

Fields and forms on ρ -algebras

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Abstract. In this paper we introduce non-commutative fields and forms on a new kind of non-commutative algebras: ρ -algebras. We also define the Frölicher–Nijenhuis bracket in the non-commutative geometry on ρ -algebras.

Keywords. Non-commutative geometry; ρ -algebras; Frölicher–Nijenhuis bracket.

1. Introduction

There are some ways to define the Frölicher–Nijenhuis bracket in non-commutative differential geometry. The Frölicher–Nijenhuis bracket on the algebra of universal differential forms of a non-commutative algebra, is presented in [2], the Frölicher–Nijenhuis bracket in several kinds of differential graded algebras are defined in [6] and the Frölicher–Nijenhuis bracket on colour commutative algebras is defined in [7]. But this notion is not defined on ρ -algebras in the context of non-commutative geometry. In this paper we introduce the Frölicher–Nijenhuis bracket on a ρ -algebra A using the algebra of universal differential forms $\Omega^*(A)$.

A ρ -algebra A over the field k (\mathbb{C} or \mathbb{R}) is a G -graded algebra (G is a commutative group) together with a twisted cocycle $\rho: G \times G \rightarrow k$. These algebras were defined for the first time in the paper [1] and are generalizations of usual algebras (the case when G is trivial) and of \mathbb{Z} (\mathbb{Z}_2)-superalgebras (the case when G is \mathbb{Z} resp. \mathbb{Z}_2). Our construction of the Frölicher–Nijenhuis bracket for ρ -algebras, in this paper, is a generalization of this bracket from [2].

In §2 we present a class of non-commutative algebras which are ρ -algebras, derivations and bimodules. In §3 we define the algebra of (non-commutative) universal differential forms $\Omega^*(A)$ of a ρ -algebra A . In §4 we present the Frölicher–Nijenhuis calculus on A , the Nijenhuis algebra of A , and the Frölicher–Nijenhuis bracket on A . We also show the naturality of the Frölicher–Nijenhuis bracket.

2. ρ -Algebras

In this section we present a class of non-commutative algebras that are ρ -algebras. For more details see [1].

Let G be an abelian group, additively written, and let A be a G -graded algebra. This implies that the vector space A has a G -grading $A = \bigoplus_{a \in G} A_a$, and that $A_a A_b \subset A_{a+b}$

$(a, b \in G)$. The G -degree of a (non-zero) homogeneous element f of A is denoted as $|f|$. Furthermore let $\rho: G \times G \rightarrow k$ be a map which satisfies

$$\rho(a, b) = \rho(b, a)^{-1}, \quad a, b \in G, \quad (1)$$

$$\rho(a + b, c) = \rho(a, c)\rho(b, c), \quad a, b, c \in G. \quad (2)$$

This implies $\rho(a, b) \neq 0$, $\rho(0, b) = 0$ and $\rho(c, c) = \pm 1$ for all $a, b, c \in G$, $c \neq 0$. We define for homogeneous elements f and g in A an expression, which is ρ -commutator of f and g as

$$[f, g]_\rho = fg - \rho(|f|, |g|)gf. \quad (3)$$

This expression as it stands make sense only for homogeneous elements f and g , but can be extended linearly to general elements. A G -graded algebra A with a given cocycle ρ will be called ρ -commutative if $fg = \rho(|f|, |g|)gf$ for all homogeneous elements f and g in A .

Examples.

- 1) Any usual (commutative) algebra is a ρ -algebra with the trivial group G .
- 2) Let $G = \mathbb{Z}$ (\mathbb{Z}_2) be the group and the cocycle $\rho(a, b) = (-1)^{ab}$, for any $a, b \in G$. In this case any ρ -(commutative) algebra is a super(commutative) algebra.
- 3) The N -dimensional quantum hyperplane [1, 3, 4] S_N^q , is the algebra generated by the unit element and N linearly independent elements x_1, \dots, x_N satisfying the relations:

$$x_i x_j = q x_j x_i, \quad i < j$$

for some fixed $q \in k$, $q \neq 0$. S_N^q is a \mathbb{Z}^N -graded algebra, i.e.,

$$S_N^q = \bigoplus_{n_1, \dots, n_N}^{\infty} (S_N^q)_{n_1 \dots n_N},$$

with $(S_N^q)_{n_1 \dots n_N}$ the one-dimensional subspace spanned by products $x^{n_1} \dots x^{n_N}$. The \mathbb{Z}^N -degree of these elements is denoted by

$$|x^{n_1} \dots x^{n_N}| = n = (n_1, \dots, n_N).$$

Define the function $\rho: \mathbb{Z}^N \times \mathbb{Z}^N \rightarrow k$ as

$$\rho(n, n') = q^{\sum_{j,k=1}^N n_j n'_k \alpha_{jk}},$$

with $\alpha_{jk} = 1$ for $j < k$, 0 for $j = k$ and -1 for $j > k$. It is obvious that S_N^q is a ρ -commutative algebra.

- 4) The algebra of matrix $M_n(\mathbb{C})$ [5] is ρ -commutative as follows:

Let

$$p = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \varepsilon & \dots & 0 \\ \dots & & & \\ 0 & 0 & \dots & \varepsilon^{n-1} \end{pmatrix} \quad \text{and} \quad q = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ \varepsilon & 0 & \dots & 0 & 0 \\ 0 & \varepsilon^2 & \dots & 0 & 0 \\ \dots & & & & \\ 0 & 0 & \dots & \varepsilon^{n-1} & 0 \end{pmatrix},$$

$p, q \in M_n(\mathbb{C})$, where $\varepsilon^n = 1$, $\varepsilon \neq 1$. Then $pq = \varepsilon qp$ and $M_n(\mathbb{C})$ is generated by the set $B = \{p^a q^b | a, b = 0, 1, \dots, n-1\}$.

It is easy to see that $p^a q^b = \varepsilon^{ab} q^b p^a$ and $q^b p^a = \varepsilon^{-ab} p^a q^b$ for any $a, b = 0, 1, \dots, n-1$. Let $G := \mathbb{Z}_n \oplus \mathbb{Z}_n$, $\alpha = (\alpha_1, \alpha_2) \in G$ and $x_\alpha := p^{\alpha_1} q^{\alpha_2} \in M_n(\mathbb{C})$. If we denote $\rho(\alpha, \beta) = \varepsilon^{\alpha_2 \beta_1 - \alpha_1 \beta_2}$ then $x_\alpha x_\beta = \rho(\alpha, \beta) x_\beta x_\alpha$, for any $\alpha, \beta \in G$, $x_\alpha, x_\beta \in B$.

It is obvious that the map $\rho: G \times G \rightarrow \mathbb{C}$, $\rho(\alpha, \beta) = \varepsilon^{\alpha_2 \beta_1 - \alpha_1 \beta_2}$ is a cocycle and that $M_n(\mathbb{C})$ is a ρ -commutative algebra.

Let α be an element of the group G . A ρ -derivation X of A , of degree α is a bilinear map $X: A \rightarrow A$ of G -degree $|\alpha|$ i.e. $X: A_* \rightarrow A_{*+|\alpha|}$, such that one has for all elements $f \in A_{|f|}$ and $g \in A$,

$$X(fg) = (Xf)g + \rho(\alpha, |f|)f(Xg). \quad (4)$$

Without any difficulties it can be obtained that if algebra A is ρ -commutative, $f \in A_{|f|}$ and X is a ρ -derivation of degree α , then fX is a ρ -derivation of degree $|f| + \alpha$ and the G -degree $|f| + |\alpha|$ i.e.

$$(fX)(gh) = ((fX)g)h + \rho(|f| + \alpha, |g|)g(fX)h$$

and $fX: A_* \rightarrow A_{*+|f|+|\alpha|}$.

We say that $X: A \rightarrow A$ is a ρ -derivation if it has degree equal to G -degree $|\alpha|$ i.e. $X: A_* \rightarrow A_{*+|\alpha|}$ and $X(fg) = (Xf)g + \rho(|\alpha|, |f|)f(Xg)$ for any $f \in A_{|f|}$ and $g \in A$.

It is known [1] that the ρ -commutator of two ρ -derivations is again a ρ -derivation and the linear space of all ρ -derivations is a ρ -Lie algebra, denoted by $\rho\text{-Der } A$.

One verifies immediately that for such an algebra A , $\rho\text{-Der } A$ is not only a ρ -Lie algebra but also a left A -module with the action of A on $\rho\text{-Der } A$ defined by

$$(fX)g = f(Xg) \quad f, g \in A, X \in \rho\text{-Der } A. \quad (5)$$

Let M be a G -graded left module over a ρ -commutative algebra A , with the usual properties, in particular $|f\psi| = |f| + |\psi|$ for $f \in A$, $\psi \in M$. Then M is also a right A -module with the right action on M defined by

$$\psi f = \rho(|\psi|, |f|)f\psi. \quad (6)$$

In fact M is a bimodule over A , i.e.

$$f(\psi g) = (f\psi)g \quad f, g \in A, \psi \in M. \quad (7)$$

Let M and N be two G -graded bimodules over the ρ -algebra A . Let $f: M \rightarrow N$ be an A -bimodule homomorphism of degree $\alpha \in G$ if $f: M_\beta \rightarrow N_{\alpha+\beta}$ such that $f(am) = \rho(\alpha, |a|)af(m)$ and $f(ma) = f(m)a$ for any $a \in A_{|a|}$ and $m \in M$. We denote by $\text{Hom}_\alpha(M, N)$ the space of A -bimodule homomorphisms of degree α and by $\text{Hom}_A^A(M, N) = \bigoplus_{\alpha \in G} \text{Hom}_\alpha(M, N)$ the space of all A -bimodule homomorphisms.

3. Differential forms on a ρ -algebra

A is a ρ -algebra as in the previous section. We denote by $\Omega_\alpha^1(A)$ the space generated by the elements: adb of G -degree $|a| + |b| = \alpha$ with the usual relations:

$$d(a+b) = d(a) + d(b), \quad d(ab) = d(a)b + ad(b) \quad \text{and} \quad d1 = 0,$$

where 1 is the unit of the algebra A .

If we denote by $\Omega^1(A) = \sum_{\alpha} \Omega_{\alpha}^1(A)$ then $\Omega^1(A)$ is an A -bimodule and satisfies the following theorem of universality.

Theorem 1. *For any A -bimodule M and for any derivation $X: A \rightarrow M$ of degree $|X|$ there is an A -bimodule homomorphism $f: \Omega^1(A) \rightarrow M$ of degree $|X|$ ($f \in \text{Hom}_{|X|}(\Omega^1(A), M)$) such that $X = f \circ d$. The homomorphism is uniquely determined and the corresponding $X \mapsto f$ establishes an isomorphism between $\rho\text{-Der}_{|X|}(A, M)$ and $\text{Hom}_{|X|}(\Omega^1(A), M)$.*

Proof. We define the map $f: \Omega^1(A) \rightarrow M$ by $f(adb) = \rho(|X|, |a|)aX(b)$ which transform the usual Leibniz rule for the operator d into the ρ -Leibniz rule for the derivation X . \square

Starting from the A -bimodule $\Omega^1(A)$ and the ρ -algebra $\Omega^0(A) = A$ we build up the algebra of differential forms over A .

This algebra will be a new $\bar{\rho}$ -algebra

$$\Omega^*(A) = \sum_{n \in \mathbb{N}, \alpha \in G} \Omega_{\alpha}^n(A)$$

graded by the group $\bar{G} = \mathbb{Z} \times G$ and generated by elements $a \in A_{|a|} = \Omega_{|a|}^0(A)$ of degree $(0, |a|)$ and their differentials $da \in \Omega_{|a|}^1(A)$ of degree $(1, |a|)$.

We will also require the universal derivation $d: A \rightarrow \Omega^1(A)$ which can be extended to a $\bar{\rho}$ -derivation of the algebra $\Omega^*(A)$ of degree $(1, 0)$ in such a way that $d^2 = 0$ and $\bar{\rho}|_{G \times G} = \rho$. Denote by $\omega \wedge \theta \in \Omega_{\alpha+\beta}^{n+m}(A)$ the product of forms $\omega \in \Omega_{\alpha}^n(A)$, $\theta \in \Omega_{\beta}^m(A)$ in the algebra $\Omega^*(A)$. Then

$$d(\omega \wedge \theta) = d\omega \wedge \theta + \bar{\rho}((1, 0), (n, \alpha))\omega \wedge d\theta,$$

and

$$d^2(\omega \wedge \theta) = \bar{\rho}((1, 0), (n+1, \alpha))d\omega \wedge d\theta + \bar{\rho}((1, 0), (n, \alpha))d\omega \wedge d\theta = 0. \quad (8)$$

Hence

$$\bar{\rho}((1, 0), (n+1, \alpha)) + \bar{\rho}((1, 0), (n, \alpha)) = 0. \quad (9)$$

From these relations it follows that

$$\bar{\rho}((1, 0), (n, \alpha)) = (-1)^n \varphi(\alpha),$$

where $\varphi: G \rightarrow U(k)$ is the group homomorphism $\varphi(\alpha) = \bar{\rho}((1, 0), (0, \alpha))$. From the properties of the cocycle ρ ,

$$\bar{\rho}((n, \alpha), (m, \beta)) = (-1)^{nm} \varphi^{-m}(\alpha) \varphi^n(\beta) \rho(\alpha, \beta) \quad (10)$$

for any $n, m \in \mathbb{Z}$ and $\alpha, \beta \in G$.

PROPOSITION 1

Let A be a ρ -algebra with the cocycle ρ . Then any cocycle $\bar{\rho}$ on the group \bar{G} with the conditions $\bar{\rho}|_{G \times G} = \rho$ and (9) are given by (10) for some homomorphism $\varphi: G \rightarrow U(k)$.

We will denote below $\Omega^*(A, \varphi)$ or simply $\Omega^*(A)$ the \overline{G} -graded algebra of forms with the cocycle $\overline{\rho}$ and the derivation $d = d_\varphi$ of degree $(1, 0)$.

Therefore for any ρ -algebra A , a group homomorphism $\varphi: G \rightarrow U(k)$ and an element $\alpha \in G$, we have the complex:

$$0 \rightarrow A_\alpha \xrightarrow{d_\varphi} \Omega_\alpha^1(A, \varphi) \xrightarrow{d_\varphi} \Omega_\alpha^2(A, \varphi) \xrightarrow{d_\varphi} \cdots \xrightarrow{d_\varphi} \Omega_\alpha^i(A, \varphi) \xrightarrow{d_\varphi} \Omega_\alpha^{i+1}(A, \varphi) \xrightarrow{d_\varphi} \cdots$$

The cohomology of this complex term $\Omega_\alpha^i(A, \varphi)$ is denoted by $H_\alpha^i(A, \varphi)$ and will be called as the *de Rham cohomology of the ρ -algebra* A .

PROPOSITION 2

Let $f: A \rightarrow B$ be a homomorphism of degree $\alpha \in G$ between the G -graded ρ -algebras. There is a natural homomorphism $\Omega(f): \Omega^*(A) \rightarrow \Omega^*(B)$ which in degree n is $\Omega(f): \Omega_\beta^n(A) \rightarrow \Omega_{\beta+(n+1)\alpha}^n(B)$ and has the G' -degree $(0, (n+1)\alpha)$ given by

$$\Omega^n(f)(a_0 da_1 \wedge \cdots \wedge da_n) = f(a_0)df(a_1) \wedge \cdots \wedge df(a_n). \quad (11)$$

4. Frölicher–Nijenhuis bracket of ρ -algebras

4.1 Derivations

Here we present the Frölicher–Nijenhuis calculus over the algebra of forms defined in the previous section.

Denote by $\text{Der}_{(k,\alpha)}(\Omega^*(A))$ the space of derivations of degree (k, α) i.e. an element $D \in \text{Der}_{(k,\alpha)}(\Omega^*(A))$ satisfies the relations:

- 1) D is linear,
- 2) the G' -degree of D is $|D| = (k, \alpha)$, and
- 3) $D(\omega \wedge \theta) = D\omega \wedge \theta + \overline{\rho}((k, \alpha), (n, \beta))\omega \wedge D\theta$ for any $\theta \in \Omega_\beta^n(A)$.

Theorem 2. *The space $\overline{\rho}\text{-Der } \Omega^*(A) = \bigoplus_{(k,\alpha) \in \overline{G}} \text{Der}_{(k,\alpha)}(\Omega^*(A))$ is a $\overline{\rho}$ -Lie algebra with the bracket $[D_1, D_2] = D_1 \circ D_2 - \overline{\rho}(|D_1|, |D_2|)D_2 \circ D_1$.*

4.2 Fields

Let us denote by $\mathcal{L}: \text{Hom}_A^A(\Omega^1(A), A) \rightarrow \rho\text{-Der}(A)$ the isomorphism from Theorem 1. We also denote by $\mathfrak{X}(A) := \text{Hom}_A^A(\Omega^1(A), A)$ the space of fields of the algebra A . Then $\mathcal{L}: \mathfrak{X}(A) \rightarrow \rho\text{-Der}(A; A)$ is an isomorphism of vector G -graded spaces. The space of ρ -derivations $\rho\text{-Der}(A)$ is a Lie ρ -algebra with the ρ -bracket $[\cdot, \cdot]$, and so we have an induced ρ -Lie bracket on $\mathfrak{X}(A)$ which is given by

$$\mathcal{L}([X, Y]) = [\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_X \mathcal{L}_Y - \rho(|X|, |Y|)\mathcal{L}_Y \mathcal{L}_X \quad (12)$$

and will be referred to as the ρ -Lie bracket of fields.

Lemma 1. *Each field $X \in \mathfrak{X}(A)$ is by definition an A -bimodule homomorphism $\Omega_1(A) \rightarrow A$ and it prolongs uniquely to a graded $\overline{\rho}$ -derivation $j(X) = j_X: \Omega(A) \rightarrow \Omega(A)$ of degree $(-1, |X|)$ by*

$$\begin{aligned} j_X(a) &= 0 \quad \text{for } a \in A = \Omega^0(A), \\ j_X(\omega) &= X(\omega) \quad \text{for } \omega \in \Omega^1(A) \end{aligned}$$

and

$$\begin{aligned} j_X(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k) &= \sum_{i=1}^{k-1} \bar{\rho} \left((-1, |X|), \left(i-1, \sum_{j=1}^{i-1} |\omega_j| \right) \right) \omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge X(\omega_i) \\ &\quad \times \omega_{i+1} \wedge \cdots \wedge \omega_k + \bar{\rho} \left((-1, |X|), \left(k-1, \sum_{j=1}^{k-1} |\omega_j| \right) \right) \\ &\quad \times \omega_1 \wedge \cdots \wedge \omega_{k-1} X(\omega_k) \end{aligned}$$

for any $\omega_i \in \Omega_{|\omega_i|}^1(A)$. The $\bar{\rho}$ -derivation j_X is called the contraction operator of the field X .

Proof. This is an easy computation. \square

With some abuse of notation we also write $\omega(X) = X(\omega) = j_X(\omega)$ for $\omega \in \Omega^1(A)$ and $X \in \mathfrak{X}(A) = \text{Hom}_A^1(\Omega^1(A), A)$.

4.2.1 Algebraic derivations: A $\bar{\rho}$ -derivation $D \in \text{Der}_{(k,\alpha)} \Omega(A)$ is called *algebraic* if $D|_{\Omega^0(A)} = 0$. Then $D(a\omega) = \bar{\rho}((k, \alpha), (0, |a|))aD(\omega)$ and $D(\omega a) = D(\omega)a$ for any $a \in A_{|a|}$ and $\omega \in \Omega(A)$. It results that D is an A -bimodule homomorphism. We denote by $\text{Hom}_\alpha(\Omega_l(A), \Omega_{k+l}(A))$ the space of A -bimodule homomorphisms from $\Omega_{(l,\alpha)}(A)$ to $\Omega_{(l+k,\alpha)}(A)$ of degree (k, α) . Then an algebraic derivation D of degree (k, α) is from $\text{Hom}_\alpha(\Omega_l(A), \Omega_{k+l}(A))$. We denote by $\bar{\rho}\text{-Der}_{(k,\alpha)}^{\text{alg}} \Omega^*(A)$ the space of all $\bar{\rho}$ -algebraic derivations of degree (k, α) from $\Omega^*(A)$. Since D is a $\bar{\rho}$ -derivation, D has the following expression on the product of 1-forms $\omega_i \in \Omega_{|\omega_i|}^1(A)$:

$$\begin{aligned} D(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k) &= \sum_{i=1}^k \bar{\rho} \left(|D|, \left(i-1, \sum_{j=1}^{i-1} |\omega_j| \right) \right) \\ &\quad \times \omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge D(\omega_i) \wedge \cdots \wedge \omega_k \end{aligned}$$

and the derivation D is uniquely determined by its restriction on $\Omega^1(A)$,

$$K := D|_{\Omega^1(A)} \in \text{Hom}_\alpha(\Omega_1(A), \Omega_{k+1}(A)). \quad (13)$$

We write $D = j(K) = j_K$ to express this dependence. Note that $j_K(\omega) = K(\omega)$ for $\omega \in \Omega_1(A)$. Next we will use the following notations:

$$\begin{aligned} \Omega_{(k,\alpha)}^1 &= \Omega_{(k,\alpha)}^1(A) := \text{Hom}_\alpha(\Omega_1(A), \Omega_k(A)), \\ \Omega_*^1 &= \Omega_*^1(A) = \bigoplus_{k \geq 0, \alpha \in G} \Omega_{(k,\alpha)}^1(A). \end{aligned}$$

Elements of the space $\Omega_{(k,\alpha)}^1$ will be called *field-valued (k, α) -forms*.

4.2.2 Nijenhuis bracket:

Theorem 3. The map $j: \Omega_{(k+1, \alpha)}^1(A) \rightarrow \bar{\rho}\text{-Der}_{(k, \alpha)}^{\text{alg}} \Omega^*(A)$, $K \mapsto j_K$ defined by

$$\begin{aligned} j_K(\omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_k) \\ = \sum_{i=1}^k \bar{\rho} \left((k+1, \alpha), \left(i-1, \sum_{j=1}^{i-1} |\omega_j| \right) \right) \\ \times \omega_1 \wedge \cdots \wedge \omega_{i-1} \wedge j_K(\omega_i) \wedge \cdots \wedge \omega_k \end{aligned}$$

is an isomorphism and satisfies the following properties:

- 1) $j_K: \Omega_{(n, \beta)}(A) \rightarrow \Omega_{(n+k, \alpha+\beta)}(A)$.
- 2) $j_K(\omega \wedge \theta) = j_K\omega \wedge \theta + \bar{\rho}((k, \alpha), (n, \beta))\omega \wedge j_K\theta$ for any $\theta \in \Omega_{\beta}^n(A)$.
- 3) $j_K(a) = 0$ and $j_K(\omega) = K(\omega)$ for any $\omega \in \Omega^1(A)$.

The module of $\bar{\rho}$ -algebraic derivations is obviously closed with respect to the $\bar{\rho}$ -commutator of derivations.

Therefore we get a $\bar{\rho}$ -Lie algebra structure on

$$\mathcal{N}_{ij}(A) = \bigoplus_{(k, \alpha) \in \bar{G}} \bar{\rho}\text{-Der}_{(k, \alpha)}^{\text{alg}} \Omega^*(A)$$

which is called the *Nijenhuis algebra* of the ρ -algebra A and its bracket is the $\bar{\rho}$ -Nijenhuis bracket.

By definition, the Nijenhuis bracket of the elements $K \in \text{Hom}_{\alpha}(\Omega^1(A), \Omega^{1+k}(A))$ and $L \in \text{Hom}_{\beta}(\Omega^1(A), \Omega^{1+l}(A))$ is given by the formula

$$[K, L]^{\Delta} = j_K \circ L - \bar{\rho}((k, \alpha), (l, \beta)) j_L \circ K$$

or

$$[K, L]^{\Delta}(\omega) = j_K(L(\omega)) - (-1)^{kl} \varphi^{-l}(\alpha) \varphi^k(\alpha) \rho(\alpha, \beta) j_L K(\omega) \quad (14)$$

for all $\omega \in \Omega^1(A)$.

4.2.3 The Frölicher–Nijenhuis bracket: The exterior derivative d is an element of $\bar{\rho}\text{-Der}_{(1,0)} \Omega^*(A)$. In the view of the formula $\mathcal{L}_X = [j_X, d]$ for fields X we define $K \in \Omega_{(k, \alpha)}^1(A)$ the Lie derivation $\mathcal{L}_K = \mathcal{L}(K) \in \bar{\rho}\text{-Der}_{(k, \alpha)} \Omega^*(A)$ by $\mathcal{L}_K := [j_K, d]$. Then the mapping $\mathcal{L}: \Omega_{*}^1 \rightarrow \bar{\rho}\text{-Der} \Omega(A)$ is injective by the universal property of $\Omega^1(A)$, since $\mathcal{L}_K(a) = j_K(da) = K(da)$ for $a \in A$.

Theorem 4. For any $\bar{\rho}$ -derivation $D \in \bar{\rho}\text{-Der}_{(k, \alpha)} \Omega^*(A)$, there are unique homomorphisms $\Omega_{(k, \alpha)}^1(A)$ and $L \in \Omega_{(k+1, \alpha)}^1(A)$ such that

$$D = \mathcal{L}_K + j_L. \quad (15)$$

We have $L = 0$ if and only if $[D, d] = 0$. D is algebraic if and only if $K = 0$.

Proof. The map $D|_A: a \mapsto D(a)$ is a ρ -derivation of degree α so $D|_A: A \rightarrow \Omega_{(k,\alpha)}(A)$ has the form $K \circ d$ for an unique $K \in \Omega_{(k,\alpha)}^1(A)$. The defining equation for K is $D(a) - j_K da = \mathcal{L}_K(a)$ for $a \in A$. Thus $D - \mathcal{L}_K$ is an algebraic derivation, so $D - \mathcal{L}_K = j_L$ for an unique $L \in \Omega_{(k+1,\alpha)}^1(A)$.

By the Jacobi identity, we have

$$0 = [j_K, [d, d]] = [[j_K, d], d] + \bar{\rho}((k, \alpha), (1, 0))[d, [j_K, d]]$$

so $2[\mathcal{L}_K, d] = 0$. It follows that $[D, d] = [j_L, d] = \mathcal{L}_L$ and using the injectivity of \mathcal{L} results that $L = 0$. \square

Let $K \in \Omega_{(k,\alpha)}^1(A)$ and $L \in \Omega_{(l,\beta)}^1(A)$. Definition of the $\bar{\rho}$ -Lie derivation results in $[[\mathcal{L}_K, \mathcal{L}_L], d] = 0$ and using the previous theorem results that is a unique element which is denoted by $[K, L] \in \Omega_{(k+l,\alpha+\beta)}^1(A)$ such that

$$[\mathcal{L}_K, \mathcal{L}_L] = \mathcal{L}_{[K,L]} \quad (16)$$

and this element will be denoted by the abstract Frölicher–Nijenhuis of K and L .

Theorem 5. *The space $\Omega_*^1(A) = \bigoplus_{(k,\alpha) \in \bar{G}} \Omega_{(k,\alpha)}^1(A)$ with the usual grading and the Frölicher–Nijenhuis is a \bar{G} -graded Lie algebra. $\mathcal{L}: (\Omega_*^1, [\cdot, \cdot]) \rightarrow \bar{\rho}\text{-Der } \Omega(A)$ is an injective homomorphism of \bar{G} -graded Lie algebras. For fields in $\text{Hom}_A^A(\Omega^1(A), A)$ the Frölicher–Nijenhuis coincides with the bracket defined in (12).*

4.3 Naturality of the Frölicher–Nijenhuis bracket

Let $f: A \rightarrow B$ be an homomorphism of degree 0 between the G -graded ρ -algebras A and B . Two forms $K \in \Omega_{(k,\alpha)}^1(A) = \text{Hom}_\alpha(\Omega_1(A), \Omega_k(A))$ and $K' \in \Omega_{(k,\alpha)}^1(B) = \text{Hom}_\alpha(\Omega_1(B), \Omega_k(B))$ are f -related or f -dependent if we have

$$K' \circ \Omega^1(f) = \Omega_k(f) \circ K: \Omega_\alpha^1(A) \rightarrow \Omega_\alpha^k(B)$$

where $\Omega_*(f): \Omega(A) \rightarrow \Omega(B)$ is the homomorphism from (11) induced by f .

Theorem 6.

- (1) *If K and K' are f -related as above then $j_{K'} \circ \Omega(f) = \Omega(f) \circ j_K: \Omega(A) \rightarrow \Omega(B)$.*
- (2) *If $j_K \circ \Omega(f)|_{d(A)} = \Omega(f) \circ j_K|_{d(A)}$, then K and K' are f -related, where $d(A) \subset \Omega^1(A)$ is the space of exact 1-forms.*
- (3) *If K_j and K'_j are f -related for $j = 1, 2$ then $j_{K_1} \circ K_2$ and $j_{K'_1} \circ K'_2$ are f -related and also $[K_1, K_2]^\Delta, [K'_1, K'_2]^\Delta$ are f -related.*
- (4) *If K and K' are f -related then $\mathcal{L}_{K'} \circ \Omega(f) = \Omega(f) \circ \mathcal{L}_K: \Omega(A) \rightarrow \Omega(B)$.*
- (5) *If $\mathcal{L}_{K'} \circ \Omega(f)|_{\Omega_0(A)} = \Omega(f) \circ \mathcal{L}_K|_{\Omega_0(A)}$ then K and K' are f -related.*
- (6) *If K_j and K'_j are f -related for $j = 1, 2$ then their Frölicher–Nijenhuis brackets $[K_1, K_2]$ and $[K'_1, K'_2]$ are also f -related.*

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