

On two functionals connected to the Laplacian in a class of doubly connected domains in space-forms

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Abstract. Let B_1 be a ball of radius r_1 in $S^n(\mathbb{H}^n)$, and let B_0 be a smaller ball of radius r_0 such that $\overline{B_0} \subset B_1$. For S^n we consider $r_1 < \pi$. Let u be a solution of the problem $-\Delta u = 1$ in $\Omega := B_1 \setminus \overline{B_0}$ vanishing on the boundary. It is shown that the associated functional $J(\Omega)$ is minimal if and only if the balls are concentric. It is also shown that the first Dirichlet eigenvalue of the Laplacian on Ω is maximal if and only if the balls are concentric.

Keywords. Eigenvalue problem; Laplacian; maximum principles.

1. Introduction

Let (M, g) be a Riemannian manifold and let D denote the Levi–Civita connection of (M, g) . For a smooth vector field X on M the divergence $\text{div}(X)$ is defined as $\text{trace}(DX)$. For a smooth function $f: M \rightarrow \mathbb{R}$, the gradient ∇f is defined by $g(\nabla f(p), v) = df(p)(v)$ ($p \in M$, $v \in T_p M$) and the Laplace–Beltrami operator Δ is defined by $\Delta f = \text{div}(\nabla f)$. Further, $\nabla^2 f$ denotes the Hessian of f . Throughout this paper, ω and dV denote the volume element of (M, g) .

Let $\Omega \subset M$ be a domain such that $\bar{\Omega}$ is a smooth compact submanifold of M . The Sobolev space $H^1(\Omega)$ is defined as the closure of $C^\infty(\bar{\Omega})$ (the space of real valued smooth functions on $\bar{\Omega}$) with respect to the Sobolev norm

$$\|f\|_{H^1(\Omega)} = \left(\int_{\Omega} \{f^2 + \|\nabla f\|^2\} dV \right)^{1/2} \quad (f \in C^\infty(\bar{\Omega})).$$

The closure of $C_0^\infty(\Omega)$ (the space of real valued smooth functions on Ω having compact support in Ω) in $H^1(\Omega)$ is denoted by $H_0^1(\Omega)$. The Sobolev space $H^2(\Omega)$ is defined as the closure of $C^\infty(\bar{\Omega})$ with respect to the Sobolev norm

$$\|f\|_{H^2(\Omega)} = \left(\int_{\Omega} \{f^2 + \|\nabla f\|^2 + \|\nabla^2 f\|^2\} dV \right)^{1/2} \quad (f \in C^\infty(\bar{\Omega})).$$

These spaces are Hilbert spaces with the corresponding norms.

Consider the Dirichlet boundary value problem on Ω :

$$\left. \begin{aligned} -\Delta u &= 1 & \text{on } \Omega, \\ u &= 0 & \text{on } \partial\Omega. \end{aligned} \right\} \quad (1.1)$$

Let $u \in H_0^1(\Omega)$ be the unique weak solution of problem (1.1). By Theorem 4.8, p. 105 of [1], $u \in C^\infty(\bar{\Omega})$.

Consider the following eigenvalue problem on Ω :

$$\left. \begin{aligned} -\Delta u &= \lambda u && \text{on } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \right\} \tag{1.2}$$

The eigenvalues of the positive Laplace–Beltrami operator $-\Delta = -\text{div}(\nabla f)$ are strictly positive. The eigenfunctions corresponding to the first eigenvalue λ_1 are proportional to each other. They belong to $C^\infty(\bar{\Omega})$ and they are either strictly positive or strictly negative on Ω . Moreover,

$$\lambda_1 = \inf \{ \|\nabla\phi\|_{L^2(\Omega)}^2 \mid \phi \in H_0^1(\Omega), \|\phi\|_{L^2(\Omega)}^2 = 1 \}$$

(cf. [1], Theorem 4.4, p. 102). Let $y := y(\Omega) \in C^\infty(\bar{\Omega})$ be the unique solution of problem (1.1). Let $y_1 := y_1(\Omega)$ be the unique solution of problem (1.2), corresponding to the first eigenvalue $\lambda_1 := \lambda_1(\Omega)$, characterized by

$$y_1 > 0 \quad \text{on } \Omega \quad \text{and} \quad \int_{\Omega} y_1^2 \, dV = 1.$$

The aim of this paper is to prove the main results of [3] for simply connected spherical and hyperbolic space-forms.

Consider the unit sphere $S^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} x_i^2 = 1\}$ with induced Riemannian metric $\langle \cdot, \cdot \rangle$ from the Euclidean space \mathbb{R}^{n+1} . Also consider the hyperbolic space $\mathbb{H}^n = \{(x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^n x_i^2 - x_{n+1}^2 = -1 \text{ and } x_{n+1} > 0\}$ with the Riemannian metric induced from the quadratic form $(x, y) := \sum_{i=1}^n x_i y_i - x_{n+1} y_{n+1}$, where $x = (x_1, x_2, \dots, x_{n+1})$ and $y = (y_1, y_2, \dots, y_{n+1})$.

Fix $0 < r_0 < r_1$. We choose $r_1 < \pi$ for the case of S^n . Let B_1 be any ball of radius r_1 in $S^n(\mathbb{H}^n)$ and B_0 be any ball of radius r_0 such that $\bar{B}_0 \subset B_1$. Consider the family $\mathcal{F} = \{B_1 \setminus \bar{B}_0\}$ of domains in $S^n(\mathbb{H}^n)$. We study the extrema of the following functionals:

$$J(\Omega) = - \int_{\Omega} \{ \|\nabla y(\Omega)\|^2 - 2y(\Omega) \} \, dV, \tag{1}$$

$$J_1(\Omega) = - \int_{\Omega} \{ \|\nabla y_1(\Omega)\|^2 - 2\lambda_1(\Omega)[y_1(\Omega)]^2 \} \, dV \tag{2}$$

on \mathcal{F} , associated to problems (1.1) and (1.2) respectively. Note here that the functionals J and J_1 are nothing but negative of the energy functional $\int_{\Omega} \|\nabla y(\Omega)\|^2 \, dV$ and the Dirichlet eigenvalue λ_1 , respectively.

We state our main results: Put $\Omega_0 = B(p, r_1) \setminus \overline{B(p, r_0)}$ for any fixed $p \in S^n(\mathbb{H}^n)$.

Theorem 1. *The functional $J(\Omega)$ on \mathcal{F} assumes minimum at Ω if and only if $\Omega = \Omega_0$, i.e., when the balls are concentric.*

Theorem 2. *The functional $J_1(\Omega)$ on \mathcal{F} assumes maximum at Ω if and only if $\Omega = \Omega_0$, i.e., when the balls are concentric.*

In §§2 and 3, following [5], we develop the ‘shape calculus’ for Riemannian manifolds for the stationary problem (1.1) and the eigenvalue problem (1.2) respectively. In §4, we prove Theorems 1 and 2 for S^n , and make the necessary remarks to carry out the proofs of Theorems 1 and 2 for \mathbb{H}^n .

2. Shape calculus for the stationary problem

Let V be a smooth vector field on M having compact support. Let $\Phi: \mathbb{R} \times M \rightarrow M$ be the smooth flow for V . For each $t \in \mathbb{R}$, denote $\Phi(t, x)$ by $\Phi_t(x)$ ($x \in M$). Let Ω be an open subset of M such that $\bar{\Omega}$ is a smooth compact submanifold of M . Put $\Omega_t := \Phi_t(\Omega)$ ($t \in \mathbb{R}$).

Let \mathcal{D} be a domain in M such that $\text{supp } V \subset \mathcal{D}$. Fix $f \in C^\infty(\mathcal{D})$. Consider the Dirichlet boundary value problem on Ω_t :

$$\left. \begin{array}{l} \Delta u = f \quad \text{on } \Omega_t, \\ u = 0 \quad \text{on } \partial\Omega_t. \end{array} \right\} \quad (2.1)$$

Let $y_t \in C^\infty(\bar{\Omega}_t)$ be the unique solution of problem (2.1) (cf. [1], Theorem 4.8, p. 105). Throughout this section $y := y(\Omega)$ denotes the unique solution of (2.1) for $t = 0$.

Denote $y_t \circ \Phi_t|_{\Omega}$ by y^t ($t \in \mathbb{R}$).

PROPOSITION 2.1

The map $t \mapsto y^t$ is a C^1 -curve in $H^2(\Omega) \cap H_0^1(\Omega)$ from a neighbourhood of 0 in \mathbb{R} .

Proof. By problem (2.1), for each $t \in \mathbb{R}$, y_t satisfies the equation

$$\int_{\Omega_t} g(\nabla y_t, \nabla \psi) \, dV = - \int_{\Omega_t} f \psi \, dV \quad \forall \psi \in C_0^\infty(\Omega_t). \quad (3)$$

There exists smooth function $\gamma_t: M \rightarrow (0, \infty)$ such that $\Phi_t^* \omega = \gamma_t \omega$ (here, $\omega := dV$, the volume element of (M, g)). Put $B_t := (D\Phi_t)^{-1}$, B_t^* = transpose of B_t (i.e., $g(B_t(x)v, w) = g(v, B_t^*(x)w) \quad \forall v \in T_x \Omega_t, w \in T_{x'} \Omega$, where $x' := \Phi_t^{-1}(x)$) and $A_t := \gamma_t B_t B_t^*$. By the change of variable $\Phi_t: \Omega \rightarrow \Omega_t$, eq. (3) can be re-written as

$$\int_{\Omega} -\text{div}(A_t \nabla(y_t \circ \Phi_t)) \psi \circ \Phi_t \, dV = - \int_{\Omega} f \circ \Phi_t \psi \circ \Phi_t \gamma_t \, dV.$$

Therefore, $y^t := y_t \circ \Phi_t: \Omega \rightarrow \mathbb{R}$ satisfies

$$\left. \begin{array}{l} -\text{div}(A_t \nabla y^t) + f \circ \Phi_t \gamma_t = 0 \quad \text{on } \Omega, \\ y^t = 0 \quad \text{on } \partial\Omega. \end{array} \right\} \quad (2.2)$$

Define $F: \mathbb{R} \times H^2(\Omega) \cap H_0^1(\Omega) \rightarrow L^2(\Omega)$ by $F(t, u) = -\text{div}(A_t \nabla u) + f \circ \Phi_t \gamma_t$. Then F is a C^1 -map. Further $D_2 F|_{(0, y)}(0, u) = -\text{div}(\nabla u)$ (recall $y = y(\Omega)$). By the standard theory of Dirichlet boundary value problem on compact Riemannian manifolds ([1], Theorem 4.8, p. 105 and [2], Theorem 7.32, p. 259),

$$D_2 F|_{(0, y)}: H^2(\Omega) \cap H_0^1(\Omega) \rightarrow L^2(\Omega)$$

is an isomorphism. By (2.2), $F(t, y^t) = 0 \quad \forall t$. Proposition 2.1 now follows by the implicit function theorem. \square

DEFINITION

$\dot{y}(\Omega, V) := \left(\frac{d}{dt} y^t \right) \Big|_{t=0} \in H_0^1(\Omega)$ is called the (strong) *material derivative* of y in the direction of V .

Consider $\Omega' \subset\subset \Omega$.

PROPOSITION 2.2

The map $t \mapsto y_t|_{\Omega'}$ is a C^1 -curve in $H^1(\Omega')$ from a neighbourhood of 0 in \mathbb{R} and $d/dt|_{t=0}(y_t|_{\Omega'}) = \{\dot{y}(\Omega, V) - g(\nabla y, V)\}|_{\Omega'}$.

Proof. There exists $\delta > 0$ such that $\Omega' \subset \Phi_t(\Omega) \forall |t| < \delta$. Then $y_t|_{\Omega'} = y^t \circ \Phi_{-t}|_{\Omega'} \forall |t| < \delta$. Proposition 2.2 now follows from Proposition 2.1 and Proposition 2.38, p. 71 of [5]. \square

DEFINITION

$y'(\Omega, V) := \dot{y}(\Omega, V) - g(\nabla y, V) \in H^1(\Omega)$ is called the *shape derivative of y* in the direction of V .

Consider the domain functional $J(\Omega_t)$ defined by $J(\Omega_t) := \int_{\Omega_t} y_t \, dV$ ($t \in \mathbb{R}$).

DEFINITION

The *Eulerian derivative* $dJ(\Omega, V)$ of $J(\Omega_t)$ at $t = 0$ is defined as

$$dJ(\Omega, V) := \lim_{t \rightarrow 0} \frac{J(\Omega_t) - J(\Omega)}{t}.$$

PROPOSITION 2.3

The function $J(\Omega_t)$ is differentiable at $t = 0$ and $dJ(\Omega, V) = \int_{\Omega} y' \, dV$.

Proof. Let $L_V \omega$ denote the Lie derivative of ω with respect to V , and $i_V \omega$ denote the interior multiplication of ω with respect to V . Then

$$\frac{d}{dt}(\Phi_t^* \omega)|_{t=0} =: L_V \omega = (d i_V + i_V d) \omega = d(i_V \omega) = \operatorname{div}(V) \omega.$$

Hence, by Propositions 2.1 and 2.2 we get

$$\begin{aligned} dJ(\Omega, V) &= \lim_{t \rightarrow 0} \int_{\Omega} \left\{ \frac{y^t \Phi_t^* \omega - y \omega}{t} \right\} = \int_{\Omega} \left(\frac{d}{dt} \{y^t \Phi_t^* \omega\} \right)_{|_{t=0}} \\ &= \int_{\Omega} \{\dot{y} + y \operatorname{div}(V)\} \, dV = \int_{\Omega} \{y' + g(\nabla y, V) + y \operatorname{div}(V)\} \, dV \\ &= \int_{\Omega} y' \, dV + \int_{\Omega} d(y i_V \omega) = \int_{\Omega} y' \, dV. \quad \square \end{aligned}$$

PROPOSITION 2.4

The shape derivative $y' = y'(\Omega, V)$ is the weak solution of the Dirichlet boundary value problem

$$\left. \begin{aligned} \Delta v &= 0 && \text{on } \Omega, \\ v|_{\partial\Omega} &= -\frac{\partial y}{\partial n} g(V, n) \end{aligned} \right\} \quad (2.3)$$

in the space $H^1(\Omega)$. (Here, n is the outward unit normal field on $\partial\Omega$).

Proof. Consider $\psi \in C_0^\infty(\Omega)$ having support in a domain $\Omega' \subset\subset \Omega$. There exists $\delta > 0$ such that $\Omega' \subset \Omega_t \forall |t| < \delta$. By problem (2.1),

$$\int_{\Omega'} g(\nabla y_t, \nabla \psi) \, dV = - \int_{\Omega'} f \psi \, dV \quad \text{for } |t| < \delta. \quad (4)$$

By Proposition 2.2, differentiation of LHS of eq. (4) with respect to t at $t = 0$ can be carried out under the integral sign. So we get

$$\int_{\Omega'} g(\nabla y', \nabla \psi) \, dV = 0.$$

Thus y' satisfies $\Delta y' = 0$ weakly on Ω .

Now $\dot{y}, y \in H^2(\Omega) \cap H_0^1(\Omega)$, and $y' = \dot{y} - g(\nabla y, V) \in H^1(\Omega)$. So by Proposition 2.39, p. 88 of [2], we get

$$y'|_{\partial\Omega} = \dot{y}|_{\partial\Omega} - g(\nabla y, V)|_{\partial\Omega} \quad \text{and} \quad \dot{y}|_{\partial\Omega} = 0.$$

Also, $y \in C^\infty(\bar{\Omega})$ and $y = 0$ on $\partial\Omega$ by (2.1). So, $g(\nabla y, V)|_{\partial\Omega} = \frac{\partial y}{\partial n} g(V, n)$. Thus, $y'|_{\partial\Omega} = -\frac{\partial y}{\partial n} g(V, n)$. \square

3. Shape calculus for the eigenvalue problem

Let (M, g) , V , Φ_t , Ω , Ω_t , γ_t , A_t be as in §2. Consider problem (1.2) posed in Ω_t :

$$\left. \begin{array}{l} -\Delta u = \lambda u \quad \text{on } \Omega_t, \\ u = 0 \quad \text{on } \partial\Omega_t. \end{array} \right\} \quad (3.1)$$

Let $\lambda_1(t) := \lambda_1(\Omega_t)$ and $y_1(t) := y_1(\Omega_t)$ be as in §1. We denote $y_1(\Omega)$ by y_1 and $\lambda_1(\Omega)$ by λ_1 throughout this section.

Denote $y_1(t) \circ \Phi_t|_\Omega$ by y_1^t ($t \in \mathbb{R}$).

PROPOSITION 3.1

The map $t \mapsto (\lambda_1(t), y_1^t)$ is a C^1 -curve in $\mathbb{R} \times H^2(\Omega) \cap H_0^1(\Omega)$ from a neighbourhood of 0 in \mathbb{R} .

Proof. By problem (3.1), for each $t \in \mathbb{R}$, $y_1(t)$ satisfies the equation

$$\int_{\Omega_t} g(\nabla y_1(t), \nabla \psi) \, dV = \int_{\Omega_t} \lambda_1(t) y_1(t) \psi \, dV \quad \forall \psi \in H_0^1(\Omega_t). \quad (5)$$

As in the proof of Proposition 2.1, eq. (5) can be re-written as

$$- \int_{\Omega} \operatorname{div}(A_t \nabla y_1^t) \psi \, dV = \int_{\Omega} \lambda_1(t) y_1^t \gamma_t \psi \, dV \quad \forall \psi \in H_0^1(\Omega). \quad (6)$$

Therefore, $t \mapsto (\lambda_1(t), y_1^t)$ satisfies

$$\left. \begin{array}{l} \operatorname{div}(A_t \nabla y_1^t) + \lambda_1(t) y_1^t \gamma_t = 0 \quad \text{on } \Omega, \\ \int_{\Omega} (y_1^t)^2 \gamma_t \, dV = 1. \end{array} \right\} \quad (3.2)$$

Let $X := \mathbb{R} \times H^2(\Omega) \cap H_0^1(\Omega)$. Define $F: \mathbb{R} \times X \longrightarrow L^2(\Omega) \times \mathbb{R}$ by $F(t, \mu, u) = (\operatorname{div}(A_t \nabla u) + \mu u \gamma_t, \int_{\Omega} u^2 \gamma_t \, dV - 1)$. Then F is a C^1 -map. Further $D_2 F|_{(0, \lambda_1, y_1)}(0, \mu, u) = (\Delta u + \lambda_1 u + \mu y_1, 2 \int_{\Omega} y_1 u \, dV)$.

Claim. $D_2 F|_{(0, \lambda_1, y_1)}: \mathbb{R} \times H^2(\Omega) \cap H_0^1(\Omega) \longrightarrow L^2(\Omega) \times \mathbb{R}$ is an isomorphism.

Let $(v, b) \in L^2(\Omega) \times \mathbb{R}$ be arbitrary. Consider the following problem:

$$\left. \begin{aligned} \Delta u + \lambda_1 u + \mu y_1 &= v \quad \text{on } \Omega, \\ 2 \int_{\Omega} y_1 u \, dV &= b. \end{aligned} \right\} \quad (3.3)$$

Now by Fredholm alternative, $\Delta u + \lambda_1 u = v - \mu y_1$ has a solution in $H^2(\Omega) \cap H_0^1(\Omega)$ if and only if $v - \mu y_1 \perp y_1$ in $L^2(\Omega)$. So, for $\mu_0 := \int_{\Omega} v y_1 \, dV$ there exists $u_1 \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $\Delta u_1 + \lambda_1 u_1 + \mu_0 y_1 = v$. Moreover, the solutions of $\Delta u + \lambda_1 u + \mu_0 y_1 = v$ are of the form $u = u_1 + a y_1, a \in \mathbb{R}$. Given $b \in \mathbb{R}$ there exists a unique $a_0 := b/2 - \int_{\Omega} y_1 u_1 \, dV \in \mathbb{R}$ such that $2 \int_{\Omega} y_1 u \, dV = b$. Put $u_0 = u_1 + a_0 y_1$. Thus for $(v, b) \in L^2(\Omega) \times \mathbb{R}$ there exists a unique $(\mu_0, u_0) \in \mathbb{R} \times H^2(\Omega) \cap H_0^1(\Omega)$ such that $D_2 F|_{(0, \lambda_1, y_1)}(0, \mu_0, u_0) = (v, b)$. This proves the claim.

By (3.2), $F(t, \lambda_1(t), y_1^t) = 0 \, \forall t$. Proposition 3.1 now follows by the implicit function theorem. \square

DEFINITION

$\dot{y}_1(\Omega, V) := ((d/dt)y_1^t)|_{t=0} \in H_0^1(\Omega)$ is called the (strong) *material derivative* of y_1 in the direction of V .

Consider $\Omega' \subset\subset \Omega$.

PROPOSITION 3.2

The map $t \longmapsto y_1(t)|_{\Omega'}$ is a C^1 -curve in $H^1(\Omega')$ from a neighbourhood of 0 in \mathbb{R} and $((d/dt)[y_1(t)|_{\Omega'}])|_{t=0} = (\dot{y}_1 - g(\nabla y_1, V))|_{\Omega'} \in H^1(\Omega')$. Further, y_1^t satisfies $y_1^t = \dot{y}_1 - g(\nabla y_1, V)$ in $H^1(\Omega)$ and $y_1^t|_{\partial\Omega} = -\frac{\partial y_1}{\partial n} g(V, n)$.

Proof. There exists $\delta > 0$ such that $\Omega' \subset \Phi_t(\Omega) \, \forall |t| < \delta$. The first part of Proposition 3.2 follows from Proposition 3.1 and Proposition 2.38, p. 71 of [5]. Now as $\dot{y}_1 \in H^1(\Omega)$ and $\nabla y_1 \in C^\infty(\bar{\Omega})$, we get $y_1^t = \dot{y}_1 - g(\nabla y_1, V) \in H^1(\Omega)$. Hence, $y_1^t|_{\partial\Omega} = \dot{y}_1|_{\partial\Omega} - g(\nabla y_1, V)|_{\partial\Omega} = -\frac{\partial y_1}{\partial n} g(V, n)$. \square

DEFINITION

The *shape derivative* of y_1 in the direction of V is the element $y_1' = y_1'(\Omega, V) \in H^1(\Omega)$ defined by $y_1' = \dot{y}_1 - g(\nabla y_1, V)$.

PROPOSITION 3.3

The shape derivative $y_1' \in H^1(\Omega)$ satisfies

$$-\Delta y_1' = \lambda_1 y_1' + \lambda_1' y_1 \quad \text{on } \Omega$$

in the sense of distributions.

Proof. Let $\psi \in C_0^\infty(\Omega)$. Let $\Omega' \subset\subset \Omega$ be a domain such that $\text{supp } \psi \subset \Omega'$. As $y_1(t)$ is a solution of problem (1.2) posed in Ω_t , for t sufficiently small we get

$$\int_{\Omega'} g(\nabla y_1(t), \nabla \psi) \, dV = \int_{\Omega'} \lambda_1(t) y_1(t) \psi \, dV. \quad (7)$$

By Propositions 3.1 and 3.2, we can differentiate with respect to t under the integral sign in eq. (7). Thus we have

$$\int_{\Omega'} g(\nabla y_1', \nabla \psi) \, dV = \int_{\Omega'} (\lambda_1 y_1' + \lambda_1' y_1) \psi \, dV.$$

Hence,

$$-\int_{\Omega} y_1' \Delta \psi \, dV = \int_{\Omega} (\lambda_1 y_1' + \lambda_1' y_1) \psi \, dV \quad \forall \psi \in C_0^\infty(\Omega). \quad \square$$

PROPOSITION 3.4

$$y_1' \in C^\infty(\bar{\Omega}).$$

Proof. By Proposition 3.2, $y_1' = \dot{y}_1 - g(\nabla y_1, V)$ on Ω . Hence it is enough to prove that $\dot{y}_1 \in C^\infty(\bar{\Omega})$. Consider $L := \Delta + \lambda_1$, a linear elliptic operator of order 2. Then $\dot{y}_1 \in H_0^1(\Omega)$ satisfies $L(\dot{y}_1) = L(y_1' + g(\nabla y_1, V)) = -\lambda_1' y_1 + L(g(\nabla y_1, V))$, by Proposition 3.3. From Proposition 3.58, p. 87 of [1], it follows that $\dot{y}_1 \in C^\infty(\bar{\Omega})$. \square

PROPOSITION 3.5

$$\lambda_1' = - \int_{\partial\Omega} \left(\frac{\partial y_1}{\partial n} \right)^2 g(V, n) \, dS.$$

Proof. We write $\lambda_1' = \lambda_1' \int_{\Omega} y_1^2 \, dV$. By Proposition 3.3, $\lambda_1' = \int_{\Omega} \{-\Delta y_1' - \lambda_1 y_1'\} y_1 \, dV$. Hence by problem (1.2) and Proposition 3.4, we get

$$\begin{aligned} \lambda_1' &= \int_{\Omega} \{-y_1 \Delta y_1' + y_1' \Delta y_1\} \, dV = \int_{\partial\Omega} \left\{ y_1' \frac{\partial y_1}{\partial n} - y_1 \frac{\partial y_1'}{\partial n} \right\} \, dS \\ &= \int_{\partial\Omega} y_1' \frac{\partial y_1}{\partial n} \, dS. \end{aligned}$$

Now the result follows by Proposition 3.2. \square

4. Proofs of Theorem 1 and Theorem 2 for S^n

Proof of Theorem 1 for S^n . We continue with the notations of §1 such as r_0, r_1, \mathcal{F} , and $y(\Omega), J(\Omega)$ for $\Omega \in \mathcal{F}$ for S^n . For $|t| < \pi$, put $p := (0, \dots, 0, 1)$ and $q(t) = (0, \dots, 0, \sin t, \cos t) \in S^n$. The Laplace–Beltrami operator Δ of (S^n, \langle, \rangle) is invariant under isometries of S^n . So we need to study the functional J only on domains $\Omega(q(t)) := B(r_1) \setminus \overline{B(q(t), r_0)}$, $0 \leq |t| < r_1 - r_0$, where $B(r_1) := B(p, r_1)$.

We define $j: (r_0 - r_1, r_1 - r_0) \rightarrow \mathbb{R}$ by $j(t) = J(\Omega(q(t)))$.

Fix t_0 such that $0 \leq t_0 < r_1 - r_0$ and put $\Omega := \Omega(q(t_0))$ and $B_0 := B(q(t_0), r_0)$. Fix r_2 such that $r_0 < r_2 < r_1 - t_0$ and consider a smooth function $\rho: S^n \rightarrow \mathbb{R}$ satisfying $\rho = 1$ on $\overline{B(q(t_0), r_2)}$ and $\rho = 0$ on $\partial B(r_1)$. Let V denote the vector field on S^n defined by $V(x) = \rho(x)(0, \dots, 0, x_{n+1}, -x_n) \forall x = (x_1, \dots, x_{n+1}) \in S^n$. Let $\{\Phi_t\}_{t \in \mathbb{R}}$ be the one-parameter family of diffeomorphisms of S^n associated with V . Then for t sufficiently close to 0, $J(\Phi_t(\Omega)) = j(t_0 + t)$. Note that $J(\Phi_t(\Omega)) = \int_{\Omega_t} y_t \, dV$, hence by Proposition 2.3, j is differentiable at t_0 .

Note that j is an even function which is differentiable at 0. Hence $j'(0) = 0$.

Now onwards we fix t_0 such that $0 < t_0 < r_1 - r_0$ and consider $\Omega := \Omega(q(t_0))$ and $B_0 := B(q(t_0), r_0)$. Let n denote the outward unit normal of Ω on $\partial\Omega$. For $x \in \partial B_0$, put $a = d(p, x)$ and $\alpha =$ the angle at p of the spherical triangle $T := [p, q(t_0), x]$ with vertices $p, q(t_0)$ and x . Then $n(x) = (q(t_0) - \cos r_0 x) / \sin r_0$ and $\langle V, n \rangle(x) = (\cos a \sin t_0 - \sin a \cos t_0 \cos \alpha) / \sin r_0$. Hence, by eq. (19) on p. 30 of [6], we get

$$\langle V, n \rangle(x) = \cos \beta(x), \quad (8)$$

where $\beta(x)$ denotes the angle at $q(t_0)$ of the spherical triangle T defined above.

By Proposition 2.3, $j'(t_0) = \int_{\Omega} y' \, dV$. Hence by Proposition 2.4 and problem (1.1),

$$\begin{aligned} \int_{\Omega} y' \, dV &= - \int_{\Omega} \{y' \Delta y - y \Delta y'\} \, dV = - \int_{\partial\Omega} \left\{ y' \frac{\partial y}{\partial n} - y \frac{\partial y'}{\partial n} \right\} \, dS \\ &= - \int_{\partial\Omega} y' \frac{\partial y}{\partial n} \, dS. \end{aligned}$$

Again by Proposition 2.4 and eq. (8) above, we get

$$j'(t_0) = \int_{x \in \partial B_0} \left(\frac{\partial y}{\partial n}(x) \right)^2 \cos \beta(x) \, dS. \quad (9)$$

Let H denote the hyperplane in \mathbb{R}^{n+1} through $(0, \dots, 0)$ having $q'(t_0)$ as a normal vector. Let r_H denote the reflection of S^n about H . Put $\mathcal{O} = \{x \in \Omega \mid \langle x, q'(t_0) \rangle > 0\}$. Then $r_H(\mathcal{O}) \subset B(r_1)$ and $r_H(\overline{B_0}) = \overline{B_0}$. For $x \in \partial B_0 \cap \partial\mathcal{O}$, let x' denote $r_H(x)$. Note that for all $x \in \partial B_0 \cap \partial\mathcal{O}$, $\cos \beta(x) < 0$ and $\cos \beta(x') = -\cos \beta(x)$. Thus eq. (9) can be re-written as

$$j'(t_0) = \int_{x \in \partial B_0 \cap \partial\mathcal{O}} \left\{ \left(\frac{\partial y}{\partial n}(x) \right)^2 - \left(\frac{\partial y}{\partial n}(x') \right)^2 \right\} \cos \beta(x) \, dS. \quad (10)$$

The Laplace–Beltrami operator Δ of S^n is uniformly elliptic on S^n and hence the maximum principle ([4], Theorem 5, p. 61) and the Hopf maximum principle ([4], Theorem 7, p. 65) are applicable on $\overline{\Omega}$. Hence, by arguments analogous to [3] at this stage, we get

$$\left| \frac{\partial y}{\partial n}(x) \right| < \left| \frac{\partial y}{\partial n}(x') \right| \quad \forall x \in \partial B_0 \cap \partial\mathcal{O}.$$

Thus from eq. (10), $j'(t_0) > 0$. This completes the proof of Theorem 1 for S^n . \square

Proof of Theorem 2 for S^n . We continue with the notations of §1 such as $\lambda_1(\Omega)$, $y_1(\Omega)$ and $J_1(\Omega)$ for $\Omega \in \mathcal{F}$. Let $p, q(t)$ be as in the proof of Theorem 1. Define $j_1: (r_0 -$

$r_1, r_1 - r_0 \longrightarrow \mathbb{R}$ by $j_1(t) = J_1(\Omega(q(t)))$. As in the proof of Theorem 1, fix t_0 such that $0 \leq t_0 < r_1 - r_0$ and put $\Omega := \Omega(q(t_0))$ and $B_0 := B(q(t_0), r_0)$. Then for t sufficiently close to 0 we have $j_1(t_0 + t) = J_1(\Phi_t(\Omega)) = \lambda_1(\Phi_t(\Omega))$. By Proposition 3.1, j_1 is differentiable at $t = t_0$ and $j_1'(t_0) = \lambda_1'(\Omega)$. As $\lambda_1(\Phi_t(\Omega)) = \lambda_1(\Phi_{-t}(\Omega))$, j_1 is an even function which is differentiable at 0. Thus $j_1'(0) = 0$.

Now onwards we fix t_0 such that $0 < t_0 < r_1 - r_0$ and put $\Omega := \Omega(q(t_0))$ and $B_0 := B(q(t_0), r_0)$. Then by Proposition 3.5 and eq. (8), we get

$$j_1'(t_0) = \lambda_1'(\Omega) = - \int_{\partial\Omega} \left(\frac{\partial y_1}{\partial n} \right)^2 \langle V, n \rangle \, dS = - \int_{\partial B_0} \left(\frac{\partial y_1}{\partial n} \right)^2 \cos \beta(x) \, dS. \tag{11}$$

As in the proof of Theorem 1, eq. (11) can be re-written as

$$j_1'(t_0) = - \int_{x \in \partial B_0 \cap \partial\mathcal{O}} \left\{ \left(\frac{\partial y_1}{\partial n}(x) \right)^2 - \left(\frac{\partial y_1}{\partial n}(x') \right)^2 \right\} \cos \beta(x) \, dS. \tag{12}$$

The Laplace–Beltrami operator Δ of S^n is uniformly elliptic on S^n . So, the Hopf maximum principle ([4], Theorem 7, p. 65) and the generalised maximum principle ([4], Theorem 10, p. 73) are applicable on $\bar{\Omega}$. Hence, by arguments analogous to [3] we get

$$\left| \frac{\partial y_1}{\partial n}(x) \right| < \left| \frac{\partial y_1}{\partial n}(x') \right| \quad \forall x \in \partial B_0 \cap \partial\mathcal{O}.$$

It follows from eq. (12) that $j_1'(t_0) < 0$. The proof of Theorem 2 is now complete for S^n . □

Remark on proofs of Theorem 1 and Theorem 2 for \mathbb{H}^n . For $t \in \mathbb{R}$, define $q(t) = (0, \dots, 0, \sinh t, \cosh t) \in \mathbb{H}^n$. Put $p := q(0)$ and $q := q(t_0)$ ($t_0 > 0$). Define the vector field V on \mathbb{H}^n by $V(x) = \rho(x) (0, \dots, 0, x_{n+1}, x_n) \forall x = (x_1, \dots, x_{n+1}) \in \mathbb{H}^n$, where $\rho: \mathbb{H}^n \longrightarrow \mathbb{R}$ is as in the proof of Theorem 1 for S^n .

Let n denote the inward unit normal of $B(q, r_0)$ on $\partial B(q, r_0)$. Then,

$$n(x) = (q - \cosh r_0 x) / \sinh r_0$$

and

$$\langle V, n \rangle(x) = (x_{n+1} \sinh t_0 - x_n \cosh t_0) / \sinh r_0 = \cos \beta(x),$$

where $\beta(x)$ denotes the angle at q of the hyperbolic triangle $[p, q, x]$ with vertices p, q and x .

Now Theorems 1 and 2 for the hyperbolic case can be proved using shape calculus of §§2 and 3 as in the case of sphere. □

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