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Inequalities for dual quermassintegrals of mixed intersection bodies

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Abstract. In this paper, we first introduce a new concept of *dual quermassintegral sum function* of two star bodies and establish Minkowski's type inequality for dual quermassintegral sum of mixed intersection bodies, which is a general form of the Minkowski inequality for mixed intersection bodies. Then, we give the Aleksandrov–Fenchel inequality and the Brunn–Minkowski inequality for mixed intersection bodies and some related results. Our results present, for intersection bodies, all dual inequalities for Lutwak's mixed prosection bodies inequalities.

Keywords. Dual mixed volumes; mixed projection bodies; mixed intersection bodies.

0. Introduction

One might say the history of intersection bodies began with the paper of Busemann [4]. Intersection bodies were first explicitly defined and named by Lutwak [11]. It was here that the duality between intersection bodies and projection bodies was first made clear. Despite considerable ingenuity of earlier attacks on the Busemann–Petty problem, it seems fair to say that the work of Lutwak [11] represents the beginning of its eventual solution. In [11], Lutwak also showed that if a convex body is sufficiently smooth and not an intersection body, then there exists a centred star body such that the conditions of Busemann–Petty problem holds, but the result inequality is reversed. Following Lutwak, the intersection body of order i of a star body is introduced by Zhang [21]. It follows from this definition that every intersection body of order i of a star body is an intersection body of a star body, and vice versa. As Zhang observes, the new definition of intersection body allows a more appealing formulation, namely: the Busemann–Petty problem has a positive answer in n-dimensional Euclidean space if and only if each centered convex body is an intersection body. The intersection body plays an essential role in Busemann's theory [5] of area in Minkowski spaces. The intersection body is also an important matter of the Brunn–Minkowski theory.

In recent years, some authors including Ball [1, 2], Bourgain [3], Gardner [6–8], Schneider [19] and Lutwak [12–18] have given considerable attention to the Brunn–Minkowski theory and their various generalizations. The purpose of this paper is to establish the Minkowski inequality for the dual quermassintegral sum, which is a generalization of the

Minkowski inequality for mixed intersection bodies. Then, the Brunn–Minkowski inequality and the Aleksandrov–Fenchel inequality for mixed intersection bodies are proved and some related results are also given. In this work we shall derive, for intersection bodies, all the analogous inequalities for Lutwak's mixed projection body inequalities [15]. Thus, this work may be seen as presenting additional evidence of the natural duality between intersection and projection bodies.

1. Notation and preliminaries

The setting for this paper is an n-dimensional Euclidean space $\mathbb{R}^n$ $(n > 2)$. Let $\mathbb{C}_n$ denote the set of non-empty convex figures (compact, convex subsets) and let $\mathcal{K}^n$ denote the subset of $\mathbb{C}_n$ consisting of all convex bodies (compact, convex subsets with non-empty interiors) in $\mathbb{R}^n$. We reserve u for unit vectors, and B for the unit ball centered at the origin. The surface of B is S^{n-1}. For $u \in S^{n-1}$, let E_u denote the hyperplane, through the origin, that is orthogonal to u. Let K^u to denote the image of K under an orthogonal projection onto the hyperplane E_u. We use $V(K)$ for the n-dimensional volume of convex body K. The support function of $K \in \mathcal{K}^n$, $h(K,\cdot)$, defined on $\mathbb{R}^n$ by $h(K,\cdot) = \max\{x \cdot y : y \in K\}$. Let δ denote the Hausdorff metric on $\mathcal{K}^n$; i.e., for $K, L \in \mathcal{K}^n$, $\delta(K, L) = |h_K - h_L|_\infty$, where $|\cdot|_\infty$ denotes the sup-norm on the space of continuous functions, $C(S^{n-1})$.

Associated with a compact subset K of $\mathbb{R}^n$, which is star-shaped with respect to the origin, its radial function $\rho(K,\cdot) : S^{n-1} \to \mathbb{R}$, defined for $u \in S^{n-1}$, by $\rho(K,u) = \max\{\lambda \geq 0 : \lambda u \in K\}$. If $\rho(K,\cdot)$ is positive and continuous, K will be called a star body. Let φ^n denote the set of star bodies in $\mathbb{R}^n$.

1.1 *Dual mixed volumes*

If $K_1, \ldots, K_r \in \varphi^n$ and $\lambda_1, \ldots, \lambda_r \in \mathbb{R}$, then the radial Minkowski linear combination, $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$, is defined by $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r = \{\lambda_1 x_1 \tilde{+} \cdots \tilde{+} \lambda_r x_r : x_i \in K_i\}$.

The following property will be used later. If $K, L \in \varphi^n$ and $\lambda, \mu \geq 0$,

$$\rho(\lambda K \tilde{+} \mu L, \cdot) = \lambda \rho(K,\cdot) + \mu \rho(L,\cdot). \tag{1.1.1}$$

For $K_1, \ldots, K_r \in \varphi^n$ and $\lambda_1, \ldots, \lambda_r \geq 0$, the volume of the radial Minkowski linear combination $\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r$ is a homogeneous nth-degree polynomial in the λ_i [19],

$$V(\lambda_1 K_1 \tilde{+} \cdots \tilde{+} \lambda_r K_r) = \sum \tilde{V}_{i_1,\ldots,i_n} \lambda_{i_1} \cdots \lambda_{i_n}, \tag{1.1.2}$$

where the sum is taken over all n-tuples $(i_1, \ldots, i_n)$ whose entries are positive integers not exceeding r. If we require the coefficients of the polynomial in (1.1.2) to be symmetric in their arguments, then they are uniquely determined. The coefficient $\tilde{V}_{i_1,\ldots,i_n}$ is non-negative and depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$. It is written as $\tilde{V}(K_{i_1}, \ldots, K_{i_n})$ and is called the *dual mixed volume* of $K_{i_1}, \ldots, K_{i_n}$. If $K_1 = \cdots = K_{n-i} = K$, $K_{n-i+1} = \cdots = K_n = L$, the dual mixed volumes is written as $\tilde{V}_i(K, L)$ and the dual mixed volumes $\tilde{V}_i(K, B)$ is written as $\tilde{W}_i(K)$.

For $K, L \in \varphi^n$ and $i \in R$, the ith dual mixed volume of K and L, $\tilde{V}_i(K, L)$, is defined by [14]

$$\tilde{V}_i(K, L) = \frac{1}{n} \int_{S^{n-1}} \rho(K,u)^{n-i} \rho(L,u)^i \mathrm{d}S(u).$$

From the above identity, if $K \in \varphi^n$, $i \in \mathbb{R}$, then

$$\tilde{W}_i(K) = \frac{1}{n}\int_{S^{n-1}} \rho(K,u)^{n-i}\mathrm{d}S(u). \tag{1.1.3}$$

If $K_i \in \varphi^n$ $(i = 1, 2, \ldots, n-1)$, then the dual mixed volume of $K_i \cap E_u$ $(i = 1, 2, \ldots, n-1)$ will be denoted by $\tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u)$. If $K_1 = \cdots = K_{n-1-i} = K$ and $K_{n-i} = \cdots = K_{n-1} = L$, then $\tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u)$ is written as $\tilde{v}_i(K \cap E_u, L \cap E_u)$. If $L = B$, then $\tilde{v}_i(K \cap E_u, B \cap E_u)$ is written as $\tilde{w}_i(K \cap E_u)$.

1.2 *Intersection bodies*

For $K \in \varphi^n$, there is a unique star body IK whose radial function satisfies for $u \in S^{n-1}$,

$$\rho(IK, u) = v(K \cap E_u). \tag{1.2.1}$$

It is called the *intersection bodies* of K. From a result of Busemann, it follows that IK is a convex if K is convex and centrally symmetric with respect to the origin. Clearly any intersection body is centred.

The volume of the intersection bodies is given by $V(IK) = \frac{1}{n}\int_{S^{n-1}} v(K \cap E_u)^n \mathrm{d}S(u)$.

The mixed intersection bodies of $K_1, \ldots, K_{n-1} \in \varphi^n$, $I(K_1, \ldots, K_{n-1})$, whose radial function is defined by

$$\rho(I(K_1, \ldots, K_{n-1}), u) = \tilde{v}(K_1 \cap E_u, \ldots, K_{n-1} \cap E_u), \tag{1.2.2}$$

where $\tilde{v}$ is $(n-1)$-dimensional dual mixed volume.

If $K \in \varphi^n$ with $\rho(K, u) \in C(S^{n-1})$, and $i \in \mathbb{R}$ is positive, the *intersection body of order* i of K is the centered star body $I_i K$ such that [3] $\rho(I_i K) = \frac{1}{n-1}\int_{S^{n-1}} \rho(K,u)^{n-i-1}\mathrm{d}S(u)$, for $u \in S^{n-1}$, where $I_i K = I(\underbrace{K, \ldots, K}_{n-i-1}, \underbrace{B, \ldots, B}_{i})$.

If $K_1 = \cdots = K_{n-i-1} = K$, $K_{n-i} = \cdots = K_{n-1} = L$, then $I(K_1, \ldots, K_{n-1})$ is written as $I_i(K, L)$. If $L = B$, then $I_i(K, L)$ is written as $I_i K$ and is called the ith intersection body of K. For $I_0 K$ simply write IK. The term is introduced by Zhang [21].

The following properties will be used later: If $K, L, M, K_1, \ldots, K_{n-1} \in \varphi^n$ and $\lambda, \mu, \lambda_1, \ldots, \lambda_{n-1} > 0$, then

$$I(\lambda K \tilde{+} \mu L, M) = \lambda I(K, M) \tilde{+} \mu I(L, M), \quad \text{where } M = (K_1, \ldots, K_{n-2}). \tag{1.2.3}$$

$$I(\lambda_1 K_1, \ldots, \lambda_{n-1} K_{n-1}) = \lambda_1 \cdots \lambda_{n-1} I(K_1, \ldots, K_{n-1}). \tag{1.2.4}$$

2. Main results

The following results will be required to prove our main Theorems.

Lemma A. If $K, L \in \varphi^n$, $0 \le i < n$ and $0 < j < n-1$, then

$$\tilde{W}_i(IK) = \frac{1}{n}\int_{S^{n-1}} v(K \cap E_u)^{n-i}\mathrm{d}S(u),$$

$$\tilde{W}_i(I_j K) = \frac{1}{n}\int_{S^{n-1}} \tilde{w}_j(K \cap E_u)^{n-i}\mathrm{d}S(u),$$

$$\tilde{W}_i(I_j(K,L)) = \frac{1}{n}\int_{S^{n-1}} \tilde{v}_j(K\cap E_u, L\cap E_u)^{n-i}\,\mathrm{d}S(u).$$

To prove this, we use (1.1.3) in conjunction with the fact (1.2.2).

Lemma B [14]. *If $K_1,\ldots,K_n \in \varphi^n$, then*

$$\tilde{V}(K_1,\ldots,K_n)^r \le \prod_{j=1}^{r} \tilde{V}(\underbrace{K_j,\ldots,K_j}_{r}, K_{r+1},\ldots,K_n)$$

with equality if and only if $K_1,\ldots,K_n$ are all dilations of each other.

We shall need the following trivial elementary inequality.

Lemma C. If $a, b \ge 0$ and $c, d > 0$, then for $0 < p < 1$,

$$(a+b)^p(c+d)^{1-p} \ge a^p c^{p-1} + b^p d^{p-1},$$

with equality if and only if $ad = bc$.

Proof. Consideration the following function

$$f(x) = (x+b)^p(c+d)^{1-p} - x^p c^{1-p}, \qquad x \ge 0.$$

Let $f'(x) = p(c+d)^{1-p}(x+b)^{p-1} - pc^{1-p}x^{p-1} = 0$, we get $x = bc/d$. If $x \in (0, \frac{bc}{d})$, then $f'(x) < 0$; if $x \in \left(\frac{bc}{d}, +\infty\right)$, then $f'(x) > 0$. It follows that

$$\min_{x\ge 0}\{f(x)\} = f\left(\frac{bc}{d}\right) = b^p d^{1-p}.$$

This completes the proof. □

2.1 *The Minkowski inequality for dual quermassintegral sum of mixed intersection bodies*

In [10], Leng introduce the concept of *i-quermassintegral difference function* of convex bodies as follows: If $K, D \in \mathcal{K}^n$ and $D \subset K$, then i-quermassintegral difference function of convex bodies K and D, $D_{w_i}(K,D)$, is defined by

$$D_{w_i}(K,D) = W_i(K) - W_i(D) \qquad (0 \le i \le n-1).$$

In the section, we first introduce a new concept, *dual quermassintegral sum function*, as follows:

If $K, D \in \varphi^n$, then the dual quermassintegral sum function of star bodies K and D, $S_{\tilde{w}_i}(K,D)$, is defined by

$$S_{\tilde{w}_i}(K,D) = \tilde{W}_i(K) + \tilde{W}_i(D) \qquad (0 \le i \le n-1).$$

When $i = 0$, we have $S_v(K,D) = V(K) + V(D)$, which is called the *dual volume sum function* of star bodies K and L.

The following Minkowski inequality for mixed intersection bodies will be established: If $K, L \in \varphi^n$, and $0 \le i < n$ and $0 < j < n-1$, then

$$\tilde{W}_i(I_j(K,L))^{n-1} \le \tilde{W}_i(IK)^{n-j-1}\tilde{W}_i(IL)^j, \tag{2.1.0}$$

with equality if and only if K and L are dilates.

This is just the special case $D = D' = \emptyset$ of the following inequality.

Theorem 2.1.1. *If $K, L, D, D' \in \varphi^n$. Let D' is a dilate copy of D, and $0 \le i < n$ and $0 < j < n-1$, then*

$$S_{\tilde{w}_i}(I_j(K,L), I_j(D,D'))^{n-1} \le S_{\tilde{w}_i}(IK, ID)^{n-j-1} S_{\tilde{w}_i}(IL, ID')^j, \quad (2.1.1)$$

with equality if and only if K and L are dilates.

Proof. In view of the special case of Lemma B, we obtain that

$$\tilde{v}_j(K \cap E_u, L \cap E_u)^{n-i} \le v(K \cap E_u)^{\frac{(n-i)(n-j-1)}{n-1}} v(L \cap E_u)^{\frac{j(n-i)}{n-1}} \quad (2.1.2)$$

with equality if and only if $K \cap E_u$ and $L \cap E_u$ are dilates. It follows if and only if K and L are dilates [20].

From Lemma A, eq. (2.1.2) and in view of Minkowski inequality for integral [9], we have for $i < n-1$,

$$\begin{aligned} n\tilde{W}_i(I_j(K,L)) &= \left(\|\tilde{v}_j(K \cap E_u, L \cap E_u)\|_{n-i}\right)^{n-i} \\ &\le \left(\|v(K \cap E_u)^{\frac{n-j-1}{n-1}} v(L \cap E_u)^{\frac{j}{n-1}}\|_{n-i}\right)^{n-i} \\ &\le (\|v(K \cap E_u\|_{n-i})^{\frac{(n-i)(n-j-1)}{n-1}} (\|v(K \cap E_u\|_{n-i})^{\frac{j(n-i)}{n-1}} \\ &= (n\tilde{W}_i(IK))^{\frac{(n-j-1)}{n-1}} (n\tilde{W}_i(IL))^{\frac{j}{n-1}} \\ &= n\tilde{W}_i(IK)^{\frac{(n-j-1)}{n-1}} \tilde{W}_i(IL)^{\frac{j}{n-1}}. \end{aligned} \quad (2.1.3)$$

In view of the conditions of (2.1.2) and Minkowski inequality for integral, it follows that the equality holds if and only if K and L are dilates.

Moreover, we consider the case of $i = n-1$ of the inequality (2.1.3). If $i = n-1$, inequality (2.1.3) reduces to

$$\tilde{W}_{n-1}(I_j(K,L))^{n-1} \le \tilde{W}_{n-1}(IK)^{n-j-1} \tilde{W}_{n-1}(IL)^j. \quad (*)$$

From Lemma A, (*) changes to

$$\begin{aligned} &\left(\int_{S^{n-1}} \tilde{v}_j(K \cap E_u, L \cap E_u) dS(u)\right)^{n-1} \\ &\quad \le \left(\int_{S^{n-1}} v(K \cap E_u) dS(u)\right)^{n-j-1} \left(\int_{S^{n-1}} v(L \cap E_u) dS(u)\right)^j. \end{aligned} \quad (**)$$

On the other hand, integrating both sides of (2.1.2) and in view of Hölder inequality for integral, we obtain

$$\begin{aligned} &\int_{S^{n-1}} \tilde{v}_j(K \cap E_u, L \cap E_u) dS(u) \\ &\quad \le \int_{S^{n-1}} v(K \cap E_u)^{\frac{n-j-1}{n-1}} v(L \cap E_u)^{\frac{j}{n-1}} dS(u) \\ &\quad \le \left(\int_{S^{n-1}} v(K \cap E_u) dS(u)\right)^{\frac{n-j-1}{n-1}} \left(\int_{S^{n-1}} v(L \cap E_u) dS(u)\right)^{\frac{j}{n-1}}. \end{aligned}$$

Moreover, from inequality (2.1.3), we obtain

$$\tilde{W}_i(I_j(K,L))^{n-1} \le \tilde{W}_i(IK)^{n-j-1}\tilde{W}_i(IL)^j,$$

with equality if and only if K and L are dilates, and

$$\tilde{W}_i(I_j(D,D'))^{n-1} = \tilde{W}_i(ID)^{n-j-1}\tilde{W}_i(ID')^j.$$

Hence, from the inequality in Lemma C, we have

$$\begin{aligned} S_{\tilde{w}_i}(I_j(K,L), I_j(D,D')) &\le \tilde{W}_i(IK)^{(n-j-1)/(n-1)}\tilde{W}_i(IL)^{j/(n-1)} \\ &\quad + \tilde{W}_i(ID)^{(n-j-1)/(n-1)}\tilde{W}_i(ID')^{j/(n-1)} \\ &\le S_{\tilde{w}}(IK, ID)^{n-j-1}S_{\tilde{w}}(IL, ID')^j. \end{aligned}$$

The proof of Theorem 2.1.1 is complete. □

Remark 2.1.1. Taking $D = D' = \emptyset$ and $j = 1$ to (2.1.1), (2.1.1) changes to

$$\tilde{W}_i(I_1(K,L))^{n-1} \le \tilde{W}_i(IK)^{n-2}\tilde{W}_i(IL),$$

with equality if and only if K and L are dilates.

This is just a dual form of the following inequality which was given by Lutwak [15].

The Minkowski inequality for mixed projection bodies. If $K, L \in \mathcal{K}^n$, and $0 \le i < n$, then

$$W_i(\Pi_1(K,L))^{n-1} \ge W_i(\Pi K)^{n-2}W_i(\Pi L),$$

with equality if and only if K and L are homothetic.

A somewhat surprising consequence of Theorem 2.1.1 is the following version.

Theorem 2.1.2. *If* $K, L \in \eta \subset \varphi^n$, *and* $0 \le i < n$, *while* $0 < j < n-1$ *and if either*

$$\tilde{W}_i(I_j(K,M)) = \tilde{W}_i(I_j(L,M)), \quad \textit{for all } M \in \eta \tag{2.1.4}$$

or

$$\tilde{W}_i(I_j(M,K)) = \tilde{W}_i(I_j(M,L)), \quad \textit{for all } M \in \eta \tag{2.1.5}$$

hold, then it follows that $K = L$, *up to translation.*

Proof. Suppose (2.1.4) holds. Take K for M in (2.1.4) and use the inequality (2.1.1). We obtain

$$\tilde{W}_i(IK) = \tilde{W}_i(I_j(L,K)) \le \tilde{W}_i(IL)^{\frac{(n-j-1)}{n-1}}\tilde{W}_i(IK)^{\frac{j}{n-1}}$$

with equality if and only if K is a dilation of L.

Hence

$$\tilde{W}_i(IK) \le \tilde{W}_i(IL)$$

with equality if and only if K is a dilation of L.

Similarly, take L for M in (2.1.4) and use again the inequality (2.1.1). We get

$$\tilde{W}_i(IK) \geq \tilde{W}_i(IL),$$

with equality if and only if K is a dilation of L.

Hence

$$\tilde{W}_i(IK) = \tilde{W}_i(IL)$$

and K is a dilation of L. In view of the fact that the intersection bodies are centered, there exist $\lambda > 0$ such that $K = \lambda L$, and $\lambda^{(n-1)(n-i)} = 1$, for $0 \leq i < n-1$. Therefore $\lambda = 1$. □

Similar sort of argument shows that condition (2.1.5) implies that K and L must be translates.

Remark 2.1.2. Theorem 2.1.2 is just the dual form of the following 'Theorem 5.4' which was given by Lutwak [15].

Theorem 5.4. *If* $K, L \in \gamma \subset \mathcal{K}^n$, *and* $0 \leq i < n$, *while* $0 < j < n-1$ *and if either*

$$W_i(\Pi_j(K, M)) = W_i(\Pi_j(L, M)), \quad \textit{for } M \in \gamma$$

or

$$W_i(\Pi_j(M, K)) = W_i(\Pi_j(M, L)), \quad \textit{for } M \in \gamma$$

hold, then it follows that $K = L$, *up to translation.*

2.2 *The Aleksandrov–Fenchel inequality for mixed intersection bodies*

The Aleksandrov–Fenchel inequality for mixed intersection bodies is as follows: If $K_1, \ldots, K_{n-1} \in \varphi^n$, then

$$V(I(K_1, \ldots, K_{n-1})) \leq \prod_{j=1}^{r} V(I(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1}))$$

with equality if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other.

This is just the special case $i = 0$ of the following.

Theorem 2.2.1. *If* $K_1, \ldots, K_{n-1} \in \varphi^n$, $0 \leq i < n$, $0 < j < n-1$ *and* $0 < r \leq n-1$, *then*

$$\tilde{W}_i(I(K_1, \ldots, K_{n-1}))^r \leq \prod_{j=1}^{r} \tilde{W}_i(I(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1})) \tag{2.2.1}$$

with equality if and only if $K_1, \ldots, K_{n-1}$ *are all dilations of each other.*

Proof. When $i = 1$, inequality (2.2.1) reduces to the inequality in Lemma B. In the following, we suppose that $i < n-1$.

From (1.1.3) and (1.2.2), we have that

$$\tilde{W}_i(I(K_1,\ldots,K_{n-1})) = \frac{1}{n}\int_{S^{n-1}} \tilde{v}(K_1\cap E_u,\ldots,K_{n-1}\cap E_u)^{n-i}\mathrm{d}S(u). \tag{2.2.2}$$

By using the inequality in Lemma B, we easily get that

$$\begin{aligned}&\tilde{v}(K_1\cap E_u,\ldots,K_{n-1}\cap E_u)^{n-i}\\ &\quad\le\left(\prod_{j=1}^{r}\tilde{v}(\underbrace{K_j\cap E_u,\ldots,K_j\cap E_u}_{r},K_{r+1}\cap E_u,\ldots,K_{n-1}\cap E_u)\right)^{\frac{n-i}{r}},\end{aligned} \tag{2.2.3}$$

with equality if and only if $K_1\cap E_u,\ldots,K_{n-1}\cap E_u$ are all dilations of each other. It follows if and only if $K_1,\ldots,K_{n-1}$ are all dilations of each other.

On the other hand, the Hölder's inequality can be stated as [9]

$$\int_{S^{n-1}}\prod_{i=1}^{m}f_i(u)\mathrm{d}S(u)\le\prod_{i=1}^{m}\left(\int_{S^{n-1}}(f_i(u))^m\mathrm{d}S(u)\right)^{1/m}, \tag{2.2.4}$$

with equality if and only if all f_i are proportional.

From (2.2.2), (2.2.3) and (2.2.4), we obtain that

$$\begin{aligned}\tilde{W}_i(I(K_1,\ldots,K_{n-1})) &= \frac{1}{n}\int_{S^{(n-1)}}\tilde{v}(K_1\cap E_u,\ldots,K_{n-1}\cap E_u)^{n-i}\mathrm{d}S(u)\\ &\le\frac{1}{n}\int_{S^u}\left(\prod_{j=1}^{r}\tilde{v}(\underbrace{K_j\cap E_u,\ldots,K_j\cap E_u}_{r},\right.\\ &\qquad\left.K_{r+1}\cap E_u,\ldots,K_{n-1}\cap E_u)\right)^{\frac{n-i}{r}}\mathrm{d}S(u)\\ &\le\left(\prod_{j=1}^{r}\frac{1}{n}\int_{S^{n-1}}\tilde{v}(\underbrace{K_j\cap E_u,\ldots,K_j\cap E_u}_{r},\right.\\ &\qquad\left.K_{r+1}\cap E_u,\ldots,K_{n-1}\cap E_u)^{n-i}\mathrm{d}S(u)\right)^{r}\\ &=\left(\prod_{j=1}^{r}\tilde{W}_i(I\,(\underbrace{K_j,\ldots,K_j}_{r},K_{r+1},\ldots,K_{n-1}))\right)^{r}.\end{aligned}$$

In view of the equality conditions (2.2.3) and (2.2.4), it follows that the equality holds if and only if $K_1,\ldots,K_{n-1}$ are all dilations of each other.

The proof is complete. □

Remark 2.2.1. The inequality (2.2.1) is just a dual form of the following inequality which was given by Lutwak [15].

The Aleksandrov–Fenchel inequality for mixed projection bodies. If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, $0 \leq i < n$, $1 < j < n-1$ and $0 < r \leq n-1$ then

$$W_i(\Pi(K_1, \ldots, K_{n-1}))^r \geq \prod_{j=1}^{r} W_i(\Pi(\underbrace{K_j, \ldots, K_j}_{r}, K_{r+1}, \ldots, K_{n-1})).$$

From the case $r = n-1$ of inequality (2.2.1), it is as follows.

COROLLARY 2.2.1

If $K_1, \ldots, K_{n-1} \in \varphi^n$, and $0 \leq i < n$ then

$$\tilde{W}_i(I(K_1, \ldots, K_{n-1}))^{n-1} \leq \tilde{W}_i(IK_1) \cdots \tilde{W}_i(IK_{n-1}), \tag{2.2.5}$$

with equality if and only if $K_1, \ldots, K_{n-1}$ are all dilations of each other.

Remark 2.2.2. Corollary (2.2.1) is similar to the following 'Theorem 5.2' which was given by Lutwak [15].

Theorem 5.2. *If $K_1, \ldots, K_{n-1} \in \mathcal{K}^n$, and $0 \leq i < n$ then*

$$W_i(\Pi(K_1, \ldots, K_{n-1}))^{n-1} \geq W_i(\Pi K_1) \cdots W_i(\Pi K_{n-1}),$$

with equality if and only if $K_1, \ldots, K_{n-1}$ are homothetic.

Taking $K_1 = \cdots = K_{n-j-1} = K$ and $K_{n-j} = \cdots = K_{n-1} = L$ to (2.2.5), (2.2.5) reduces to (2.1.0). Taking $K_1 = \cdots = K_r = K$, $K_r = L$, and $K_{r+1} = \cdots = K_{n-1} = B$ to (2.2.1), (2.2.1) changes to the following.

COROLLARY 2.2.2

If $K, L \in \varphi^n$, and $0 \leq i < n$ and $0 \leq j < n-1$, then

$$\tilde{W}_i(I(\underbrace{K, \ldots, K}_{n-j-2}, \underbrace{B, \ldots, B}_{j}, L))^{n-j-1} \leq \tilde{W}_i(I_j K)^{n-j-2} \tilde{W}_i(I_j L),$$

with equality if and only if K and L are dilates.

A somewhat surprising consequence of Corollary 2.2.2 is the following version for mixed intersection bodies.

Theorem 2.2.2. *If $K, L \in \eta \subset \varphi^n$, $0 \leq i < n-1$, $0 \leq j < n-1$ and if either*

$$\tilde{W}_i(I(\underbrace{K, \ldots, K}_{n-j-1}, \underbrace{B, \ldots, B}_{j}, M))$$

$$= \tilde{W}_i(I(\underbrace{L, \ldots, L}_{n-j-1}, \underbrace{B, \ldots, B}_{j}, M)), \qquad \textit{for all } M \in \eta, \tag{2.2.6}$$

or

$$\tilde{W}_i(I(\underbrace{M,\ldots,M}_{n-j-1},\underbrace{B,\ldots,B}_{j},K)) = \tilde{W}_i(I(\underbrace{M,\ldots,M}_{n-j-1},\underbrace{B,\ldots,B}_{j},L)), \qquad \textit{for all } M\in\eta, \tag{2.2.7}$$

hold, then it follows that $K=L$, up to translation.

Proof. Suppose (2.2.6) holds, take K for M, use Corollary 2.2.2, and get

$$\tilde{W}_i(I_jK) = \tilde{W}_i(I(\underbrace{L,\ldots,L}_{n-j-1},\underbrace{B,\ldots,B}_{j},K)) \le \tilde{W}_i(I_jL)^{\frac{n-j-2}{n-j-1}}\tilde{W}_i(I_jK)^{\frac{1}{n-j-1}},$$

with equality if and only if K and L are dilates.

Hence

$$\tilde{W}_i(I_jK) \le \tilde{W}_i(I_jL),$$

with equality if and only if K and L are dilates.

On the other hand, take L for M, use Corollary 2.2.2 again, and get

$$\tilde{W}_i(I_jK) \ge \tilde{W}_i(I_jL),$$

with equality if and only if K and L are dilates.

Therefore

$$\tilde{W}_i(I_jK) = \tilde{W}_i(I_jL),$$

where K and L are dilates and in view of the fact that the intersection bodies are centered, there exists $\lambda > 0$ such that $K = \lambda L$. From (1.2.4), we have $\lambda^{(n-j-1)(n-i)}\tilde{W}_i(I_jL) = \tilde{W}_i(I_jL)$, hence $\lambda = 1$.

Similar argument shows that condition (2.2.7) implies K=L, up to tanslation. □

Remark 2.2.3. Taking $j = 0$ to Theorem 2.2.2, it reduces to the following:

If $K, L \in \eta \subset \varphi^n$, $0 \le i < n$ and if either

$$\tilde{W}_i(I_1(K,M)) = \tilde{W}_i(I_1(L,M)), \qquad \textit{for all } M\in\eta$$

or

$$\tilde{W}_i(I_1(M,K)) = \tilde{W}_i(I_1(M,L)), \qquad \textit{for all } M\in\eta$$

hold, then it follows that $K = L$, up to translation.

This is just the special case $j = 1$ of Theorem 2.1.2.

2.3 *The Brunn–Minkowski inequality for mixed intersection bodies*

The Brunn–Minkowski inequality for intersection bodies, which will be established is: If $K, L \in \varphi^n$, then

$$V(I(K\tilde{+}L))^{1/n(n-1)} \le V(IK)^{1/n(n-1)} + V(IL)^{1/n(n-1)},$$

with equality if and only if K and L are dilates.

This is just the special case $i = 0$ and $\alpha = 1$ of the following.

Theorem 2.3.1. *If $K, L \in \varphi^n$, and $0 \le i < n$, then for $0 \le \alpha \le 1$,*

$$\tilde{W}_i(I(K\tilde{+}L))^{1/(n-i)(n-1)} \le \tilde{W}_i(I(\alpha K\tilde{+}(1-\alpha)L))^{1/(n-i)(n-1)} + \tilde{W}_i(I((1-\alpha)K\tilde{+}\alpha L))^{1/(n-i)(n-1)}, \quad (2.3.1)$$

with equality if and only if $(\alpha K\tilde{+}(1-\alpha)L)$ and $(1-\alpha)K\tilde{+}\alpha L$ are dilates.

Proof. Let $M = (L_1, \dots, L_{n-2})$, from (1.1.1), (1.1.3), (1.2.3) and in view of the Minkowski inequality for integral [9], we obtain that

$$\begin{aligned}
\tilde{W}_i(I(K\tilde{+}L, M))^{1/(n-i)} &= n^{-1/(n-i)}\|\rho(I(K\tilde{+}L, M), u)\|_{n-i} \\
&= n^{-1/(n-i)}\|\rho(I(K, M)\tilde{+}I(L, M), u)\|_{n-i} \\
&= n^{-1/(n-i)}\|\rho(I(K, M), u) + \rho(I(L, M), u)\|_{n-i} \\
&\le n^{-1/(n-i)}(\|\alpha\rho(I(K, M), u) \\
&\quad + (1-\alpha)\rho(I(L, M), u)\|_{n-i} + \|(1-\alpha) \\
&\quad \times\rho(I(K, M), u) + \alpha\rho(I(L, M), u)\|_{n-i}) \\
&= n^{-1/(n-i)}\big(\|\rho(\alpha\cdot I(K, M)\tilde{+}(1-\alpha) \\
&\quad \times I(L, M), u)\|_{n-i} + \|\rho((1-\alpha)\cdot I(K, M) \\
&\quad \tilde{+}\alpha I(L, M), u)\|_{n-i}\big) \\
&= n^{-1/(n-i)}\big(\|\rho(I(\alpha\cdot K\tilde{+}(1-\alpha)L, M), u)\|_{n-i} \\
&\quad +\|\rho(I((1-\alpha)\cdot K\tilde{+}\alpha L, M), u)\|_{n-i}\big) \\
&= \tilde{W}_i(I(\alpha\cdot K\tilde{+}(1-\alpha)L, M))^{1/(n-i)} \\
&\quad + \tilde{W}_i(I(1-\alpha)\cdot K\tilde{+}\alpha L, M))^{1/(n-i)}. \qquad (2.3.2)
\end{aligned}$$

On the other hand, taking $L_1 = \cdots = L_{n-2} = K\tilde{+}L$ to (2.3.2) and apply the inequality (2.1.0) twice, we get

$$\begin{aligned}
\tilde{W}_i(I(K\tilde{+}L))^{1/(n-i)} &\le \tilde{W}_i(I_{n-2}(\alpha\cdot K\tilde{+}(1-\alpha)L, K\tilde{+}L))^{1/(n-i)} \\
&\quad + \tilde{W}_i(I_{n-2}((1-\alpha)\cdot K\tilde{+}\alpha L, K\tilde{+}L))^{1/(n-i)} \\
&\le \tilde{W}_i(I(\alpha\cdot K\tilde{+}(1-\alpha)L))^{1/(n-1)(n-i)} \\
&\quad \times \tilde{W}_i(I(K\tilde{+}L))^{(n-2)/(n-1)(n-i)} \\
&\quad + \tilde{W}_i(I((1-\alpha)\cdot K\tilde{+}\alpha L))^{1/(n-1)(n-i)} \\
&\quad \times \tilde{W}_i(I(K\tilde{+}L))^{(n-2)/(n-1)(n-i)}, \qquad (2.3.3)
\end{aligned}$$

with equality if and only if $\alpha \cdot K\tilde{+}(1-\alpha)L$, $(1-\alpha)\cdot K\tilde{+}\alpha L$ and $M = K\tilde{+}L$ are dilates. Combining this with the equality condition of (2.3.2), it follows that the condition holds if and only if K and L are dilates.

Dividing both sides of (2.3.3) by $\tilde{W}_i(I(K\tilde{+}L))^{(n-2)/(n-1)(n-i)}$, we get the inequality (2.3.1).

The proof is complete. □

Taking $\alpha = 1$ in inequality (2.3.1), we have the following.

COROLLARY 2.3.1

If $K, L \in \varphi^n$, and $0 \le i < n$, then

$$\tilde{W}_i(I(K\tilde{+}L))^{1/(n-i)(n-1)} \le \tilde{W}_i(IK)^{1/(n-i)(n-1)} + \tilde{W}_i(IL)^{1/(n-i)(n-1)}, \tag{2.3.4}$$

with equality if and only if K and L are dilates.

Remark 2.3.1. From inequalities (2.3.4) and (1.2.4), we obtain that

$$\tilde{W}_i(I(\alpha K\tilde{+}(1-\alpha)L))^{1/(n-i)(n-1)} \le \alpha\tilde{W}_i(IK)^{1/(n-i)(n-1)} + (1-\alpha)\tilde{W}_i(IL)^{1/(n-i)(n-1)} \tag{2.3.5}$$

and

$$\tilde{W}_i(I((1-\alpha)K\tilde{+}\alpha L))^{1/(n-i)(n-1)} \le (1-\alpha)\tilde{W}_i(IK)^{1/(n-i)(n-1)} + \alpha\tilde{W}_i(IL)^{1/(n-i)(n-1)}. \tag{2.3.6}$$

From (2.3.5) and (2.3.6), we obtain that

$$\tilde{W}_i(I(\alpha K\tilde{+}(1-\alpha)L))^{1/(n-i)(n-1)} + \tilde{W}_i(I((1-\alpha)K\tilde{+}\alpha L))^{1/(n-i)(n-1)} \le \tilde{W}_i(IK)^{1/(n-i)(n-1)} + \tilde{W}_i(IL)^{1/(n-i)(n-1)}.$$

This shows that inequality (2.3.1) is a strengthened form of inequality (2.3.4).

Remark 2.3.2. Inequality (2.3.4) is just a dual form of the following inequality which was given by Lutwak [15].

The Brunn–Minkowski inequality for mixed projection bodies. If $K, L \in \mathcal{K}^n$, and $0 \le i < n$, then

$$W_i(\Pi(K+L))^{1/(n-i)(n-1)} \ge W_i(\Pi K)^{1/(n-i)(n-1)} + W_i(\Pi L)^{1/(n-i)(n-1)},$$

with equality if and only if K and L are homothetic.

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