

Irreducibility results for compositions of polynomials in several variables

ANCA IULIANA BONCIOCAT^{1,2} and
ALEXANDRU ZAHARESCU³

¹Universität Bonn, Institut für Angewandte Mathematik, Abt. für Stochastik,
53115 Bonn, Wegelerstr. 6, Deutschland

²Institute of Mathematics “Simion Stoilow” of the Romanian Academy,
P.O. Box 1-764, RO-70700 Bucharest, Romania

³Department of Mathematics, University of Illinois at Urbana-Champaign,
Altgeld Hall, 1409 W. Green Street, Urbana, IL 61801, USA

E-mail: anca@wiener.iam.uni-bonn.de; Anca.Bonciocat@imar.ro;
zaharesc@math.uiuc.edu

MS received 28 September 2004; revised 15 January 2005

Abstract. We obtain explicit upper bounds for the number of irreducible factors for a class of compositions of polynomials in several variables over a given field. In particular, some irreducibility criteria are given for this class of compositions of polynomials.

Keywords. Composition of polynomials; irreducibility results.

1. Introduction

In connection with Hilbert’s irreducibility theorem, Cavachi proved in [3] that for any relatively prime polynomials $f(X), g(X) \in \mathbb{Q}[X]$ with $\deg f < \deg g$, the polynomial $f(X) + pg(X)$ is irreducible over \mathbb{Q} for all but finitely many prime numbers p . Sharp explicit upper bounds for the number of factors over \mathbb{Q} of a linear combination $n_1 f(X) + n_2 g(X)$, covering also the case $\deg f = \deg g$, have been derived in [2]. In [1], we realized that by using technics similar to those employed in [4] and [2], upper bounds for the number of factors and irreducibility results can also be obtained for a class of compositions of polynomials of one variable with integer coefficients. More specifically, the following result is proved in [1].

Let $f(X) = a_0 + a_1 X + \cdots + a_m X^m$ and $g(X) = b_0 + b_1 X + \cdots + b_n X^n \in \mathbb{Z}[X]$ be nonconstant polynomials of degree m and n respectively, with $a_0 \neq 0$, and let $L_1(f) = |a_0| + \cdots + |a_{m-1}|$. Assume that d_1 is a positive divisor of a_m and d_2 a positive divisor of b_n such that

$$|a_m| > d_1^{mn} d_2^{m^2 n} L_1(f).$$

Then the polynomial $f \circ g$ has at most $\Omega(a_m/d_1) + m\Omega(b_n/d_2)$ irreducible factors over \mathbb{Q} , where $\Omega(k)$ is the total number of prime factors of k , counting multiplicities. The same conclusion holds in the wider range

$$|a_m| > d_1^n d_2^{mn} L_1(f),$$

provided that f is irreducible over \mathbb{Q} .

In the present paper we provide explicit upper bounds for the number of factors, and irreducibility results for a class of compositions of polynomials in several variables over a given field. We will deduce this result from the corresponding result for polynomials in two variables X, Y over a field K . We use the following notation. For any polynomial $f \in K[X, Y]$ we denote by $\deg_Y f$ the degree of f as a polynomial in Y , with coefficients in $K[X]$. Then we write any polynomial $f \in K[X, Y]$ in the form

$$f = a_0(X) + a_1(X)Y + \cdots + a_d(X)Y^d,$$

with a_0, a_1, \dots, a_d in $K[X]$, $a_d \neq 0$, and define

$$H_1(f) = \max\{\deg a_0, \dots, \deg a_{d-1}\}.$$

Finally, for any polynomial $f \in K[X]$ we denote by $\Omega(f)$ the number of irreducible factors of f , counting multiplicities ($\Omega(c) = 0$ for $c \in K$). We will prove the following theorem.

Theorem 1. *Let K be a field and let $f(X, Y) = a_0 + a_1Y + \cdots + a_mY^m$, $g(X, Y) = b_0 + b_1Y + \cdots + b_nY^n$, with $a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n \in K[X]$, $a_0a_mb_n \neq 0$. If d_1 is a factor of a_m and d_2 a factor of b_n such that*

$$\deg a_m > mn \deg d_1 + m^2n \deg d_2 + H_1(f),$$

then the polynomial $f(X, g(X, Y))$ has at most $\Omega(a_m/d_1) + m\Omega(b_n/d_2)$ irreducible factors over $K(X)$. The same conclusion holds in the wider range

$$\deg a_m > n \deg d_1 + mn \deg d_2 + H_1(f),$$

provided that f is irreducible over $K(X)$.

Theorem 1 provides, in particular, bounds for the number of irreducible factors of $f(X, Y)$ over $K(X)$, by taking $g(X, Y) = Y$.

COROLLARY 1

Let K be a field and let $f(X, Y) = a_0 + a_1Y + \cdots + a_mY^m$, with $a_0, a_1, \dots, a_m \in K[X]$, $a_0a_m \neq 0$. If d is a factor of a_m such that

$$\deg a_m > m \deg d + H_1(f),$$

then the polynomial $f(X, Y)$ has at most $\Omega(a_m/d)$ irreducible factors over $K(X)$.

Under the assumption that a_m has an irreducible factor over K of large enough degree, we have the following irreducibility criteria.

COROLLARY 2

Let K be a field and let $f(X, Y) = a_0 + a_1Y + \cdots + a_mY^m$, with $a_0, a_1, \dots, a_m \in K[X]$, $a_0a_m \neq 0$. If $a_m = pq$ with $p, q \in K[X]$, p irreducible over K , and

$$\deg p > (m - 1) \deg q + H_1(f),$$

then the polynomial $f(X, Y)$ is irreducible over $K(X)$.

COROLLARY 3

Let K be a field and let $f(X, Y) = a_0 + a_1Y + \cdots + a_mY^m$, $g(X, Y) = b_0 + b_1Y + \cdots + b_nY^n$, with $a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n \in K[X]$, $a_0a_mb_n \neq 0$, and f irreducible over $K(X)$. If $a_m = pq$ with $p, q \in K[X]$, p irreducible over K , and

$$\deg p > (n - 1) \deg q + mn \deg b_n + H_1(f),$$

then the polynomial $f(X, g(X, Y))$ is irreducible over $K(X)$.

COROLLARY 4

Let K be a field and let $f(X, Y) = a_0 + a_1Y + \cdots + a_mY^m$, $g(X, Y) = b_0 + b_1Y + \cdots + b_nY^n$, with $a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n \in K[X]$, $a_0a_mb_n \neq 0$. If $a_m = pq$ with $p, q \in K[X]$, p irreducible over K , and

$$\deg p > \max\{(m - 1) \deg q, (n - 1) \deg q + mn \deg b_n\} + H_1(f),$$

then the polynomial $f(X, g(X, Y))$ is irreducible over $K(X)$.

Another consequence of Theorem 1 is the following corresponding result for polynomials in $r \geq 2$ variables X_1, X_2, \dots, X_r over K . In this case, for any polynomial $f \in K[X_1, \dots, X_r]$, $\Omega(f)$ will stand for the number of irreducible factors of f over $K(X_1, \dots, X_{r-1})$, counting multiplicities. Then, for any polynomial $f \in K[X_1, \dots, X_r]$ and any $j \in \{1, \dots, r\}$ we denote by $\deg_{X_j} f$ the degree of f as a polynomial in X_j with coefficients in $K[X_1, \dots, X_{j-1}, X_{j+1}, \dots, X_r]$. We also write any polynomial $f \in K[X_1, \dots, X_r]$ in the form

$$f = a_0(X_1, \dots, X_{r-1}) + a_1(X_1, \dots, X_{r-1})X_r + \cdots + a_d(X_1, \dots, X_{r-1})X_r^d,$$

with $a_0, a_1, \dots, a_d \in K[X_1, \dots, X_{r-1}]$, $a_0 \neq 0$, and for any $j \in \{1, \dots, r - 1\}$ we let

$$H_j(f) = \max\{\deg_{X_j} a_0, \deg_{X_j} a_1, \dots, \deg_{X_j} a_{d-1}\}.$$

Then one has the following result.

COROLLARY 5

Let K be a field, $r \geq 2$, and let $f(X_1, \dots, X_r) = a_0 + a_1X_r + \cdots + a_mX_r^m$, $g(X_1, \dots, X_r) = b_0 + b_1X_r + \cdots + b_nX_r^n$, with $a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n \in K[X_1, \dots, X_{r-1}]$, $a_0a_mb_n \neq 0$. If d_1 is a factor of a_m and d_2 a factor of b_n such that for an index $j \in \{1, \dots, r - 1\}$,

$$\deg_{X_j} a_m > mn \deg_{X_j} d_1 + m^2n \deg_{X_j} d_2 + H_j(f),$$

then the polynomial $f(X_1, \dots, X_{r-1}, g(X_1, \dots, X_r))$ has at most $\Omega(a_m/d_1) + m\Omega(b_n/d_2)$ irreducible factors over the field $K(X_1, \dots, X_{r-1})$. The same conclusion holds in the wider range

$$\deg_{X_j} a_m > n \deg_{X_j} d_1 + mn \deg_{X_j} d_2 + H_j(f),$$

provided that f is irreducible over $K(X_1, \dots, X_{r-1})$.

In particular we have the following irreducibility criterion.

COROLLARY 6

Let K be a field, $r \geq 2$, and let $f(X_1, \dots, X_r) = a_0 + a_1 X_r + \dots + a_m X_r^m$, $g(X_1, \dots, X_r) = b_0 + b_1 X_r + \dots + b_n X_r^n$, with $a_0, a_1, \dots, a_m, b_0, b_1, \dots, b_n \in K[X_1, \dots, X_{r-1}]$, $a_0 a_m b_n \neq 0$. If $a_m = p \cdot q$ with p a prime element of the ring $K[X_1, \dots, X_{r-1}]$ such that for an index $j \in \{1, \dots, r-1\}$,

$$\deg_{X_j} p > \max\{(m-1) \deg_{X_j} q, (n-1) \deg_{X_j} q + mn \deg_{X_j} b_n\} + H_j(f),$$

then the polynomial $f(X_1, \dots, X_{r-1}, g(X_1, \dots, X_r))$ is irreducible over the field $K(X_1, \dots, X_{r-1})$.

Corollary 5 follows from Theorem 1 by writing Y for X_r and X for X_j , where j is any index for which

$$\deg_{X_j} a_m > mn \deg_{X_j} d_1 + m^2 n \deg_{X_j} d_2 + H_j(f),$$

and by replacing the field K with the field generated by K and the variables X_1, X_2, \dots, X_{r-1} except for X_j .

The reader may naturally wonder how sharp the above results are. In this connection, we discuss a couple of examples in the next section.

2. Examples

Let $K = \mathbb{Q}$, choose integers $m, d \geq 2$, select polynomials $a_0(X), a_1(X), \dots, a_{m-1}(X) \in \mathbb{Q}[X]$ with $a_0(X) \neq 0$, and consider the polynomial in two variables $f(X, Y)$ given by

$$f(X, Y) = a_0(X) + a_1(X)Y + \dots + a_{m-1}(X)Y^{m-1} + (X^d + 5X + 5)Y^m.$$

Under these circumstances, in terms of the degrees of the polynomials $a_0(X), a_1(X), \dots, a_{m-1}(X)$, can we be sure that the polynomial $f(X, Y)$ is irreducible over $\mathbb{Q}(X)$? The polynomial $p(X) = X^d + 5X + 5$ is an Eisensteinian polynomial with respect to the prime number 5, and hence it is irreducible over \mathbb{Q} . We may then apply Corollary 2, with $q = 1$, in order to conclude that $f(X, Y)$ is irreducible over $\mathbb{Q}(X)$ as long as $H_1(f) < d$, that is, as long as each of the polynomials $a_0(X), a_1(X), \dots, a_{m-1}(X)$ has degree less than or equal to $d-1$. We remark that for any choice of $m, d \geq 2$ this bound is the best possible, in the sense that there are polynomials $a_0(X), a_1(X), \dots, a_{m-1}(X) \in \mathbb{Q}[X]$, $a_0(X) \neq 0$, for which

$$\max\{\deg_X a_0(X), \deg_X a_1(X), \dots, \deg_X a_{m-1}(X)\} = d,$$

such that the corresponding polynomial $f(X, Y)$ is reducible over $\mathbb{Q}(X)$. Indeed, one may choose for instance $a_0(X), a_1(X), \dots, a_{m-2}(X)$ to be any polynomials with coefficients in \mathbb{Q} , with $a_0(X) \neq 0$, of degrees less than or equal to d , and define $a_{m-1}(X)$ by the equality

$$a_{m-1}(X) = -X^d - 5X - 5 - \sum_{0 \leq i \leq m-2} a_i(X).$$

Then, on the one hand, we will have $\max\{\deg_X a_0, \dots, \deg_X a_{m-1}\} = d$ and on the other hand, the corresponding polynomial $f(X, Y)$ will be reducible over $\mathbb{Q}(X)$, being divisible by $Y - 1$.

In the next example, let us slightly modify the polynomial $f(X, Y)$, and choose a polynomial $g(X, Y)$ of arbitrary degree, say

$$\begin{aligned} f(X, Y) &= a_0 + a_1 Y + \dots + a_{m-1} Y^{m-1} + (X^n + 5X + 5)^2 Y^m, \\ g(X, Y) &= b_0 + b_1 Y + \dots + b_{n-1} Y^{n-1} + Y^n, \end{aligned}$$

where $a_0, a_1, \dots, a_{m-1}, b_0, b_1, \dots, b_{n-1} \in \mathbb{Q}[X], a_0(X) \neq 0$. We may apply Theorem 1, with $d_1 = d_2 = 1$, in order to conclude that $f(X, g(X, Y))$ has at most two irreducible factors over $\mathbb{Q}(X)$ as long as $H_1(f) < 2n$, that is, as long as each of the polynomials a_0, a_1, \dots, a_{m-1} has degree less than or equal to $2n - 1$. This bound too is the best possible, as there exist polynomials $a_0, a_1, \dots, a_{m-1} \in \mathbb{Q}[X], a_0(X) \neq 0, g \in \mathbb{Q}[Y], g$ monic, for which

$$\max\{\deg a_0, \deg a_1, \dots, \deg a_{m-1}\} = 2n,$$

such that the corresponding polynomial $f(X, g(X, Y))$ has at least three irreducible factors over $\mathbb{Q}(X)$. For instance, one may take $g(Y) = Y^2$, choose polynomials $a_0(X), a_1(X), \dots, a_{m-2}(X)$ with coefficients in \mathbb{Q} , with $a_0(X) \neq 0$, of degrees less than or equal to $2n$, and define $a_{m-1}(X)$ by the equality

$$a_{m-1}(X) = -(X^n + 5X + 5)^2 - \sum_{0 \leq i \leq m-2} a_i(X).$$

Then we will have $\max\{\deg a_0, \dots, \deg a_{m-1}\} = 2n$, while the corresponding polynomial $f(X, g(X, Y))$ will have at least three irreducible factors over $\mathbb{Q}(X)$, being divisible by $Y^2 - 1$.

3. Proof of Theorem 1

Let $K, f(X, Y), g(X, Y), d_1$ and d_2 be as in the statement of the theorem. Let $b \in K[X]$ denote the greatest common divisor of b_0, b_1, \dots, b_n , and define the polynomial $\bar{g}(X, Y) \in K[X, Y]$ by the equality

$$g(X, Y) = b_0 + b_1 Y + \dots + b_n Y^n = b \bar{g}(X, Y).$$

Next, let $a \in K[X]$ denote the greatest common divisor of the coefficients of $f(X, g(X, Y))$ viewed as a polynomial in Y , and define the polynomial $F(X, Y) \in K[X, Y]$ by the equality

$$f(X, g(X, Y)) = a F(X, Y).$$

If we assume that $f(X, g(X, Y))$ has $s > \Omega(a_m/d_1) + m\Omega(b_n/d_2)$ irreducible factors over $K(X)$, then the polynomial $F(X, Y)$ will have a factorization $F(X, Y) = F_1(X, Y) \cdots F_s(X, Y)$, with $F_1(X, Y), \dots, F_s(X, Y) \in K[X, Y], \deg_Y F_1(X, Y) \geq 1, \dots, \deg_Y F_s(X, Y) \geq 1$. Let $t_1, \dots, t_s \in K[X]$ be the leading coefficients of $F_1(X, Y), \dots, F_s(X, Y)$ respectively, viewed as polynomials in Y . By comparing the

leading coefficients in the equality

$$a_0 + a_1g(X, Y) + \cdots + a_mg^m(X, Y) = aF_1(X, Y) \cdots F_s(X, Y)$$

we obtain the following equality in $K[X]$:

$$at_1 \cdots t_s = a_m b_n^m = d_1 d_2^m \cdot \frac{a_m}{d_1} \cdot \left(\frac{b_n}{d_2} \right)^m. \quad (1)$$

Then, in view of (1), it follows easily that at least one of the t_i 's, say t_1 , will divide $d_1 d_2^m$. As a consequence, one has

$$\deg t_1 \leq \deg d_1 + m \deg d_2. \quad (2)$$

We now consider the polynomial

$$\begin{aligned} h(X, Y) &= f(X, g(X, Y)) - a_m(X)g(X, Y)^m \\ &= a_0(X) + a_1(X)g(X, Y) + \cdots + a_{m-1}(X)g(X, Y)^{m-1}. \end{aligned}$$

Recall that a_0 and \bar{g} are relatively prime, and $a_0(X) \neq 0$. It follows that the polynomials $\bar{g}^m(X, Y)$ and $h(X, Y)$ are relatively prime. Therefore $\bar{g}^m(X, Y)$ and $F_1(X, Y)$ are relatively prime. As a consequence, the resultant $R(\bar{g}^m, F_1)$ of $\bar{g}^m(X, Y)$ and $F_1(X, Y)$, viewed as polynomials in Y with coefficients in $K[X]$, will be a nonzero element of $K[X]$. We now introduce a nonarchimedean absolute value $|\cdot|$ on $K(X)$, as follows. We fix a real number ρ , with $0 < \rho < 1$, and for any polynomial $F(X) \in K[X]$ we define $|F(X)|$ by the equality

$$|F(X)| = \rho^{-\deg F(X)}. \quad (3)$$

We then extend the absolute value $|\cdot|$ to $K(X)$ by multiplicativity. Thus for any $L(X) \in K(X)$, $L(X) = \frac{F(X)}{G(X)}$, with $F(X), G(X) \in K[X]$, $G(X) \neq 0$, let $|L(X)| = \frac{|F(X)|}{|G(X)|}$. Let us remark that for any non-zero element u of $K[X]$ one has $|u| \geq 1$. In particular, $R(\bar{g}^m, F_1)$ being a non-zero element of $K[X]$, we have

$$|R(\bar{g}^m, F_1)| \geq 1. \quad (4)$$

Next, we estimate $|R(\bar{g}^m, F_1)|$ in a different way. Let $\overline{K(X)}$ be a fixed algebraic closure of $K(X)$, and let us fix an extension of the absolute value $|\cdot|$ to $\overline{K(X)}$, which we will also denote by $|\cdot|$. Consider now the factorizations of $\bar{g}(X, Y)$, $\bar{g}^m(X, Y)$ and $F_1(X, Y)$ over $\overline{K(X)}$. Say

$$\begin{aligned} \bar{g}(X, Y) &= \bar{b}_n(Y - \xi_1) \cdots (Y - \xi_n), \\ \bar{g}^m(X, Y) &= \bar{b}_n^m(Y - \xi_1)^m \cdots (Y - \xi_n)^m \end{aligned}$$

and

$$F_1(X, Y) = t_1(Y - \theta_1) \cdots (Y - \theta_r),$$

with $\xi_1, \dots, \xi_n, \theta_1, \dots, \theta_r \in \overline{K(X)}$. Here $1 \leq r \leq mn - 1$, by our assumption that $\deg_Y F_1(X, Y) \geq 1$ and $\deg_Y F_2(X, Y) \geq 1$. Then

$$|R(\overline{g}^m, F_1)| = \left| \frac{b_n^{mr} t_1^{mn}}{b_n^{mr} t_1^{mn}} \prod_{1 \leq i \leq n} \prod_{1 \leq j \leq r} (\xi_i - \theta_j)^m \right| = |t_1|^{mn} \prod_{1 \leq j \leq r} |\overline{g}^m(X, \theta_j)|. \quad (5)$$

The fact that $F_1(X, \theta_j) = 0$ for any $j \in \{1, \dots, r\}$ also implies that $f(X, g(X, \theta_j)) = 0$, and so

$$\begin{aligned} h(X, \theta_j) &= f(X, g(X, \theta_j)) - a_m(X)g^m(X, \theta_j) \\ &= -a_m(X)b^m(X)\overline{g}^m(X, \theta_j). \end{aligned}$$

Since $b(X)$ is a non-zero element of $K[X]$, one has $|b(X)| \geq 1$. We deduce that

$$|\overline{g}^m(X, \theta_j)| = \frac{|h(X, \theta_j)|}{|a_m(X)b^m(X)|} \leq \frac{|h(X, \theta_j)|}{|a_m(X)|}. \quad (6)$$

By combining (5) and (6) we find that

$$|R(\overline{g}^m, F_1)| \leq \frac{|t_1|^{mn}}{|a_m|^r} \prod_{1 \leq j \leq r} |h(X, \theta_j)|. \quad (7)$$

We now proceed to find an upper bound for $|h(X, \theta_j)|$, for $1 \leq j \leq r$. In order to do this, we first use the identity

$$h(X, Y) = a_0(X) + a_1(X)g(X, Y) + \dots + a_{m-1}(X)g(X, Y)^{m-1}$$

to obtain

$$\begin{aligned} |h(X, \theta_j)| &= |a_0(X) + a_1(X)g(X, \theta_j) + \dots + a_{m-1}(X)g(X, \theta_j)^{m-1}| \\ &\leq \max_{0 \leq k \leq m-1} |a_k(X)| \cdot |g(X, \theta_j)|^k, \end{aligned} \quad (8)$$

for $1 \leq j \leq r$. Next, we consider the factorization of $f(X, Y)$ over $\overline{K(X)}$, say

$$f(X, Y) = a_m(X)(Y - \lambda_1) \cdots (Y - \lambda_m),$$

with $\lambda_1, \dots, \lambda_m \in \overline{K(X)}$. For any $i \in \{1, \dots, m\}$ one has

$$0 = f(X, \lambda_i) = a_0(X) + a_1(X)\lambda_i + \dots + a_m(X)\lambda_i^m. \quad (9)$$

By (9) we see that

$$\begin{aligned} |a_m(X)| \cdot |\lambda_i^m| &= |a_0(X) + a_1(X)\lambda_i + \dots + a_{m-1}(X)\lambda_i^{m-1}| \\ &\leq \max_{0 \leq c \leq m-1} |a_c(X)| \cdot |\lambda_i|^c. \end{aligned} \quad (10)$$

For any $i \in \{1, \dots, m\}$ let us select an index $c_i \in \{0, \dots, m-1\}$ for which the maximum is attained on the right side of (10). We then have $|a_m(X)| \cdot |\lambda_i|^m \leq |a_{c_i}(X)| \cdot |\lambda_i|^{c_i}$, and so

$$|\lambda_i| \leq \left(\frac{|a_{c_i}(X)|}{|a_m(X)|} \right)^{1/(m-c_i)}. \quad (11)$$

We now return to (8). Fix a $j \in \{1, \dots, r\}$. In order to provide an upper bound for $|h(X, \theta_j)|$, it is sufficient to find an upper bound for $|g(X, \theta_j)|$. Recall that $f(X, g(X, \theta_j)) = 0$. Therefore there exists an $i \in \{1, \dots, m\}$, depending on j , for which $g(X, \theta_j) = \lambda_i$. Then, by (11) we obtain

$$|g(X, \theta_j)| \leq \left(\frac{|a_{c_i}(X)|}{|a_m(X)|} \right)^{1/(m-c_i)} \leq \max_{1 \leq v \leq m} \left(\frac{|a_{m-v}(X)|}{|a_m(X)|} \right)^{1/v}. \quad (12)$$

Inserting (12) in (8) we conclude that, uniformly for $1 \leq j \leq r$, one has

$$|h(X, \theta_j)| \leq \max_{\substack{1 \leq v \leq m \\ 0 \leq k \leq m-1}} |a_k| \left(\frac{|a_{m-v}|}{|a_m|} \right)^{k/v}. \quad (13)$$

Combining (13) with (7) we derive the inequality

$$|R(\bar{g}^m, F_1)| \leq \frac{|t_1|^{mn}}{|a_m|^r} \max_{\substack{1 \leq v \leq m \\ 0 \leq k \leq m-1}} \frac{|a_k|^r \cdot |a_{m-v}|^{rk/v}}{|a_m|^{rk/v}},$$

which may be written as

$$|R(\bar{g}^m, F_1)| \leq |t_1|^{mn} \left(\max_{\substack{1 \leq v \leq m \\ 0 \leq k \leq m-1}} \frac{|a_k| \cdot |a_{m-v}|^{k/v}}{|a_m|^{1+k/v}} \right)^r. \quad (14)$$

In what follows we are going to prove that

$$|t_1|^{mn} \max_{\substack{1 \leq v \leq m \\ 0 \leq k \leq m-1}} \frac{|a_k| \cdot |a_{m-v}|^{k/v}}{|a_m|^{1+k/v}} < 1, \quad (15)$$

which by (14) will contradict (4), since $r \geq 1$. Using the definition of the absolute value $|\cdot|$, we write the inequality (15) in the form

$$\max_{\substack{1 \leq v \leq m \\ 0 \leq k \leq m-1}} \rho^{(1+\frac{k}{v}) \deg a_m - \deg a_k - \frac{k}{v} \deg a_{m-v}} < \rho^{mn \deg_X t_1},$$

which is equivalent to

$$\min_{\substack{1 \leq v \leq m \\ 0 \leq k \leq m-1}} \left\{ \left(1 + \frac{k}{v} \right) \deg a_m - \deg a_k - \frac{k}{v} \deg a_{m-v} \right\} > mn \deg t_1. \quad (16)$$

By combining (16) with (2), it will be sufficient to prove that

$$\begin{aligned} & \min_{\substack{1 \leq v \leq m \\ 0 \leq k \leq m-1}} \left\{ \left(1 + \frac{k}{v} \right) \deg a_m - \deg a_k - \frac{k}{v} \deg a_{m-v} \right\} \\ & > mn \deg d_1 + m^2 n \deg d_2, \end{aligned}$$

or equivalently,

$$\min_{\substack{1 \leq v \leq m \\ 0 \leq k \leq m-1}} \left(1 + \frac{k}{v}\right) \cdot \left\{ \deg a_m - \frac{\deg a_k + \frac{k}{v} \deg a_{m-v}}{1 + \frac{k}{v}} \right\} > mn \deg d_1 + m^2 n \deg d_2. \quad (17)$$

By our assumption on the size of $\deg a_m$ we have

$$\deg a_m - H_1(f) > mn \deg d_1 + m^2 n \deg d_2,$$

from which (17) follows, since

$$\max_{\substack{1 \leq v \leq m \\ 0 \leq k \leq m-1}} \frac{\deg a_k + \frac{k}{v} \deg a_{m-v}}{1 + \frac{k}{v}} \leq H_1(f). \quad (18)$$

This completes the proof of the first part of the theorem. Assuming now that f is irreducible over $K(X)$, the proof goes as in the first part, except that now we have $\deg_Y F_1 = r \geq m$, since by Capelli's Theorem [5], the degree in Y of every irreducible factor of $f(X, g(X, Y))$ must be a multiple of m . Therefore, instead of (15) one has to prove that

$$|t_1|^n \max_{\substack{1 \leq v \leq m \\ 0 \leq k \leq m-1}} \frac{|a_k| \cdot |a_{m-v}|^{k/v}}{|a_m|^{1+k/v}} < 1,$$

which is equivalent to

$$\min_{\substack{1 \leq v \leq m \\ 0 \leq k \leq m-1}} \left\{ \left(1 + \frac{k}{v}\right) \deg a_m - \deg a_k - \frac{k}{v} \deg a_{m-v} \right\} > n \deg t_1.$$

By combining this inequality with (2), it will be sufficient to prove that

$$\min_{\substack{1 \leq v \leq m \\ 0 \leq k \leq m-1}} \left(1 + \frac{k}{v}\right) \cdot \left\{ \deg a_m - \frac{\deg a_k + \frac{k}{v} \deg a_{m-v}}{1 + \frac{k}{v}} \right\} > n \deg d_1 + mn \deg d_2. \quad (19)$$

Finally, our assumption that $\deg a_m - H_1(f) > n \deg d_1 + mn \deg d_2$ together with (18) imply (19), which completes the proof of the theorem. \square

We end by noting that in the statement of Theorem 1, the assumption on the size of $\deg a_m$, and the bound $\Omega(a_m/d_1) + m\Omega(b_n/d_2)$ exhibited for the number of factors do not depend on the first n coefficients of g . So these bounds remain the same once we fix $n, b_n(X)$ and $d_2(X)$, and let $b_0(X), \dots, b_{n-1}(X)$ vary independently.

Acknowledgements

This work was partially supported by the CERES Program 3-28/2003 and the CNCSIS Grant 40223/2003 code 1603 of the Romanian Ministry of Education and Research.

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