{I}_{i_1} & 0 \\ 0 & t\mathbb{I}_{r-i_1} \end{bmatrix} \cdot x(t) \right).$$

From the commutativity of the diagram

$$\begin{array}{ccccc} (k((s)) \times k((t)))^{[1,r]} & \xrightarrow{(H^1(s), H^2(t))} & (k((s)) \times k((t)))^{[1,r]} & \xleftarrow{(1),(3)} & \mathcal{E}_\xi \otimes_{\mathcal{O}_{\tilde{C}}} \widehat{K}_p \\ \downarrow \rho' & & t \mapsto s \downarrow s \mapsto t & & \| \\ (k((s)) \times k((t)))^{[1,r]} & \xrightarrow{(\tilde{H}^1(s), \tilde{H}^2(t))} & (k((s)) \times k((t)))^{[1,r]} & \xleftarrow{(1'),(3)} & \mathcal{E}_\xi \otimes_{\mathcal{O}_{\tilde{C}}} \widehat{K}_p \end{array},$$

it finally follows that the Gieseker vector bundle $(\tilde{C}' \to C, \tilde{\mathcal{F}})$ constructed from the data ξ, $(1')$, (3), $(\tilde{a}_1, \dots, \tilde{a}_r)$, $\tilde{H}^1(s)$ and $\tilde{H}^2(t)$ is isomorphic to the Gieseker vector bundle $(C' \to C, \mathcal{F})$ constructed from the data ξ, (1), (3), $(a_1, \dots, a_r)$, $H^1(s)$ and $H^2(t)$.

7. Independence of the isomorphisms (3) and (4)

Independence of (3) is immediate, since if we change it by an automorphism of $\mathrm{GL}_r \times \mathrm{Spec}(\widehat{K}_p)$ (which can be written as an element in $\mathrm{GL}_r(k((s))) \times \mathrm{GL}_r(k((t)))$), then $(H^1(s), H^2(t))$ is changed by that same matrix.

Two isomorphisms (4) differ by a matrix $A = (A_{i,j}) \in \mathrm{GL}_r(\widehat{\mathcal{O}}_q)$ such that

$$A = \mathrm{diag}(\zeta^{-\alpha_1}, \dots, \zeta^{-\alpha_r}) \cdot \gamma(A) \cdot \mathrm{diag}(\zeta^{\alpha_1}, \dots, \zeta^{\alpha_r}). \tag{16}$$

After identifying $\widehat{\mathcal{O}}_q$ with the ring $k[[u, v]]/(u \cdot v)$ via the isomorphism (1), we can write

$$A = A^0 + u \cdot A^1(u) + v \cdot A^2(v)$$

with uniquely determined matrices $A^0 \in \mathrm{GL}_r(k)$, $A^1(u) \in M(r \times r, k[[u]])$ and $A^2(v) \in M(r \times r, k[[v]])$. Condition (16) implies that A^0 is a block matrix of the form

$$A^0 = \begin{bmatrix} A^0_1 & & 0 \\ & \ddots & \\ 0 & & A^0_m \end{bmatrix}, \tag{17}$$

where A^0_i are blocks of size $\sharp D_i$ for $i = 1, \dots, m$. Condition (16) implies furthermore that there are matrices $B^1(s) = (B^1_{i,j}(s)) \in \mathrm{GL}_r(k[[s]])$ and $B^2(t) = (B^2_{i,j}(t)) \in \mathrm{GL}_r(k[[t]])$ such that

$$A^1(u) = u^{-1}\mathrm{diag}(u^{a_1}, \dots, u^{a_r}) \cdot B^1(u^e) \cdot \mathrm{diag}(u^{-a_1}, \dots, u^{-a_r}),$$

$$A^2(v) = v^{-1}\mathrm{diag}(v^{-a_1}, \dots, v^{-a_r}) \cdot B^2(v^e) \cdot \mathrm{diag}(v^{a_1}, \dots, v^{a_r})$$

and such that

$$B^1_{i,j}(0) = 0 \quad \text{for } a_i - a_j \le 0,$$

$$B^2_{i,j}(0) = 0 \quad \text{for } a_j - a_i \le 0. \tag{18}$$

The change of (4) by the matrix A means that we have to replace F by the matrix

$$\tilde{F} = F \cdot A$$

and that consequently we have to replace the matrices $H^1(s)$ and $H^2(t)$ by the matrices

$$\tilde{H}^1(s) = H^1(s) \cdot (A^0 + B^1(s))$$

and

$$\tilde{H}^2(t) = H^2(t) \cdot (A^0 + B^2(t))$$

respectively.

The pair of matrices $(A^0 + B^1(s), A^0 + B^2(t))$ defines an automorphism of $\mathcal{V}$ which induces the automorphisms $A^0 + B^1(0)$ and $A^0 + B^2(0)$ on the special fibers V and W respectively. From (17) and (18) it follows that the induced automorphism of $\mathrm{KGL}(V, W)$ maps the generalized isomorphism Φ to itself.

It follows that the pair $(\tilde{\mathcal{E}}, \tilde{\Phi})$ obtained by the glueing datum $(\tilde{H}^1, \tilde{H}^2)$ is isomorphic to the pair $(\mathcal{E}, \Phi)$ obtained by the glueing datum (H^1, H^2). Therefore the induced Gieseker vector bundles are also isomorphic.

8. Surjectivity

Let $(C' \to C, \mathcal{F})$ be a Gieseker vector bundle on C. By definition, C' is either isomorphic to C, or it is the union of the normalization $\widetilde{C}$ of C and a chain R of projective lines which intersects $\widetilde{C}$ in the two points p_1 and p_2 lying above the singularity $p \in C$. In the first case we let $p' := p$, and in the second case we let $p' = p_2$. Then the triple

$$(\widetilde{C}' \to \widetilde{C}, \tilde{\mathcal{F}}', p')$$

is a Gieseker vector bundle datum in the sense of Definition 3.3. By Proposition 3.4 such a datum induces a vector bundle $\mathcal{E}$ on the curve $\widetilde{C}$ together with a generalized isomorphism Φ from $V := \mathcal{E}[p_1]$ to $W := \mathcal{E}[p_2]$.

More precisely, Φ is a k-valued point of $\mathrm{KGL}(V, W)$ which lies in the stratum $\mathbf{O}_{I,J}$ for some $I \subseteq [0, r-1]$ and $J = \emptyset$. As we have recalled in §3, such a point is given by a tupel

$$((F_\bullet(V), F_\bullet(W)), \overline{\varphi}_1, \dots, \overline{\varphi}_{m-1}, \Phi'),$$

where $m := |I| + 1$,

$$0 = F_0(V) \subseteq F_1(V) \subsetneq F_2(V) \subsetneq \cdots \subsetneq F_{m-1}(V) \subsetneq F_m(V) = V,$$
$$0 = F_0(W) \subsetneq F_1(W) \subsetneq F_2(W) \subsetneq \cdots \subsetneq F_{m-1}(W) \subseteq F_m(W) = W$$

are flags in V and W respectively, $\overline{\varphi}_i$ is the homothety class of an isomorphism

$$\varphi_\nu \colon F_{m-\nu}(W)/F_{m-\nu-1}(W) \xrightarrow{\sim} F_{\nu+1}(V)/F_\nu(V)$$

and Φ' denotes an isomorphism $F_1(V)/F_0(V) \xrightarrow{\sim} F_m(W)/F_{m-1}(W)$.

There is a basis $v_1, \dots, v_r$ of V and $w_1, \dots, w_r$ of W and a partition

$$[1, r] = D_1 \sqcup D_2 \sqcup \cdots \sqcup D_m$$

with the property that

(1) $i \in D_\nu$, $j \in D_{\nu'}$ and $i < j$ implies $\nu \leq \nu'$.
(2) $F_\nu(V)$ is generated by $\{v_i \mid i \in D_1 \sqcup \cdots \sqcup D_\nu\}$ and $F_\nu(W)$ is generated by $\{w_i \mid i \in D_{m-\nu+1} \sqcup \cdots \sqcup D_m\}$.
(3) For $i \in D_{\nu+1}$ the isomorphism φ_ν sends the residue class of w_i mod $F_{m-\nu-1}(W)$ to the residue class of v_i mod $F_\nu(V)$.
(4) For $i \in D_1$ the isomorphism Φ' sends v_i to the residue class of w_i mod $F_{m-1}(W)$.

For $i = 1, 2$ we choose an isomorphism

$$\mathcal{E} \otimes_{\mathcal{O}_{\widetilde{C}}} \widehat{\mathcal{O}}_{p_i} \xrightarrow{\sim} \widehat{\mathcal{O}}_{p_i}^r \tag{19}$$

which induces the isomorphism $V \to k^r$, $v_i \to e_i$ and $W \to k^r$, $w_i \to e_i$ from the fibres at p_1 and p_2 respectively, where $e_1, \dots, e_r$ is the canonical basis of k^r. Let ξ be the prinicpal GL_r-bundle on C_{gen} of local frames of the restriction of the vector bundle $\mathcal{E}$ to $C_{\mathrm{gen}} = \widetilde{C} \backslash \{p_1, p_2\}$. Then the isomorphism (19) induces the isomorphism

$$\xi \times_{C_{\mathrm{gen}}} \mathrm{Spec}(\widehat{K}_p) \xrightarrow{\sim} \mathrm{GL}_r \times \mathrm{Spec}(\widehat{K}_p). \tag{20}$$

Lemma 8.1. *There is a morphism $f\colon U \to C$, an integer $e \geq m$ prime to the characteristic of k and an operation of $\Gamma := \mathbb{Z}/e\mathbb{Z}$ on U such that*

(1) *Γ leaves f invariant and the induced morphism $U/\Gamma \to C$ is étale,*
(2) *U has exactly one singular point q and $f^{-1}(p) = \{q\}$,*
(3) *the action of Γ on $f^{-1}(C_{\text{gen}})$ is free.*

Proof. Since p is an ordinary double point of C, there exists a diagram of pointed schemes and étale morphisms as follows:

$$(C, p) \xleftarrow{\text{étale}} (U_0, q_0) \xrightarrow{\text{étale}} (V_0, y_0) := ((\operatorname{Spec}(k[s,t]/(s \cdot t)), (s,t)) .$$

After removing from U_0 the points $\neq q_0$ in the fiber of $U_0 \to C$ we may assume that q_0 is the only point lying above p. Choose $e \in \mathbb{Z}$ prime to $\operatorname{char}(k)$ with $e \geq m$. Let

$$(V, y) := (\operatorname{Spec}(k[u,v]/(u \cdot v)), (u,v))$$

and let $(V, y) \to (V_0, y_0)$ be defined by $s \mapsto u^e$, $t \mapsto v^e$. Let γ be a generator of $\Gamma = \mathbb{Z}/e\mathbb{Z}$ and let $\zeta \in k$ be a primitive e-th root of unity. We define an action of Γ on (V, y) by letting $\gamma(u) = \zeta u$ and $\gamma(v) = \zeta^{-1} v$. Now we set

$$U := U_0 \times_{V_0} V$$

and let $f\colon U \to C$ be the composition $U \to U_0 \to C$. From V the scheme U inherits an action of the group Γ. Since $V/\Gamma = V_0$ and $U_0 \to V_0$ is flat we have $U/\Gamma = U_0$ which by construction is étale over C. The only point in the fiber of f over p is the point $q = (q_0, y) \in U$. Since $U_0 \to V_0$ is étale and $V \to V_0$ is smooth outside the point y, it follows that the fiber product $U = U_0 \times_{V_0} V$ is regular outside q. Furthermore, since the action of Γ on $V \setminus \{y\}$ is free the same holds for the action of Γ on $U \setminus \{q\}$. □

In what follows we will construct a chart (U, η, Γ) for ξ where $U \to C$ and Γ are chosen as in the lemma and the GL_r-bundle η with Γ-operation is glued together from an object η_{gen} over U_{gen} and an object $\widehat{\eta}_q$ over the completion of U at the singular point q.

To fix the notation, let $\widehat{\mathcal{O}}_q$ be the completion of the local ring $\mathcal{O}_{U,q}$ and let γ be a generator of Γ. There exists an isomorphism

$$\widehat{\mathcal{O}}_q \xrightarrow{\sim} k[[u,v]]/(u \cdot v) \tag{21}$$

and a primitive e-th root of unity ζ such that the automorphism $\gamma\colon \widehat{\mathcal{O}}_q \xrightarrow{\sim} \widehat{\mathcal{O}}_q$ translates into the automorphism $u \mapsto \zeta u$, $v \mapsto \zeta^{-1} v$ of $k[[u,v]]/(u \cdot v)$ (see [2], §2.1.2).

Let $a_i \in [0, e-1]$ $(i \in [1, r])$ be chosen such that

$$\begin{aligned} a_i &= 0 \quad && \text{for } i \in D_1, \\ a_i &< a_j && \text{for } i \in D_\nu,\ j \in D_{\nu'},\ \nu < \nu', \\ a_i &= a_j && \text{for } i, j \in D_\nu,\ \nu \in [1, m]. \end{aligned}$$

Let $\widehat{\eta}_q := \mathrm{GL}_r \times \operatorname{Spec} \widehat{\mathcal{O}}_q$ together with the Γ-operation be defined by

$$\operatorname{diag}(\zeta^{a_1}, \dots, \zeta^{a_r}) \times \gamma\colon \mathrm{GL}_r \times \operatorname{Spec} \widehat{\mathcal{O}}_q \xrightarrow{\sim} \mathrm{GL}_r \times \operatorname{Spec} \widehat{\mathcal{O}}_q .$$

Let $\eta_{\text{gen}} := \xi \times_{C_{\text{gen}}} U_{\text{gen}}$ together with the Γ-operation be given by

$$\text{id} \times \gamma \colon \xi \times_{C_{\text{gen}}} U_{\text{gen}} \xrightarrow{\sim} \xi \times_{C_{\text{gen}}} U_{\text{gen}}.$$

Now we glue together $\widehat{\eta}_q$ and η_{gen} along $\operatorname{Spec} \widehat{K}_q \cong k((u)) \times k((v))$ via the isomorphism

$$\begin{aligned}\widehat{\eta}_q \times_{\widehat{\mathcal{O}}_q} \operatorname{Spec}(\widehat{K}_q) &\xrightarrow{(20)} \mathrm{GL}_r \times \operatorname{Spec}(\widehat{K}_q) \xrightarrow{F^1 \times F^2} \mathrm{GL}_r \times \operatorname{Spec}(\widehat{K}_q) \\ &= \eta_{\text{gen}} \times_{U_{\text{gen}}} \operatorname{Spec}(\widehat{K}_q),\end{aligned}$$

where

$$F^1 = \operatorname{diag}(u^{a_1}, \dots, u^{a_r}) \quad \text{and} \quad F^2 = \operatorname{diag}(v^{-a_1}, \dots, v^{-a_r}).$$

This gives a principal GL_r-bundle η on U. From the commutativity of the diagram

$$\begin{array}{ccc} \widehat{\eta}_q \times_{\widehat{\mathcal{O}}_q} \operatorname{Spec}(\widehat{K}_q) & \xrightarrow{\cong} & \eta_{\text{gen}} \times_{U_{\text{gen}}} \operatorname{Spec}(\widehat{K}_q) \\ \downarrow{\scriptstyle \operatorname{diag}(\zeta^{a_1},\dots,\zeta^{a_r}) \times \gamma} & & \downarrow{\scriptstyle \text{id} \times \gamma} \\ \widehat{\eta}_q \times_{\widehat{\mathcal{O}}_q} \operatorname{Spec}(\widehat{K}_q) & \xrightarrow{\cong} & \eta_{\text{gen}} \times_{U_{\text{gen}}} \operatorname{Spec}(\widehat{K}_q) \end{array},$$

it follows that the Γ-operation on $\widehat{\eta}_q$ and η_{gen} induces a Γ-operation on η. It is clear from the construction that the triple (U, η, Γ) forms a chart for ξ.

There is a chart (U_1, η_1, Γ_1) for ξ, where $U_1 := C_{\text{gen}}$, $\eta := \xi$, $\Gamma := (1)$. This chart together with the chart (U, η, Γ) make up a balanced atlas $\mathcal{A}$ for ξ. It is clear by construction that the twisted G-bundle $(\xi, C \to \operatorname{Spec}(k), \mathcal{A})$ is mapped to the Gieseker vector bundle $(C' \to C, \mathcal{F})$.

9. Further directions

The relationship between twisted GL_r-bundles and Gieseker vector bundles should be further investigated since it might lead to a clue as to what the right notion of stable maps to the classifying stack of a reductive group are. The next step would be to try to extend the mapping given in Theorem 4.2 so that it works for families.

For example let $A := \mathbb{C}[[t]]$, $S := \operatorname{Spec} A$ and $C \to S$ be a stable curve over S. Let

$$\left(\begin{array}{ccc} C' & \longrightarrow & C \\ & \searrow \;\; S \;\; \swarrow & \end{array}, \mathcal{F} \right)$$

be a Gieseker vector bundle of rank r on C.

Assume in particular that the generic fiber of $C \to S$ is smooth and that its special fiber is irreducible with one double point p. Then it can be shown that there is a twisted GL_r-bundle $(\xi, C \to S, \mathcal{A})$ such that if we apply the mappings from Theorems 4.1 and 4.2 to the isomorphism class of the generic and special fiber of $(\xi, C \to S, \mathcal{A})$, then we obtain the generic and special fiber of the Gieseker vector bundle $(C' \to C, \mathcal{F})$ respectively.

Indeed, in the neighbourhood of p one may chose a chart (U, η, Γ) for ξ, where $U \to C$ étale locally looks like

$$\begin{array}{ccc} \operatorname{Spec} A[u, v]/(uv - t) & \longrightarrow & \operatorname{Spec} A[x, y]/(xy - t^e), \\ u^e & \longleftarrow & x, \\ v^e & \longleftarrow & y. \end{array}$$

On the other hand, assume $C = C_0 \times S$, where C_0 is an irreducible curve with one ordinary double point p, and assume that $C' \to C$ induces an isomorphism of the generic fibers and the morphism $C_1 \to C_0$ on the special fibers. For simplicity let us assume furthermore that the rank r of the Gieseker bundle is one. In this situation it would be interesting to know, whether there is a twisted GL_1-bundle $(\xi, C \to S, \mathcal{A})$ such that the map of Theorem 4.2 maps the generic and the special fiber of $(\xi, C \to S, \mathcal{A})$ to the generic and special fiber of $(C' \to C, \mathcal{F})$ respectively.

Acknowledgements

The author would like to thank Seshadri for generously imparting his ideas. This paper owes very much to long discussions which the author had with Nagaraj in November and December 2002. The author would also like to thank the Institute of Mathematical Sciences in Chennai, whose hospitality made these discussions possible. This work was partially supported by the DFG.

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