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Localization of tight closure in two-dimensional rings

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Abstract. It is shown that tight closure commutes with localization in any two-dimensional ring R of prime characteristic if either R is a Nagata ring or R possesses a weak test element. Moreover, it is proved that tight closure commutes with localization at height one prime ideals in any ring of prime characteristic.

Keywords. Tight closure; localization; test elements.

1. Introduction

Throughout this paper, R is a commutative Noetherian ring (with identity) of prime characteristic p. The theory of tight closure was introduced by Hochster and Huneke [2]. There are many applications for this notion in both commutative algebra and algebraic geometry. However, there are many basic open questions concerning tight closure. One of the essential questions is whether tight closure commutes with localization. For an expository account on tight closure, we refer the reader to [3] or [8].

In the sequel, R^o denotes the set of elements of R which are not contained in any minimal prime ideal of R. We use the letter q for non-negative powers p^e of p. Let I be an ideal of R and $I^{[q]}$ the ideal generated by q-th powers of elements of I. Then I^*, *tight closure* of I is the set of all elements $x \in R$ for which there exists $c \in R^o$ such that $cx^q \in I^{[q]}$ for all $q \gg 0$. Also, for a non-negative power q' of p an element $c \in R^o$ is called q'-weak test element, if for any ideal I of R and any element $x \in I^*$, we have $cx^q \in I^{[q]}$ for all $q \geq q'$.

We say that tight closure commutes with localization for the ideal I, if for any multiplicative system W in R, $I^*R_W = (IR_W)^*$. It is conjectured that tight closure commutes localization in a general situation. There are some related conjectures that a positive answer to each of them will yield solution to the localization problem. For example, it is an open question that for any ideal I of a domain R, $I^* = IR^+ \cap R$, where R^+ denote the integral closure of R in an algebraic closure of its fraction field. A positive solution to this problem implies a solution to the localization problem (see [3]). Also, if the Frobenius powers of the proper ideal I of R have *linear growth of primary decompositions* [7], then tight closure of I commutes with localization at a multiplicative system consisting of the powers of a single element of R.

Tight closure commutes with localization in several important special cases. For example, it is known that tight closure commutes with localizations on principal ideals and also on ideals generated by regular sequences (see for e.g. [3]). We refer the reader to [8] for a survey of results on localization problem. Also, Smith [6] proved that tight closure commutes with localization in affine rings which are quotients of a polynomial ring over a field, by a binomial ideal. She proved this result, by showing that if for any minimal prime ideals $\mathfrak{p}$ of the ring R, the quotient $R/\mathfrak{p}$ has a finite extension domain in which tight closure commutes with localization, then tight closure commutes with localization in R itself. This fact is essential in the proof of our main result.

Theorem 1.1. *Let R be a Noetherian ring of prime characteristic. Then tight closure commutes with localization in R in the following cases*:

(i) $\dim R < 2$.
(ii) *R is a two-dimensional local ring.*
(iii) $\dim R = 2$ *and either R is a Nagata ring or R possesses a q'-weak test element.*

Note that, by ([4], Theorem 78 and Definition 34.A), any excellent ring is Nagata. It seems that the solution of localization problem in dimension two has been basically known (at least in the case of excellent rings) by most experts who have attacked the problem. However, since the solution did not appear in any article, according to our knowledge, the main achievement of this paper is that it fills a gap in the literature.

2. The proof

To prove the theorem, we proceed through the following lemmas, some of which may be of independent interest.

Lemma 2.1. *Let I be an ideal of R and W a multiplicative system in R. Suppose there exists $w \in W$ such that $w^q(I^{[q]}R_W \cap R) \subseteq I^{[q]}$ for all $q \gg 0$. Then $I^*R_W = (IR_W)^*$.*

Proof. Clearly $I^*R_W \subseteq (IR_W)^*$. Now, let $x/1 \in (IR_W)^*$. Then there is $c \in (R_W)^\circ$ such that $c(x/1)^q \in (IR_W)^{[q]}$ for all $q \gg 0$. It is easy to see that in fact we can chose c in R°. Hence $cx^q \in I^{[q]}R_W \cap R$ and so $w^q cx^q = c(wx)^q \in I^{[q]}$ for all $q \gg 0$. Thus $wx \in I^*$, and therefore $x \in I^*R_W$, as required. ■

By ([2], Proposition 4.14), for any maximal ideal and any $\mathfrak{m}$-primary ideal I of R, $I^*R_\mathfrak{m} = (IR_\mathfrak{m})^*$. The following extend this fact.

Lemma 2.2. *Let I be an ideal of R. Let $\mathfrak{p}$ be a maximal ideal of R which is minimal over I. Then $I^*R_\mathfrak{p} = (IR_\mathfrak{p})^*$. In particular, if R/I is an Artinian ring, then tight closure commutes with localization for I.*

Proof. Since $IR_\mathfrak{p}$ is $\mathfrak{p}R_\mathfrak{p}$-primary, it follows that $IR_\mathfrak{p}$ contains some power of $\mathfrak{p}R_\mathfrak{p}$. Suppose $\mathfrak{p}^kR_\mathfrak{p} \subseteq IR_\mathfrak{p}$. Then $s\mathfrak{p}^k \subseteq I$ for some $s \in R\backslash\mathfrak{p}$. Assume that I is generated by t elements and let $w = s^t$ and $n = tk$. Then w^q multiplies the nq-th power of $\mathfrak{p}$ into $I^{[q]}$ for all q. By Lemma 2.1, it suffices to show that

$$w^q(I^{[q]}R_\mathfrak{p} \cap R) \subseteq I^{[q]},$$

for all q. To this end, let $x \in (I^{[q]}R_{\mathfrak{p}} \cap R)$ for some q. Then there exists $s_q \in R \backslash \mathfrak{p}$ such that $s_q x \in I^{[q]}$. Since $Rs_q + \mathfrak{p}^{nq} = R$, there are α_q in R and β_q in $\mathfrak{p}^{nq}$ such that $\alpha_q s_q + \beta_q = 1$. Then

$$w^q x = \alpha_q s_q w^q x + \beta_q w^q x \in I^{[q]}.$$

Therefore $w^q(I^{[q]}R_{\mathfrak{p}} \cap R) \subseteq I^{[q]}$.

For the second assertion, first note that, by ([1], Lemma 3.5(a)), it is enough to treat only the localization at prime ideals of R. Also, note that for any prime ideal I which does not contain I, we have $I^* R_{\mathfrak{p}} = (IR_{\mathfrak{p}})^* = R_{\mathfrak{p}}$. Hence the claim follows by the first part of the lemma. ■

It follows from Lemma 2.2 that tight closure commutes with localization in any domain of dimension less than or equal to one. Thus by using ([6], Lemma 1), we deduce the following result.

COROLLARY 2.3

Assume R is a ring with $\dim R \leq 1$. *Then tight closure commutes with localization in R.*

Lemma 2.4. *Let I be an ideal of the integral domain R. Then $I^* R_{\mathfrak{p}} = (IR_{\mathfrak{p}})^*$ for any height one prime ideal $\mathfrak{p}$ of R.*

Proof. Let $\mathfrak{p}$ be a height one prime ideal of R. Let R' be the integral closure of R in its field of fractions. It is known that the normalization of any one-dimensional Noetherian domain is Noetherian. Since $(R')_{\mathfrak{p}}$ is the integral closure of the domain $R_{\mathfrak{p}}$ in its field of fractions, it follows that $(R')_{\mathfrak{p}}$ is a Noetherian normal domain of dimension one. Hence $(R')_{\mathfrak{p}}$ is a regular ring. This implies that every ideal of $(R')_{\mathfrak{p}}$ is tightly closed.

Now, let $x/1 \in (IR_{\mathfrak{p}})^*$. Then there is a non-zero element c in R such that $c(x/1)^q \in (IR_{\mathfrak{p}})^{[q]}$ for all q. But for each q, $(IR_{\mathfrak{p}})^{[q]} \subseteq (I(R')_{\mathfrak{p}})^{[q]}$, and so $x/1 \in (I(R')_{\mathfrak{p}})^* = I(R')_{\mathfrak{p}}$. Hence there is $s \in R \backslash \mathfrak{p}$ such that $sx \in IR'$. This implies that $x/1 \in I^* R_{\mathfrak{p}}$, because $IR' \cap R \subseteq I^*$, by ([3], p. 15). ■

Lemma 2.4 has the following interesting conclusion.

COROLLARY 2.5

Let R be a ring and I an ideal of R. For any height one prime ideal $\mathfrak{p}$ of R, we have $I^* R_{\mathfrak{p}} = (IR_{\mathfrak{p}})^*$.

Proof. Let $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ be the set of minimal prime ideals of R, which are contained in $\mathfrak{p}$. Fix $1 \leq i \leq n$. By Lemma 2.4, we have

$$(I + \mathfrak{p}_i/\mathfrak{p}_i)^*(R/\mathfrak{p}_i)_{\mathfrak{p}/\mathfrak{p}_i} = ((I + \mathfrak{p}_i/\mathfrak{p}_i)(R/\mathfrak{p}_i)_{\mathfrak{p}/\mathfrak{p}_i})^*.$$

Now, by following the argument of ([6], Lemma 1), it turns out that $I^* R_{\mathfrak{p}} = (IR_{\mathfrak{p}})^*$. ■

The following is the only technical tool remaining in order to prove Theorem 1.1. Some tricks in the argument of the following result is very close to those which are used in ([9], Proposition 1.2).

Lemma 2.6. *Let R be a two-dimensional normal ring. Let I be a height one ideal of R and $\mathfrak{p}$ a height two prime ideal of R containing I. Then $I^* R_{\mathfrak{p}} = (IR_{\mathfrak{p}})^*$.*

Proof. If $\mathfrak{p}$ is minimal over I, then the assertion follows by Lemma 2.2. Thus in the sequel, we assume that $\mathfrak{p}$ is not minimal over I. Suppose that $\{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_n\}$ is the set of minimal prime ideals of I, which are contained in $\mathfrak{p}$. Let $I^{[q]} = \cap_{i=1}^{t_q} Q_{iq}$ be a minimal primary decomposition of the ideal $I^{[q]}$, with $\operatorname{Rad}(Q_{iq}) = \mathfrak{p}_{iq}$. Then

$$I^{[q]}R_{\mathfrak{p}} \cap R = \bigcap_{\mathfrak{p}_{iq}\subseteq \mathfrak{p}} Q_{iq} \subseteq \bigcap_{i=1}^{n} (I^{[q]}R_{\mathfrak{p}_i} \cap R).$$

Since R is reduced, it follows that each associated prime ideal of R is minimal. Hence, by prime avoidance theorem, we can deduce that I can be generated by regular elements in R. Because each $R_{\mathfrak{p}_i}$ is a DVR, there are $a_1, a_2, \ldots, a_n$ in I such that each a_i is regular in R and $IR_{\mathfrak{p}_i} = a_i R_{\mathfrak{p}_i}$. It follows that $\mathfrak{p}_i$ is a minimal over $a_i R$. Let $\mathfrak{p}_{i2}, \mathfrak{p}_{i3} \ldots, \mathfrak{p}_{in_i}$ be the other associated prime ideals of the ideal $a_i R$. Take $z_i \in \bigcap_{j=2}^{n_i} \mathfrak{p}_{ij} \backslash \mathfrak{p}_i$. For an ideal J of R, we denote $\bigcup_{k\in\mathbb{N}} J : z_i^k$, by $J : \langle z_i \rangle$. Since a_i is a regular element of R, it follows that for each q, $\operatorname{Ass}_R(R/a_i^q R) = \operatorname{Ass}_R(R/a_i R)$, and so one can easily check that

$$a_i^q R_{\mathfrak{p}_i} \cap R = a_i^q R : \langle z_i \rangle \,.$$

By ([3], Exercise 4.2), there exists an integer c_i such that $a_i^q R : \langle z_i \rangle = a_i^q R : z_i^{c_i q}$ for all q. Therefore

$$I^{[q]}R_{\mathfrak{p}} \cap R \subseteq \bigcap_{i=1}^{n} (a_i^q R : z_i^{c_i q}),$$

for all q. Because the ideal $\sum_{i=1}^{n} z_i^{c_i} R$ is not contained in the union of $\mathfrak{p}_i$'s, $i = 1, 2, \ldots, n$, there are elements $r_1, r_2, \ldots, r_n$ in R such that

$$\alpha = r_1 z_1^{c_1} + r_2 z_2^{c_2} + \cdots + r_n z_n^{c_n} \notin \bigcup_{i=1}^{n} \mathfrak{p}_i .$$

Let $x \in I^{[q]}R_{\mathfrak{p}} \cap R$ for some q. Then for each $i = 1, 2, \ldots, n$, we have $z_i^{c_i q} x \in a_i^q R \subseteq I^{[q]}$. Thus $\alpha^q x \in I^{[q]}$. If $\alpha \notin \mathfrak{p}$, then by Lemma 2.1, it turns out that $I^* R_{\mathfrak{p}} = (IR_{\mathfrak{p}})^*$.

Now, assume that $\alpha \in \mathfrak{p}$. Since $\alpha \notin \cup_{i=1}^{n} \mathfrak{p}_i$, it turns out that $\mathfrak{p}$ is minimal over $\alpha R + I$. Using the similar argument as in the proof of Lemma 2.2, we can deduce that there are $s \in R \backslash \mathfrak{p}$ and $l \in \mathbb{N}$ such that $s^q \mathfrak{p}^{lq} \subseteq \alpha^q R + I^{[q]}$ for all q. Since $x \in I^{[q]}R_{\mathfrak{p}} \cap R$, there is $w_q \in R \backslash \mathfrak{p}$ such that $w_q x \in I^{[q]}$. Because $\mathfrak{p}^{lq} + R w_q = R$, there are $\beta_q \in \mathfrak{p}^{lq}$ and $\gamma_q \in R$ such that $1 = \beta_q + \gamma_q w_q$, and so

$$s^q x = \beta_q s^q x + s^q \gamma_q w_q x \in I^{[q]}.$$

Hence $s^q (I^{[q]}R_{\mathfrak{p}} \cap R) \subseteq I^{[q]}$ for all q and so the claim follows by using Lemma 2.1 again. ■

Recall that an element $c \in R^\circ$ is called q'-weak test element, if for any ideal I of R and any element $x \in I^*$, we have $cx^q \in I^{[q]}$ for all $q \geq q'$. The following improves ([8], Proposition 6.5).

Lemma 2.7. *Let $R \subseteq S$ be an integral extension of Noetherian domains. Suppose that R possesses a q'-weak test element. Then for any ideal I of R, $(IS)^* \cap R \subseteq I^*$.*

Proof. Let $x \in (IS)^* \cap R$. Then there is a non-zero element d of S such that $dx^q \in (IS)^{[q]}$ for all $q \gg 0$. Since S is an integral over R, there are $a_0, a_1, \ldots, a_n \in R$ such that

$$d^n + a_1 d^{n-1} + \cdots + a_0 = 0.$$

We may and do assume that a_0 is non-zero. Then $a_0 x^q \in I^{[q]}S \cap R$ for all $q \gg 0$. Therefore, by ([3], p. 15), $a_0 x^q \in (I^{[q]})^*$ for all $q \gg 0$. Now it follows from ([2], Lemma 8.16), that $x \in I^*$. ■

Now, we are ready to conclude Theorem 1.1.

Proof of Theorem 1.1. If R has a q'-weak test element, then it follows by ([1], Lemma 2.10(a)), that for any minimal prime ideal $\mathfrak{p}$, the domain $R/\mathfrak{p}$ has also a q'-weak test element. Thus, in view of ([6], Lemma 1), we can assume that R is a domain. Also, by ([1], Lemma 3.5(a)), it suffices to consider only localization at prime ideals.

The case (i) holds by Corollary 2.3 and the case (ii) follows from Lemma 2.4.

Now suppose that either R is a Nagata ring or possesses a q'-weak test element. By ([5], Theorem 33.12), the integral closure of a Noetherian two-dimensional domain in its field of fractions is Noetherian. Let R' denote the integral closure of R in its field of fractions. Let I be a non-zero ideal of R. If R possesses a q'-weak test element, then $(IR')^* \cap R \subseteq I^*$, by Lemma 2.7. If R is a Nagata ring, then R' is a finite extension of R, and so $(IR')^* \cap R \subseteq I^*$, by ([3], Theorem 1.7). Therefore in both case, by adopting the argument of ([6], Lemma 2), we can deduce that tight closure commutes with localization, if the same happens in R'. Hence, in the sequel, we suppose that R is a Noetherian normal domain of dimension two. If $\mathrm{ht}(I) = 2$, then by Lemma 2.2, tight closure commutes with localization for I. Hence we may assume that $\mathrm{ht}(I) = 1$. Let $\mathfrak{p}$ be a prime ideal of R. If I is not contained in $\mathfrak{p}$, then $I^* R_{\mathfrak{p}} = (IR_{\mathfrak{p}})^* = R_{\mathfrak{p}}$. Therefore the proof is complete by Lemmas 2.4 and 2.6. ■

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