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Isometric multipliers of $L^p(G, X)$

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Abstract. Let G be a locally compact group with a fixed right Haar measure and X a separable Banach space. Let $L^p(G, X)$ be the space of X-valued measurable functions whose norm-functions are in the usual L^p. A left multiplier of $L^p(G, X)$ is a bounded linear operator on $L^p(G, X)$ which commutes with all left translations. We use the characterization of isometries of $L^p(G, X)$ onto itself to characterize the isometric, invertible, left multipliers of $L^p(G, X)$ for $1 \le p < \infty$, $p \neq 2$, under the assumption that X is not the ℓ^p-direct sum of two non-zero subspaces. In fact we prove that if T is an isometric left multiplier of $L^p(G, X)$ onto itself then there exists a $y \in G$ and an isometry U of X onto itself such that $Tf(x) = U(R_y f)(x)$. As an application, we determine the isometric left multipliers of $L^1 \bigcap L^p(G, X)$ and $L^1 \bigcap C_0(G, X)$ where G is non-compact and X is not the ℓ^p-direct sum of two non-zero subspaces. If G is a locally compact abelian group and H is a separable Hilbert space, we define $A^p(G, H) = \{f \in L^1(G, H)\colon \hat{f} \in L^p(\Gamma, H)\}$ where Γ is the dual group of G. We characterize the isometric, invertible, left multipliers of $A^p(G, H)$, provided G is non-compact. Finally, we use the characterization of isometries of $C(G, X)$ for G compact to determine the isometric left multipliers of $C(G, X)$ provided X^* is strictly convex.

Keywords. Locally compact group; Haar measure; Banach space-valued measurable functions; isometric multipliers.

1. Introduction

Let G be a locally compact group with right Haar measure μ. Suppose X is a separable Banach space. If $1 \le p < \infty$, let $L^p(G, X)$ be the space of X-valued measurable functions F such that $\int_G \|F(x)\|^p \,\mathrm{d}\mu < \infty$. The p-norm of F is defined by $[\int_G \|F\|^p \,\mathrm{d}\mu]^{1/p}$. In case X is a one-dimensional complex Banach space, $L^p(G, X)$ is denoted by $L^p(G)$.

The left and right translation operators L_g and R_g are defined by $(L_g F)(x) = F(gx)$ and $(R_g F)(x) = F(xg)$. A left multiplier of $L^p(G, X)$ is a bounded linear operator on $L^p(G, X)$ which commutes with all left translations. The main result of this paper gives a characterization of the isometric, invertible, left multipliers of $L^p(G, X)$ for $1 \le p < \infty$, $p \neq 2$, under the assumption that X is not the ℓ^p-direct sum of two non-zero subspaces. More precisely we shall prove the following theorem.

Theorem 1. *Let G be a locally compact group and T an isometric, invertible, left multiplier on $L^p(G, X)$ for $1 \le p < \infty$, $p \neq 2$. Suppose that X is not the ℓ^p-direct sum of two non-zero subspaces. Then there exists an isometry U of X onto itself and $y \in G$ such that T is of the form*

$$(TF)(x) = U R_y F(x).$$

Wendel [7] proved this result for $L^1(G)$ in 1952. Later Strichartz [6] and Parrot [4] proved it for $L^p(G)$ if $1 \leq p < \infty$, $p \neq 2$.

Let G be a non-compact locally compact group. If $f \in L^1 \bigcap L^p(G, X)$, we define $\|f\| = \|f\|_1 + \|f\|_p$. Then $L^1 \bigcap L^p(G, X)$ is a Banach space with this norm. Similarly for $f \in L^1 \bigcap C_0(G, X)$, we define $\|f\| = \|f\|_1 + \|f\|_\infty$. Then $L^1 \bigcap C_0(G, X)$ is a Banach space. In both cases, we shall show that if T is an isometric, invertible, left multiplier, then T is of the form

$$(Tf)(x) = UR_y f(x).$$

If G is a locally compact abelian group and H is a separable Hilbert space, we define $A^p(G, H) = \{f \in L^1(G, H)\colon \hat{f} \in L^p(\Gamma, H)\}$ where Γ is the dual group of G. For $f \in A^p(G, H)$, we define $\|f\| = \|f\|_1 + \|\hat{f}\|_p$. $A^p(G, H)$ is a Banach space with this norm. We will prove that if T is an isometric, invertible, left multiplier, then T is of the form

$$(Tf)(x) = UR_y f(x).$$

Let G be a compact group and X be a separable Banach space. $C(G, X)$ denotes the Banach space of continuous X-valued functions. Using the characterization of isometries of $C(G, X)$, we will prove that if T is an isometric, invertible, left multiplier, then T is of the form

$$(TF)(x) = UR_y F(x).$$

provided X^* is strictly convex.

2. Preliminaries

Let (Ω, Σ, μ) be a measure space. Suppose Σ' is the σ-ring generated by the sets of σ-finite measure. A mapping Φ of Σ' onto itself, defined modulo null sets, is said to be a regular set isomorphism if

1. $\Phi(A \backslash A') = \Phi(A) \backslash \Phi(A')$ for $A, A' \in \Sigma'$.
2. $\Phi(\bigcup_{n=1}^{\infty} A_n) = (\bigcup_{n=1}^{\infty} \Phi(A_n)$, where $\{A_n\}$ is a sequence of disjoint sets in Σ'.
3. $\mu(\Phi(A)) = 0$ iff $\mu(A) = 0$.

A regular set isomorphism induces a linear map on X-valued measurable functions. If $A \in \Sigma'$ and $x \in X$, define $\Phi(\chi_A)(x) = \chi_{\Phi(A)} x$ where χ_A is the characterstic function of A. This extends linearly to simple functions. Let f be an X-valued measurable function. Then there exists a sequence $\{f_n\}$ of simple functions converging to f in measure. Then $\{\Phi(f_n\}$ is a Cauchy sequence in measure and hence converges to a measurable function $\Phi(f)$. It is easy to show that $\Phi(f)$ depends only on f and not on the particular sequence $\{f_n\}$.

We also note that any Σ'-measurable function is also Σ-measurable and any Σ-measurable function with σ-finite support is Σ'-measurable. Thus the spaces of Σ' and Σ measurable functions with σ-finite support coincide.

If Φ is a regular set isomorphism, define a measure ν by $\nu(A) = \mu(\Phi^{-1}(A))$. The measure ν is absolutely continuous with respect to μ. Let $h = ((\mathrm{d}\nu)/\mathrm{d}\mu)^{1/p}$. It is easy to see that h is a function on Ω whose restriction to any measurable set of σ-finite measure is measurable. Further, if $f \in L^p(\Omega, X)$, then $h\Phi(f) \in L^p(\Omega, X)$ and $\|h\Phi(f)\|_p = \|f\|_p$.

We say that a Banach space X is the ℓ^p-direct sum of two Banach spaces X_1 and X_2 if X is isometrically isomorphic to $X_1 \bigoplus X_2$ where the norm on the direct sum is given by $\|x_1 \bigoplus x_2\| = \{\|x_1\|^p + \|x_2\|^p\}^{1/p}$.

Our main tool for the proof of the main result is a theorem of Sourour [5]. We state it in a form slightly different from that of [5], but virtually no modification of the proof given there is necessary. The assumption that Ω is σ-finite is not needed for our conclusion because every function in $L^p(\Omega, X)$ has σ-finite support.

Theorem S. *Let (Ω, Σ, μ) be a measure space and T be an isometry of $L^p(\Omega, X)$ onto itself. Suppose X is not the ℓ^p-direct sum of two non-zero Banach spaces. Then there exists a regular set isomorphism Φ of Σ' onto itself, a measurable function h on Ω and a strongly measurable map S of Ω into the Banach space of bounded linear maps of X into X with $S(t)$ a surjective isometry of X for almost all $t \in \Omega$, such that*

$$T(F(t)) = S(t)h(t)\Phi(F)(t)$$

for $F \in L^p(\Omega, X)$ and almost all $t \in \Omega$.

3. Isometric multipliers of $L^p(G, X)$

In this section we characterize the isometric, invertible, left multipliers of $L^p(G, X)$.

Proof of Theorem 1. Let T be an isometric, invertible, left multiplier of $L^p(G, X)$. It follows from Theorem S that

$$TF(t) = h(t)S(t)\Phi(F)(t) \quad \text{a.e.}$$

for every $F \in L^p(G, X)$.

Let $A(t) = h(t)S(t)\ \forall t \in G$. Fix $s \in G$. We will show that $L_s A(t) = A(t)$. If this is not true, then there exists a set E of positive finite measure such that $A(st) \neq A(t)\ \forall t \in E$.

The sets $s\Phi^{-1}(E)$ and $\Phi^{-1}(sE)$ may be of σ-finite measure. But by choosing a suitable subset E still of positive finite measure, we can assume that $s\Phi^{-1}(E)$ and $\Phi^{-1}(sE)$ are of positive finite measure. Having done this, let $F = s\Phi^{-1}(E) \bigcup \Phi^{-1}(sE)$. Then $\forall t \in E$, $st \in sE \subseteq \Phi(F)$ and $E \subseteq \Phi(s^{-1}F)$. Now for $t \in E$ and $x \in X$,

$$\begin{aligned} L_s(T\chi_F x)(t) &= T(\chi_F x)(st) \\ &= \chi_{\Phi(F)}(st)A(st)(x) \\ &= A(st)(x). \end{aligned}$$

Also,

$$\begin{aligned} T(L_s\chi_F x)(t) &= T(\chi_{s^{-1}F}x)(t) \\ &= \chi_{\Phi(s^{-1}F)}(t)A(t)(x) \\ &= A(t)(x). \end{aligned}$$

Since $L_sT = TL_s$, it follows that $A(st)(x) = A(t)(x)$ for almost all $t \in E$. By choosing a countable dense set $\{x_n\}_{n=1}^{\infty}$ in X, we conclude that

$$A(st)(x) = A(t)(x)$$

for almost all $t \in E$ and all $x \in X$. But this is a contradiction. Hence

$$A(st) = A(t)$$

for almost all $t \in G$. Therefore for each $x \in X$,

$$h(t)S(t)(x) = h(st)S(st)(x)$$

for almost all $t \in G$. Since $S(t)$ is an isometry of X onto itself and $h(t) \geq 0$, we have

$$h(st) = h(t)$$

for almost all $t \in G$. This implies that h is a constant, say k. It also follows that

$$S(st) = S(t)$$

for almost all $t \in G$. Hence S is also a constant operator, say V. Therefore, T is an isometric multiplier of $L^p(G, X)$ onto itself for all p, in particular for $p = 1$. Now fix $x \in X$ such that $\|x\| = 1$. Then for $f \in L^1(G)$,

$$L_s T(fx) = L_s kV\Phi(f)x = L_s\Phi(f)kV(x)$$

and

$$TL_s(fx) = kV\Phi(L_s f)x = \Phi(L_s f)kV(x).$$

Hence $L_s\Phi(f) = \Phi(L_s(f))$. This implies that the map $f \longrightarrow k\Phi(f)$ is an isometric multiplier of $L^1(G)$ onto itself. Hence by Wendel's characterization there exists an $s \in G$ and a scalar c such that $|c| = 1$ for which we have

$$k\Phi(f)(t) = cf(ts).$$

Let $U = kV$. Then U is an isometry of X onto itself such that $T = U \circ R_s$ and

$$(TF)(t) = UF(ts)$$

for almost all $t \in G$ and all $F \in L^p(G, X)$. This completes the proof of the theorem. □

We shall now show that the condition that X is not an ℓ^p-direct sum is a necessary (as well as a sufficient) condition for the conclusion of Theorem 1 to hold. In fact, we prove the following theorem.

Theorem 2. *Let X be a separable Banach space which is ℓ^p-direct sum of two non-zero subspaces of X. Then there exists an isometric, invertible, left multiplier T of $L^p(G, X)$ which is not of the form $U \circ R_y$ for any isometry U of X and $y \in G$.*

Proof. Suppose $X = X_1 \oplus_p X_2$. Then

$$L^p(G, X) = L^p(G, X_1) \oplus_p L^p(G, X_2).$$

Choose $z \in G$ where z is not the identity element of G. Define T by

$$T(f_1 \oplus f_2) = f_1 \oplus R_z f_2.$$

Then it is easy to verify that T is an isometric, invertible, left multiplier of $L^p(G, X)$ which is not of the form $U \circ R_y$ for any isometry U of X and $y \in G$. □

4. Isometric multipliers of $L^1 \cap L^p(G, X)$ and $L^1 \cap C_0(G, X)$

In this section we assume that G is non-compact and X is not an ℓ^p-direct sum of two non-zero subspaces of X. We will prove that if T is an isometric, invertible, left multiplier of $L^1 \cap L^p(G, X)$ or $L^1 \cap C_0(G, X)$ then T is of the form $U \circ R_y$ for some isometry U of X and $y \in G$.

The proof of the following proposition is similar to the proof of Theorems 3.5.1 and 3.5.2 in [2] and hence omitted.

PROPOSITION 3

Suppose G is non-compact. If T is a left multiplier of $L^1 \cap L^p(G, X)$ or $L^1 \cap C_0(G, X)$ then T has a unique extension to $L^1(G, X)$ as a left multiplier such that $\|Tf\|_1 \leq \|T\| \|f\|_1$, where $\|T\|$ is the norm of T as an operator on $L^1 \cap L^p(G, X)$ or $L^1 \cap C_0(G, X)$.

We now prove the characterization of an isometric, invertible, left multiplier of $L^1 \cap L^p(G, X)$ or $L^1 \cap C_0(G, X)$.

Theorem 4. *Suppose G is non-compact and X is not ℓ^p-direct sum of two non-zero subspaces of X. If T is an isometric, invertible, left multiplier of $L^1 \cap L^p(G, X)$ or $L^1 \cap C_0(G, X)$ then T is of the form $U \circ R_y$ for some isometry U of X and $y \in G$.*

Proof. Since T and T^{-1} are both isometric multipliers of $L^1 \cap L^p(G, X)$ or $L^1 \cap C_0(G, X)$, it follows from Proposition 3 that T extends to $L^1(G, X)$ as an isometric left multiplier. Therefore by Theorem 1, there exists an isometry of X onto itself and $y \in G$ such that $T = U \circ R_y$. □

5. Isometric multipliers of $A^p(G, H)$

Let G be a locally compact Abelian group and H be a separable Hilbert space. We define the Fourier transform of $f \in L^1(G, H)$ by

$$\hat{f}(\gamma) = \int_G \overline{\gamma(x)} f(x) \mathrm{d}x,$$

where $\gamma \in \Gamma$, the dual group of G. Given a Haar measure on G there exists a unique Haar measure on Γ such that the map $f \to \hat{f}$ is an isometry of $L^1 \cap L^2(G, H)$ into $L^2(\Gamma, H)$ and extends to an isometry of $L^2(G, H)$ onto $L^2(\Gamma, H)$. The Fourier–Plancherel formula $\|\hat{f}\|_2 = \|f\|_2$ holds for $f \in L^2(G, H)$, see [1].

For $f \in A^p(G, H)$, we define $\|f\| = \|f\|_1 + \|\hat{f}\|_p$. Then $A^p(G, H)$ is a Banach space. We note that left and right translates mean the same for Abelian groups. Suppose G is non-compact. We will prove that if T is an isometric and invertible multiplier of $A^p(G, H)$ then $T = U \circ R_y$, where U is an isometry of H onto itself and $y \in G$.

The proof of the following Proposition is similar to the argument in the proof of Theorem 6.3.1 in [2] where it is shown that if T is a multiplier of $A^p(G)$ then $\|Tf\|_1 \leq \|T\| \|f\|_1$ for $f \in A^p(G)$, where $\|T\|$ denotes the operator norm of T. The necessary modifications are easy and hence we omit the details.

PROPOSITION 5

Let G be a non-compact locally compact Abelian group and $1 \leq p < \infty$. Suppose T is a multiplier of $A^p(G, H)$ then $\|Tf\|_1 \leq \|T\| \|f\|_1$ for $f \in A^p(G, H)$.

We now prove the characterization of isometric multipliers of $A^p(G, H)$.

Theorem 6. *Let G be a non-compact locally compact Abelian group and* $1 \leq p < \infty$. *Suppose T is an isometric multiplier of* $A^p(G, H)$. *Then there exists a unique* $y \in G$ *and an isometry U of H onto itself such that* $T = U \circ R_y$.

Proof. Let T be an isometric multiplier of $A^p(G, H)$. Then T^{-1} is also an isometric multiplier and we conclude from Proposition 5 that $\|Tf\|_1 = \|f\|_1$ for every $f \in A^p(G, H)$. It follows that T extends to $L^1(G, H)$ as an isometric multiplier of $L^1(G, H)$. Hence, by Theorem 1, there exists an isometry U of H onto itself and $y \in G$ such that $T = U \circ R_y$. □

6. Isometric multipliers of $C(G, X)$

In this section we describe the isometric, invertible, left multipliers of $C(G, X)$ where G is a compact group and X^* is strictly convex. The space $C(G, X)$ consists of all continuous X-valued function and is a Banach space under the supremum norm. The norm of $f \in C(G, X)$ will be denoted by $\|f\|_\infty$. For the space X, we denote the set of isometries of X onto itself by $I(X)$. The isometries of $C(G, X)$ were characterized by Lau [3]. He has shown that if T is an isometry of $C(G, X)$ onto itself, then there exists a homeomorphism ϕ of G onto itself and a continuous map $\lambda: X \to I(X)$ (with the strong operator topology) such that

$$Tf(t) = \lambda(t) f(\phi(t)).$$

Using this characterization of isometries of $C(G, X)$, we prove the following:

Theorem. *Let T be an isometric, invertible, left multiplier of* $C(G, X)$. *Then there exists an isometry U of X onto itself and* $y \in G$ *such that* $T = U \circ R_y$.

Proof. Since T is an isometry of $C(G, X)$, there exists a continuous map $\lambda: X \to I(X)$ and a homeomorphism ϕ of G onto itself such that

$$Tf(s) = \lambda(s) f(\phi(s)) \quad \forall s \in G.$$

Fix $x \in X$ and let $f(s) = x$ $\forall s \in G$. Then

$$TL_t f(s) = \lambda(s) f(t(\phi(s)) \tag{1}$$

and

$$L_t Tf(s) = \lambda(ts) f(\phi(ts)). \tag{2}$$

Since $TL_t = L_t T$, it follows that $\lambda(s)(x) = \lambda(ts)(x)$. Since $x \in X$ is arbitrary, we conclude that $\lambda(ts) = \lambda(s)$ $\forall s, t \in G$. Hence there exists an isometry U of X onto itself such that $\lambda(s) = U$ $\forall s \in G$. Therefore

$$Tf(s) = Uf(\phi(s)) \quad \forall f \in C(G, X).$$

Let $g \in C(G)$ and $x \in X$. Define f by $f(s) = g(s)x$ $\forall s \in G$. Then (1) and (2) imply that

$$g(t\phi(s)) = g(\phi(ts)) \quad \forall g \in C(G).$$

Since $C(G)$ separates points, we conclude that $t\phi(s) = \phi(ts)$ $\forall s, t \in G$. Let s be the identity element of G. Then $\phi(t) = t\phi(e)$. Let us denote $\phi(e)$ by y. Then we have $Tf(s) = Uf(sy)$ $\forall f \in C(G, X)$ and $s \in G$. Therefore we have $T = U \circ R_y$. □

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