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Moduli for decorated tuples of sheaves and representation spaces for quivers

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Abstract. We extend the scope of a former paper to vector bundle problems involving more than one vector bundle. As the main application, we obtain the solution of the well-known moduli problems of vector bundles associated with general quivers.

Keywords. Quiver; representation; decoration; split sheaf; moduli space; Hitchin map.

1. Introduction

If we are given a projective manifold X, a reductive linear algebraic group G, and a representation $\rho\colon\ G \longrightarrow \mathrm{GL}(V)$, we may associate to every principal G-bundle $\mathcal{P}$ over X a vector bundle $\mathcal{P}(V)$ with fibre V. The objects of interest are pairs $(\mathcal{P}, \tau)$ where $\mathcal{P}$ is an algebraic principal G-bundle and $\tau\colon\ X \longrightarrow \mathcal{P}(V)$ is a section of the associated vector bundle. Motivated, e.g., by the quest for differentiable invariants of 4-manifolds, one associates to the data of G, ρ, and a fixed differentiable principal G-bundle P certain vortex equations. By the so-called Kobayashi–Hitchin correspondence, the solutions of these vortex equations have an interpretation as pairs $(\mathcal{P}, \tau)$ as above, satisfying certain stability conditions which may be understood in purely algebraic terms. Here, $\mathcal{P}$ is an algebraic structure on the bundle P. The Kobayashi–Hitchin correspondence first arose in the context of vector bundles, i.e., when no representation is given (see [14]), and was then considered in various special cases before Banfield [2] gave it a unified treatise. It was afterwards widely extended to more general contexts [1, 3, 15, 19, 21]. In order to apply the machinery of algebraic geometry to the gauge theoretic moduli space for the pairs $(\mathcal{P}, \tau)$ with $\mathcal{P}$ of topological type P satisfying the stability conditions, one must equip it with an algebraic structure and find a suitable compactification. One is therefore led to a purely algebro–geometric moduli problem. Another motivation to study this kind of moduli problems that comes from within algebraic geometry is the fact that many interesting classification problems for projective manifolds may be encoded by data of the above type. We will give an example below and refer the reader to [24] for further discussions. A first sufficiently general solution of this kind of moduli problems was given by the author in the case that X is a projective curve, $G = \mathrm{GL}(r)$, and ρ is a homogeneous representation [24]. Later, Gómez and Sols [8] established this case on higher dimensional base manifolds X.

The aim of the present paper is to extend these results to the case when the reductive group is a product of general linear groups, $G = \mathrm{GL}(r_1) \times \cdots \times \mathrm{GL}(r_l)$, ρ belongs to the class of homogeneous representations (which comprises all irreducible representations), and X is a base manifold of arbitrary dimension. However, we will not repeat the detailed constructions of [24], but rather introduce several non-trivial 'tricks' which will enable us to adapt the proofs in that paper to the more general situation studied here.

A nice example of a classification problem which can be formulated in our context is provided by the work of Casnati and Ekedahl [4]. Let X be a projective manifold. Then, any integral Gorenstein cover $\pi\colon\ Y \longrightarrow X$ of degree 4 can be obtained from locally free $\mathcal{O}_X$-modules $\mathcal{E}$ and $\mathcal{F}$ of rank 3 and 2, respectively, such that $\det(\mathcal{E}) \cong \det(\mathcal{F})$, and a section $s \in H^0(X, \mathcal{F}^\vee \otimes S^2\mathcal{E}) = \mathrm{Hom}\,(\mathcal{F}, S^2\mathcal{E})$. The construction is as follows: If $\pi\colon\ \mathbb{P}(\mathcal{E}^\vee) := \mathrm{Proj}\,(S^*\mathcal{E}) \longrightarrow X$ is the projection, then

$$\begin{aligned}\mathrm{Hom}\,(\pi^*(\det\,\mathcal{E})(-4), \pi^*(\mathcal{F})(-2)) &= \mathrm{Hom}\,(\pi^*(\mathcal{F}^\vee \otimes \det(\mathcal{E})), \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}(2))\\ &= \mathrm{Hom}\,(\mathcal{F}^\vee \otimes \det(\mathcal{E}), S^2\mathcal{E})\\ &= \mathrm{Hom}\,(\mathcal{F}, S^2\mathcal{E}).\end{aligned}$$

Here, $\mathcal{F}^\vee \otimes \det(\mathcal{E}) \cong \mathcal{F}^\vee \otimes \det(\mathcal{F}) \cong \mathcal{F}$, because $\mathcal{F}$ has rank 2. Thus, any section $s \in H^0(X, \mathcal{F}^\vee \otimes S^2\mathcal{E})$ yields an exact sequence

$$\begin{aligned}0 \longrightarrow \pi^* \det(\mathcal{E})(-4) \stackrel{s}{\longrightarrow} \pi^*(\mathcal{F})(-2) \longrightarrow \mathcal{O}_{\mathbb{P}(\mathcal{E}^\vee)}\\ \longrightarrow \mathcal{O}_Y \longrightarrow 0.\end{aligned}$$

Hence, the moduli problem for degree 4 covers of X is included in the moduli problem associated with the group $\mathrm{GL}(3) \times \mathrm{GL}(2)$ and its representation on $\mathrm{Hom}\,(\mathbb{C}^2, S^2\mathbb{C}^3)$. Similarly, degree five covers $\rho\colon\ Y \longrightarrow X$ are determined by locally free sheaves $\mathcal{E}$ and $\mathcal{F}$ of rank 5 and 4, respectively, and a homomorphism $\varphi\colon\ \mathcal{E} \longrightarrow \bigwedge^2 \mathcal{F} \otimes \det(\mathcal{E})$ [5].

Another interesting moduli problem which we will treat with our methods comes from the representation theory of finite dimensional algebras (see [12] and [22] for introductions to this topic). Let $Q = (V, A, t, h)$ be a quiver with vertex set $V = \{v_1, \ldots, v_t\}$ and $\underline{\mathcal{G}} = (\mathcal{G}_a, a \in A)$ a collection of coherent $\mathcal{O}_X$-modules on the projective manifold X. This defines a *twisted path algebra* $\mathcal{B} = \mathcal{B}(Q, \underline{\mathcal{G}})$ (see [1] and [7]). Modules over $\mathcal{B}$ can now be described by *representations* of Q, i.e., tuples $(\mathcal{E}_v, v \in V; f_a, a \in A)$ composed of $\mathcal{O}_X$-modules $\mathcal{E}_v$, $v \in V$, and twisted homomorphisms $f_a\colon\ \mathcal{G}_a \otimes \mathcal{E}_{t(a)} \longrightarrow \mathcal{E}_{h(a)}$, $a \in A$. Numerous famous special cases of this construction have been studied in the literature, such as the Higgs bundles. Recent research has focussed on more general aspects of this theory. Gothen and King [7] have developed the homological algebra of these representations and Álvarez–Cónsul and García-Prada [1] formulated a semistability concept for the representations of Q and proved a Kobayashi–Hitchin correspondence. The semistability concept depends on additional parameters $\underline{\sigma} = (\sigma_v \in \mathbb{Z}_{>0}, v \in V)$ and $\underline{\chi} = (\chi_v \in \mathbb{Q}[x], v \in V)$ where the polynomials χ_v have degree at most $\dim X - 1$, $v \in V$, and an ample line bundle $\mathcal{O}_X(1)$ on X. For any coherent sheaf $\mathcal{A}$ on X, the Hilbert polynomial with respect to $\mathcal{O}_X(1)$ is denoted by $P(\mathcal{A})$. We set

$$P_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v \in V) := \sum_{v\in V}(\sigma_v P(\mathcal{E}_v) - \chi_v \mathrm{rk}\,\mathcal{E}_v)$$

and

$$\mathrm{rk}_{\underline{\sigma}}(\mathcal{E}_v, v \in V) := \sum_{v\in V} \sigma_v \mathrm{rk}\,\mathcal{E}_v.$$

A representation $(\mathcal{E}_v, v \in V; f_a, a \in A)$ is then called *(semi)stable*, if (a) the sheaves $\mathcal{E}_v$, $v \in V$, are torsion free and (b) for any collection of saturated subsheaves $\mathcal{F}_v \subset \mathcal{E}_v$, $v \in V$, (i.e., $\mathcal{E}_v/\mathcal{F}_v$ is again torsion free, $v \in V$) not all trivial and not all equal to $\mathcal{E}_v$, such that $f_a(\mathcal{G}_a \otimes \mathcal{F}_{t(a)}) \subset \mathcal{F}_{h(a)}$ for all arrows a, one has

$$\frac{P_{\underline{\sigma},\underline{\chi}}(\mathcal{F}_v, v \in V)}{\mathrm{rk}_{\underline{\sigma}}(\mathcal{F}_v, v \in V)} \quad (\preceq) \quad \frac{P_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v \in V)}{\mathrm{rk}_{\underline{\sigma}}(\mathcal{E}_v, v \in V)}.$$

The notation $(\preceq)$ means that $\prec$ is used for defining 'stable' and $\preceq$ for defining 'semistable', and $\prec$ and $\preceq$ refer to the lexicographic ordering of polynomials. Finally, $(\mathcal{E}_v, v \in V; f_a, a \in A)$ is called *polystable*, if it is a direct sum of stable representations $(\mathcal{E}^i_v, v \in V; f^i_a, a \in A)$, $i = 1, \dots, s$, with

$$\frac{P_{\underline{\sigma},\underline{\chi}}(\mathcal{E}^i_v, v \in V)}{\mathrm{rk}_{\underline{\sigma}}(\mathcal{E}^i_v, v \in V)} = \frac{P_{\underline{\sigma},\underline{\chi}}(\mathcal{E}^j_v, v \in V)}{\mathrm{rk}_{\underline{\sigma}}(\mathcal{E}^j_v, v \in V)}, \quad \text{for all } i, j = 1, \dots, s.$$

As one of the interesting and important applications of the main result of this paper, we will prove the following theorem.

Theorem. *Fix Hilbert polynomials $\underline{P} = (P_v, v \in V)$, the sheaves $\underline{\mathcal{G}}$, as well as the parameters $\underline{\sigma}$, and $\underline{\chi}$ with $\chi_v = \eta_v \cdot \delta$ for some polynomial $\delta \in \mathbb{Q}[x]$ and rational numbers η_v, $v \in V$.*

i) *There exists a quasi-projective moduli space $\mathcal{D} := \mathcal{D}(Q)^{(\underline{\sigma},\underline{\chi})-\mathrm{ss}}_{\underline{P},\underline{\mathcal{G}}}$ for polystable representations $(\mathcal{E}_v, v \in V; f_a, a \in A)$ with $P(\mathcal{E}_v) = P_v$, $v \in V$. The points corresponding to stable representations form an open subset $\mathcal{D}^{\mathrm{s}}$.*
ii) *There are a vector space $\mathbb{D}$ and a projective morphism $H\colon \mathcal{D} \longrightarrow \mathbb{D}$, the* generalized Hitchin map.

Remark. Álvarez-Cónsul and García-Prada define semistability with respect to parameters $\underline{\sigma} = (\sigma_v \in \mathbb{Q}_{>0}, v \in V)$ and $\underline{\tau} = (\tau_v \in \mathbb{Q}, v \in V)$. To be precise, for a representation $(\mathcal{E}_v, v \in V; f_a, a \in A)$, they set

$$\deg_{\underline{\sigma},\underline{\tau}}(\mathcal{E}_v, v \in V) := \sum_{v \in V} (\sigma_v \deg(\mathcal{E}_v) - \tau_v \mathrm{rk}(\mathcal{E}_v))$$

$$\mu_{\underline{\sigma},\underline{\tau}}(\mathcal{E}_v, v \in V) := \frac{\deg_{\underline{\sigma},\underline{\tau}}(\mathcal{E}_v, v \in V)}{\mathrm{rk}_{\underline{\sigma}}(\mathcal{E}_v, v \in V)}.$$

Álvarez-Cónsul and García-Prada say that $(\mathcal{E}_v, v \in V; f_a, a \in A)$ is *(semi)stable*, if (a) the sheaves $\mathcal{E}_v$, $v \in V$, are torsion free and (b) for any collection of saturated subsheaves $\mathcal{F}_v \subset \mathcal{E}_v$, $v \in V$, such that $f_a(\mathcal{G}_a \otimes \mathcal{F}_{t(a)}) \subset \mathcal{F}_{h(a)}$ for all arrows a, one has

$$\mu_{\underline{\sigma},\underline{\tau}}(\mathcal{F}_v, v \in V) \ (\leq)\ \mu_{\underline{\sigma},\underline{\tau}}(\mathcal{E}_v, v \in V).$$

Since multiplying all the parameters with a common positive factor does not alter the semistability condition, we may assume that the σ_v, $v \in V$, be positive integers. We may choose $\delta := x^{\dim X-1}$ and $\chi_v := \tau_v \cdot \delta$, $v \in V$. Then, $\deg_{\underline{\sigma},\underline{\tau}}(\mathcal{E}_v, v \in V)$ is essentially given as the coefficient of $x^{\dim X-1}/(\dim X - 1)!$ in $P_{\underline{\sigma},\underline{\chi}}$ (see [10], Definition 1.2.11,

p. 13, for the precise statement). In particular, we have the following implications: $(\mathcal{E}_v, v \in V;\ f_a, a \in A)$ stable in the sense Álvarez-Cónsul and García-Prada $\Longrightarrow$ stable in our sense $\Longrightarrow$ semistable in our sense $\Longrightarrow$ semistable in the sense Álvarez-Cónsul and García-Prada.

Therefore, the above theorem gives a quasi-projective moduli space for the stable objects (in either sense) together with a Gieseker-type compactification.

In future, we hope to extend the techniques introduced in this paper to treat the case of other reductive groups. This provides another motivation for studying the more general and abstract moduli problems introduced here.

2. Notation

X will be a fixed projective manifold over the complex numbers, and $\mathcal{O}_X(1)$ a fixed ample line bundle on X. For any coherent sheaf $\mathcal{E}$, $\deg \mathcal{E}$ is the degree of $\mathcal{E}$ with respect to $\mathcal{O}_X(1)$, and $P(\mathcal{E})$ with $P(\mathcal{E})(l) := \chi(\mathcal{E}(l))$, for all $l \in \mathbb{N}$, is the Hilbert polynomial of $\mathcal{E}$ with respect to $\mathcal{O}_X(1)$. In order to avoid excessive occurrences of the symbol '$^\vee$', we define $\mathbb{P}(\mathcal{V})$ as the projective bundle of lines in the fibres of the vector bundle $\mathcal{V}$. For any scheme S, h_S denotes its functor of points $T \longrightarrow \mathrm{Mor}(T, S)$.

In the appendix, we have stated two auxiliary results which will be used on several occasions.

3. Background, formal set-up and statement of the main results

3.1 *Homogeneous representations*

Let $V = \{v_1, \dots, v_l\}$ be a finite index set, $\underline{r} := (r_v, v \in V)$ a tuple of positive integers, and define

$$\mathrm{GL}(V, \underline{r}) := \mathop{\times}_{v \in V} \mathrm{GL}(\mathbb{C}^{r_v}).$$

A (finite dimensional, rational) representation $\rho\colon\ \mathrm{GL}(V, \underline{r}) \longrightarrow \mathrm{GL}(A)$ is said to be *homogeneous* (*of degree* α), if there is an integer α, such that

$$\rho(z, \dots, z) = z^\alpha \cdot \mathrm{id}_A, \quad \text{for all } z \in \mathbb{C}^*.$$

Example 3.1.1. Every irreducible representation is homogeneous. For any tuple $\underline{\sigma} = (\sigma_1, \dots, \sigma_l)$ of positive integers and $a, b, c \in \mathbb{Z}_{\geq 0}$, we define

$$W(\underline{\sigma}, \underline{r}) := \mathbb{C}^{\sum_{i=1}^l \sigma_i r_i}$$

and the $\mathrm{GL}(V, \underline{r})$-module

$$W(\underline{\sigma}, \underline{r})_{a,b,c} := (W(\underline{\sigma}, \underline{r})^{\otimes a})^{\oplus b} \otimes \left(\bigwedge^{\sum_{i=1}^l \sigma_i r_i} W(\underline{\sigma}, \underline{r}) \right)^{\otimes -c}.$$

The corresponding representation

$$\varrho_{a,b,c}\colon \mathrm{GL}(V, \underline{r}) \longrightarrow \mathrm{GL}(W(\underline{\sigma}, \underline{r})_{a,b,c})$$

is homogeneous.

PROPOSITION 3.1.2

Let $\underline{\sigma} = (\sigma_1, \ldots, \sigma_l)$ *be a tuple of positive integers and* $\rho\colon \mathrm{GL}(V, \underline{r}) \longrightarrow \mathrm{GL}(A)$ *a homogeneous representation. Then, there are non-negative integers* a, b *and* c, *such that the module* A *is a direct summand of the module* $W(\underline{\sigma}, \underline{r})_{a,b,c}$.

Proof. First, there are integers s_i, t_i, $i = 1, \ldots, k$, such that A is a direct summand of the $\mathrm{GL}(V, \underline{r})$-module

$$\bigoplus_{i=1}^{k} W(\underline{\sigma}, \underline{r})^{\otimes s_i} \otimes \mathrm{Hom}\,(W(\underline{\sigma}, \underline{r}), \mathbb{C})^{\otimes t_i}.$$

This is a consequence of ([6], Proposition 3.1(a), p. 40). Since ρ is assumed to be homogeneous, we have

$$s_i - t_i = s_j - t_j, \quad \text{for all } 1 \le i \le j \le k.$$

The assertion now follows from Corollary 1.2 in [24]. □

Remark 3.1.3. The tuple $\underline{\sigma}$ will be a natural parameter in our theory.

Example 3.1.4. Let $Q = (V, A, t, h)$ be a quiver with vertex set $V = \{v_1, \ldots, v_l\}$, arrow set $A = \{a_1, \ldots, a_n\}$, tail map $t\colon A \longrightarrow V$, and head map $h\colon A \longrightarrow V$. Fix a dimension vector $\underline{r} = (r_v, v \in V)$ as well as another tuple $\underline{\alpha} = (\alpha_v, v \in V)$ of positive integers with $\alpha_{h(a)} - \alpha_{t(a)} = \alpha_{h(a')} - \alpha_{t(a')} = \alpha$ for all $a, a' \in A$. Then, the $\mathrm{GL}(V, \underline{r})$-module

$$\bigoplus_{a \in A} \mathrm{Hom}\,((\mathbb{C}^{r_{t(a)}})^{\otimes \alpha_{t(a)}}, (\mathbb{C}^{r_{h(a)}})^{\otimes \alpha_{h(a)}})$$

is homogeneous of degree α.

3.2 *V-split vector spaces*

First, let W be a finite dimensional $\mathbb{C}$-vector space. A *weighted flag in* W is a pair $(W^\bullet, \underline{\gamma})$ with

$$W^\bullet\colon\ 0 \subsetneq W_1 \subsetneq \cdots \subsetneq W_s \subsetneq W$$

a — not necessarily complete — flag in W and $\underline{\gamma} = (\gamma_1, \ldots, \gamma_{s+1})$ a vector of integers with $\gamma_1 < \cdots < \gamma_{s+1}$.

Remark 3.2.1. In our context, weighted flags arise in the following way: Let $\lambda\colon \mathbb{C}^* \longrightarrow \mathrm{GL}(W)$ be a one-parameter subgroup and $\chi_1, \ldots, \chi_{s+1}$, $s \ge 0$, the characters of $\mathbb{C}^*$ with non-trivial eigenspace in W. Then, $\chi_i(z) = z^{\gamma_i}$ with $\gamma_i \in \mathbb{Z}$. Let $W^{\chi_i} \subset W$ be the corresponding eigenspace. We number the characters in such a way that $\gamma_1 < \cdots < \gamma_{s+1}$. This yields the weight vector $\underline{\gamma}$. The flag $W^\bullet$ is obtained by setting

$$W_i := \bigoplus_{j=1}^{i} W^{\chi_j}, \quad i = 1, \ldots, s.$$

Let $V = \{v_1, \ldots, v_t\}$ be an index set. A *V-split vector space* is a collection $(W_v, v \in V)$ of vector spaces indexed by V. Note that V-split vector spaces form in a natural way an Abelian category. A *weighted flag* $(W_\bullet, \underline{\gamma})$ *in the V-split vector space* $(W_v, v \in V)$ is a pair $(W_\bullet, \underline{\gamma})$ with

$$W_\bullet\colon\ 0 \subsetneq (W_1^v, v \in V) \subsetneq \cdots \subsetneq (W_s^v, v \in V) \subsetneq (W_v, v \in W)$$

a filtration of $(W_v, v \in V)$ by V-split subspaces and

$$\underline{\gamma} = (\gamma_1, \ldots, \gamma_{s+1})$$

a vector of integers. We then have the equivalent notions of

(a) Tuples $((\widehat{W}_\bullet^v, \underline{\gamma}^v), v \in V)$ of weighted flags in the $W_v, v \in V$,

$$\widehat{W}_\bullet^v\colon\ 0 \subsetneq \widehat{W}_1^v \subsetneq \cdots \subsetneq \widehat{W}_{s_v}^v \subsetneq W_v,$$
$$\underline{\gamma}^v = (\gamma_1^v, \ldots, \gamma_{s_v+1}^v), \quad v \in V.$$

Here, $s_v = 0$ is permitted.

(b) Weighted flags $(W_\bullet, \underline{\gamma})$ in the V-split vector space $(W_v, v \in V)$.

Indeed, suppose we are given a tuple as in (a). Let $\gamma_1 < \cdots < \gamma_{s+1}$ be the different weights occurring among the γ_j^v, $v \in V$, $j = 1, \ldots, s_v + 1$. Then, we define $(W_j^v, v \in V)$ by $W_j^v := \widehat{W}_{\iota_v(j)}^v$ with

$$\iota_v(j) := \max\{\iota = 1, \ldots, s_v + 1 \mid \gamma_\iota^v \le \gamma_j\},$$

$v \in V$, $j = 1, \ldots, s+1$. Conversely, given a weighted flag $(W_\bullet, \underline{\gamma})$ in the V-split vector space $(W_v, v \in V)$, we get a weighted flag $(\widehat{W}_\bullet^v, \underline{\gamma}^v)$ in W_v by just projecting onto W_v, $v \in V$. These two operations are clearly inverse to each other.

A weight formula

Next, we fix $\underline{\sigma} = (\sigma_v \in \mathbb{Z}_{>0}, v \in V)$ and set $M := \bigoplus_{v \in V} W_v^{\oplus \sigma_v}$. Suppose we are given a V-split vector space $(W_v, v \in V)$ and a weighted flag $(W_\bullet, \underline{\gamma})$. Define $M_j := \bigoplus_{v \in V} (W_j^v)^{\oplus \sigma_v}$ in order to obtain a weighted flag $(M_\bullet, \underline{\gamma})$ in M. Assume, furthermore, that we are given quotients $k_v\colon\ W_v \longrightarrow \mathbb{C}^{t_v}$, $v \in V$, and set $k := \bigoplus_{v \in V} k_v^{\oplus \sigma_v}\colon\ M \longrightarrow \mathbb{C}^t$, $t := \sum_{v \in V} \sigma_v \cdot t_v$. The central formula that we need later is contained in the following.

PROPOSITION 3.2.2

Suppose that, in the above situation, we are given a tuple $((\widehat{W}_\bullet^v, \underline{\gamma}^v), v \in V)$ of weighted flags in the W_v, $v \in V$. Let $(W_\bullet, \underline{\gamma})$ be the corresponding weighted flag in $(W_v, v \in V)$ and $(M_\bullet, \underline{\gamma})$ the resulting weighted flag in M. Then, the following identity holds *true*:

$$\sum_{j=1}^{s} \frac{\gamma_{j+1}-\gamma_j}{r}(r \cdot \dim k(M_j) - t \cdot \dim M_j)$$
$$= \sum_{v\in V} \sigma_v \left(\sum_{j=1}^{s_v} \frac{\gamma_{j+1}^v - \gamma_j^v}{r_v} (r_v \cdot \dim k_v(\widehat{W}_j^v) - t_v \cdot \dim \widehat{W}_j^v) \right)$$
$$- \sum_{v\in V} \sigma_v \cdot \left(\frac{t_v}{r_v} - \frac{t}{r} \right) \cdot \left(\sum_{j=1}^{s_v+1} \gamma_j^v (\dim \widehat{W}_j^v - \dim \widehat{W}_{j-1}^v) \right).$$

Here, $r_v := \dim W_v$, $v \in V$, *and* $r = \sum_{v\in V} \sigma_v \cdot r_v$.

Proof. From the definitions, the formula

$$\sum_{v\in V} \sigma_v \cdot \left(\sum_{j=1}^{s_v+1} \gamma_j^v (\dim \widehat{W}_j^v - \dim \widehat{W}_{j-1}^v) \right) = \sum_{j=1}^{s+1} \gamma_j (\dim M_j - \dim M_{j-1})$$

follows immediately. Therefore, the assertion is equivalent to the following equation:

$$\sum_{j=1}^{s} \frac{\gamma_{j+1}-\gamma_j}{r}(r \cdot \dim k(M_j) - t \cdot \dim M_j)$$
$$- \frac{t}{r} \cdot \sum_{j=1}^{s+1} \gamma_j \cdot (\dim M_j - \dim M_{j-1})$$
$$= \sum_{v\in V} \sigma_v \left(\sum_{j=1}^{s_v} \frac{\gamma_{j+1}^v - \gamma_j^v}{r_v} (r_v \cdot \dim k_v(\widehat{W}_j^v) - t_v \cdot \dim \widehat{W}_j^v) \right)$$
$$- \sum_{v\in V} \sigma_v \cdot \frac{t_v}{r_v} \cdot \left(\sum_{j=1}^{s_v+1} \gamma_j^v (\dim \widehat{W}_j^v - \dim \widehat{W}_{j-1}^v) \right).$$

Now,

$$- \frac{t}{r} \cdot \sum_{j=1}^{s} (\gamma_{j+1} - \gamma_j) \dim M_j$$
$$= \left(\frac{t}{r} \sum_{j=1}^{s} \gamma_j \dim M_j \right) - \left(\frac{t}{r} \sum_{j=1}^{s} \gamma_{j+1} \dim M_j \right)$$
$$\overset{M_0=\{0\}}{=} \left(\frac{t}{r} \sum_{j=1}^{s} \gamma_j \dim M_j \right) - \left(\frac{t}{r} \sum_{j=1}^{s+1} \gamma_j \dim M_{j-1} \right)$$
$$= -\frac{t}{r} \cdot \gamma_{s+1} \dim M + \sum_{j=1}^{s+1} \gamma_j (\dim M_j - \dim M_{j-1}).$$

Therefore, the left-hand side simplifies to

$$-\frac{t}{r}\cdot\gamma_{s+1}\dim M+\sum_{j=1}^{s}(\gamma_{j+1}-\gamma_j)\cdot\dim k(M_j).$$

With the same argument as before, we see

$$\sum_{j=1}^{s}(\gamma_{j+1}-\gamma_j)\cdot\dim k(M_j)$$
$$=\gamma_{s+1}\dim k(M)-\sum_{j=1}^{s+1}\gamma_j(\dim k(M_j)-\dim k(M_{j-1})).$$

Since $\dim M=r$ and $\dim k(M)=t$, the left-hand side finally takes the form

$$-\sum_{j=1}^{s+1}\gamma_j(\dim k(M_j)-\dim k(M_{j-1})). \tag{1}$$

Likewise, the right-hand side becomes

$$-\sum_{v\in V}\sigma_v\cdot\left(\sum_{j=1}^{s_v+1}\gamma_j^v(\dim k_v(\widehat{W}_j^v)-\dim k(\widehat{W}_{j-1}^v))\right). \tag{2}$$

The equality of (1) and (2) is now clear from the definitions. □

Remark 3.2.3. The conceptual way to see the above formula which explains how it will arise later is the following: Denote by $\mathbb{G}_v$ the Grassmannian of t_v-dimensional quotients of W_v, $v\in V$. Let $\mathfrak{k}_v\colon\ W_v\otimes\mathcal{O}_{\mathbb{G}_v}\longrightarrow Q_v$ be the universal quotient and $\mathcal{O}_{\mathbb{G}_v}(1):=\det(Q_v)$, $v\in V$. Likewise, we let $\mathbb{G}$ be the Grassmannian of t-dimensional quotients of M, $\mathfrak{k}\colon\ M\otimes\mathcal{O}_{\mathbb{G}}\longrightarrow Q$ the universal quotient, and $\mathcal{O}_{\mathbb{G}}(1):=\det(Q)$. On $\times_{v\in V}\mathbb{G}_v$, we have the quotient

$$\mathfrak{k}':=\bigoplus_{v\in V}\pi_{\mathbb{G}_v}^*\mathfrak{k}_v^{\oplus\sigma_v}\colon\ M\otimes\mathcal{O}_{\times_{v\in V}\mathbb{G}_v}\longrightarrow\bigoplus_{v\in V}\pi_{\mathbb{G}_v}^*Q_v^{\oplus\sigma_v}.$$

This quotient defines a $(\times_{v\in V}\mathrm{GL}(W_v))$-equivariant embedding

$$h\colon\ \mathop{\times}_{v\in V}\mathbb{G}_v\hookrightarrow\mathbb{G},$$

such that

$$h^*\mathcal{O}_{\mathbb{G}}(1)=\mathcal{O}_{\times_{v\in V}\mathbb{G}_v}(\sigma_{v_1},\ldots,\sigma_{v_t}).$$

Let $\lambda_v\colon\ \mathbb{C}^*\longrightarrow\mathrm{GL}(W_v)$ be a one-parameter subgroup which induces the weighted flag $(\widehat{W}_\bullet^v,\underline{\gamma}^v)$ in W_v, $v\in V$, see Remark 3.2.1. Then, the expression (2) is just

$$\mu(\lambda,(k_{v_1},\ldots,k_{v_l})),\quad \lambda:=(\lambda_{v_1},\ldots,\lambda_{v_l}),$$

with respect to the linearization of the $\mathrm{GL}(V,\underline{r})$-action in $\mathcal{O}_{\times_{v\in V}\mathbb{G}_v}(\sigma_{v_1},\ldots,\sigma_{v_t})$. Now, we can view λ as a one-parameter subgroup of $\mathrm{GL}(M)$. The induced weighted filtration of M is then $(M_\bullet,\underline{\gamma})$, and (1) agrees with

$$\mu(\lambda,k)$$

with respect to the linearization in $\mathcal{O}_{\mathbb{G}}(1)$. Obviously, we must have

$$\mu(\lambda,k)=\mu(\lambda,(k_{v_1},\ldots,k_{v_l})).$$

3.3 *V-split sheaves*

We fix a finite index set $V=\{v_1,\ldots,v_t\}$. A *V-split sheaf* is simply a tuple $(\mathcal{E}_v, v\in V)$ of coherent sheaves on X. Likewise, a *homomorphism between V-split sheaves* $(\mathcal{E}_v, v\in V)$ and $(\mathcal{E}'_v, v\in V)$ is a collection $(f_v, v\in V)$ of homomorphisms $f_v\colon\ \mathcal{E}_v\longrightarrow\mathcal{E}'_v$, $v\in V$. In this way, the V-split sheaves on X form an Abelian category. The *type of the V-split sheaf* $(\mathcal{E}_v, v\in V)$ is the tuple $\underline{P}(\mathcal{E}_v, v\in V)=(P(\mathcal{E}_{v_1}),\ldots,P(\mathcal{E}_{v_t}))$.

Remark 3.3.1. The datum of a V-split vector bundle is equivalent to the datum of a principal $\mathrm{GL}(V,\underline{r})$-bundle. Thus, a V-split sheaf can be seen as the natural 'singular' version of a principal $\mathrm{GL}(V,\underline{r})$-bundle.

Now, we fix additional parameters $\underline{\sigma}=(\sigma_v\in\mathbb{Z}_{>0}, v\in V)$ and $\underline{\chi}=(\chi_v\in\mathbb{Q}[x], v\in V)$ where the polynomials χ_v have degree at most $\dim X-1$, $v\in V$. We denote by $\overline{\chi}_v$ the coefficient of $x^{\dim X-1}$ in χ_v, $v\in V$. Then, we define for any V-split sheaf $(\mathcal{E}_v, v\in V)$ the following quantities.
The $(\underline{\sigma},\underline{\chi})$-degree:

$$\deg_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v\in V):=\sum_{v\in V}(\sigma_v\deg\mathcal{E}_v-\overline{\chi}_v\mathrm{rk}\,\mathcal{E}_v).$$

The $(\underline{\sigma},\underline{\chi})$-Hilbert polynomial:

$$P_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v\in V):=\sum_{v\in V}(\sigma_v P(\mathcal{E}_v)-\chi_v\mathrm{rk}\,\mathcal{E}_v).$$

The $\underline{\sigma}$-rank:

$$\mathrm{rk}_{\underline{\sigma}}(\mathcal{E}_v, v\in V):=\sum_{v\in V}\sigma_v\mathrm{rk}\,\mathcal{E}_v.$$

The $(\underline{\sigma},\underline{\chi})$-slope:

$$\mu_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v\in V):=\frac{\deg_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v\in V)}{\mathrm{rk}_{\underline{\sigma}}(\mathcal{E}_v, v\in V)}.$$

Note that the $(\underline{\sigma},\underline{\chi})$-degree, $(\underline{\sigma},\underline{\chi})$-Hilbert polynomial, and $\underline{\sigma}$-rank all behave additively on short exact sequences. Thus, the $(\underline{\sigma},\underline{\chi})$-slope will have all the formal properties of the usual slope. More specifically, we call a V-split sheaf $(\mathcal{E}_v, v\in V)$ *$(\underline{\sigma},\underline{\chi})$-slope (semi)stable*, if $\mathcal{E}_v$ is torsion free, $v\in V$, and, for all non-trivial, proper V-split subsheaves $(\mathcal{F}_v, v\in V)$, the inequality

$$\mu_{\underline{\sigma},\underline{\chi}}(\mathcal{F}_v, v\in V)(\le)\mu_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v\in V)$$

is satisfied.

Remark 3.3.2. Let $(\mathcal{E}_v, v \in V)$ be a $(\underline{\sigma}, \underline{\chi})$-slope semistable V-split sheaf. Let v_0 be a vertex with $\mathcal{E}_{v_0} \neq 0$ and $0 \subsetneq \mathcal{F} \subseteq \mathcal{E}_{v_0}$ a subsheaf. Set $\mathcal{F}_v := 0$ for $v \neq v_0$ and $\mathcal{F}_{v_0} := \mathcal{F}$. Then, we get $\mu_{\underline{\sigma},\underline{\chi}}(\mathcal{F}_v, v \in V) = \mu(\mathcal{F}) - (\overline{\chi}_{v_0}/\sigma_{v_0})$, so that the semistability condition yields

$$\mu(\mathcal{F}) \leq \mu_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v \in V) + \frac{\overline{\chi}_{v_0}}{\sigma_{v_0}}.$$

Likewise, we find for every non-trivial quotient $\mathcal{Q}$ of $\mathcal{E}_{v_0}$,

$$\mu(\mathcal{Q}) \geq \mu_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v \in V) + \frac{\overline{\chi}_{v_0}}{\sigma_{v_0}}.$$

If we apply this to $\mathcal{F} = \mathcal{E}_{v_0} = \mathcal{Q}$, we see that $\mathcal{E}_{v_0}$ must be a semistable sheaf with slope $\mu_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v \in V) + \overline{\chi}_{v_0}/\sigma_{v_0}$. Note that this forces $\sum_{v\in V} r_v \cdot \overline{\chi}_v = 0$.

Recall that any homomorphism $f\colon\ \mathcal{E} \longrightarrow \mathcal{E}'$ between the semistable sheaves $\mathcal{E}$ and $\mathcal{E}'$ will be zero, if $\mu(\mathcal{E}) > \mu(\mathcal{E}')$. Therefore, we deduce the following.

PROPOSITION 3.3.3

Let $(\mathcal{E}_v, v \in V)$ and $(\mathcal{E}'_v, v \in V)$ be $(\underline{\sigma}, \underline{\chi})$-slope semistable V-split sheaves. Assume

$$\mu_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v \in V) + \min_{v\in V}\left\{\frac{\overline{\chi}_v}{\sigma_v}\right\} > \mu_{\underline{\sigma},\underline{\chi}}(\mathcal{E}'_v, v \in V) + \max_{v\in V}\left\{\frac{\overline{\chi}_v}{\sigma_v}\right\}.$$

Then, for any choice of exponents $s_v, s'_v > 0$, $v \in V$, any homomorphism $f\colon\ \bigoplus_{v\in V} \mathcal{E}_v^{\oplus s_v} \longrightarrow \bigoplus_{v\in V} \mathcal{E}'^{\oplus s'_v}_v$ is zero.

Finally, we have the following Proposition.

PROPOSITION 3.3.4 (Harder–Narasimhan filtration)

Let $(\mathcal{E}_v, v \in V)$ be any torsion-free V-split sheaf. Then, $(\mathcal{E}_v, v \in V)$ possesses a unique filtration by $(\underline{\sigma}, \underline{\chi})$-destabilizing V-split subsheaves

$$\begin{aligned} 0 &= (\mathcal{F}^v_0, v \in V) \subsetneq (\mathcal{F}^v_1, v \in V) \subsetneq \cdots \subsetneq (\mathcal{F}^v_s, v \in V) \subsetneq (\mathcal{F}^v_{s+1}, v \in V) \\ &= (\mathcal{E}_v, v \in V), \end{aligned}$$

such that

1. *the V-split sheaf $(\mathcal{F}^v_i/\mathcal{F}^v_{i-1}, v \in V)$ is $(\underline{\sigma}, \underline{\chi})$-slope semistable, $i = 1, \ldots, s+1$.*
2. *$\mu_{\underline{\sigma},\underline{\chi}}(\mathcal{F}^v_i/\mathcal{F}^v_{i-1}, v \in V) > \mu_{\underline{\sigma},\underline{\chi}}(\mathcal{F}^v_{i+1}/\mathcal{F}^v_i, v \in V)$, $i = 1, \ldots, s$.*

The weight formula for sheaves

Let $(\mathcal{E}_v, v \in V)$ be a V-split sheaf. As before, the following data are equivalent:

(a) Tuples $((\widehat{\mathcal{E}}^v_\bullet, \underline{\gamma}^v), v \in V)$ of weighted filtrations of the $\mathcal{E}_v$, $v \in V$,

$$\begin{aligned} &\widehat{\mathcal{E}}^v_\bullet\colon\ 0 \subsetneq \widehat{\mathcal{E}}^v_1 \subsetneq \cdots \subsetneq \widehat{\mathcal{E}}^v_{s_v} \subsetneq \mathcal{E}_v, \\ &\underline{\gamma}^v = (\gamma^v_1, \ldots, \gamma^v_{s_v+1}), \quad v \in V. \end{aligned}$$

(b) Weighted filtrations $(\mathcal{E}_\bullet, \underline{\gamma})$ of the V-split sheaf $(\mathcal{E}_v, v \in V)$.

Moreover, given $\underline{\sigma} = (\sigma_v \in \mathbb{Z}_{>0}, v \in V)$, a V-split sheaf $(\mathcal{E}_v, v \in V)$, and a weighted filtration $(\mathcal{E}_\bullet, \underline{\gamma})$, we set $\mathcal{E}^{\text{total}} := \bigoplus_{v\in V} \mathcal{E}_v^{\oplus\sigma_v}$ and

$$\mathcal{E}_\bullet^{\text{total}}\colon\ 0 \subsetneq \mathcal{E}_1^{\text{total}} := \bigoplus_{v\in V} \mathcal{E}_1^{v,\oplus\sigma_v} \subsetneq \cdots \subsetneq \mathcal{E}_s^{\text{total}} := \bigoplus_{v\in V} \mathcal{E}_s^{v,\oplus\sigma_v} \subsetneq \mathcal{E}^{\text{total}}.$$

PROPOSITION 3.3.5

Suppose that, in the above situation, we are given a tuple $((\widehat{\mathcal{E}}_\bullet^v, \underline{\gamma}^v), v \in V)$ *of weighted filtrations of the* $\mathcal{E}_v$, $v \in V$. *Let* $(\mathcal{E}_\bullet^{\text{total}}, \underline{\gamma})$ *be the resulting weighted filtration of* $\mathcal{E}^{\text{total}}$. *Then, for all* $l \gg 0$,

$$\begin{aligned}
&\sum_{j=1}^{s} \frac{\gamma_{i+1}-\gamma_i}{P(\mathcal{E}^{\text{total}})(l)}\left(P(\mathcal{E}^{\text{total}})(l)\cdot \operatorname{rk}\mathcal{E}_j^{\text{total}} - P(\mathcal{E}_j^{\text{total}})(l)\cdot \operatorname{rk}\mathcal{E}^{\text{total}}\right)\\
&\quad = \sum_{v\in V}\sigma_v\left(\sum_{j=1}^{s_v}\frac{\gamma_{j+1}^v-\gamma_j^v}{P(\mathcal{E}_v)(l)}\left(P(\mathcal{E}_v)(l)\cdot\operatorname{rk}\widehat{\mathcal{E}}_j^v - P(\widehat{\mathcal{E}}_j^v)(l)\cdot\operatorname{rk}\mathcal{E}_v\right)\right)\\
&\qquad - \sum_{v\in V}\sigma_v\cdot\left(\frac{\operatorname{rk}\mathcal{E}_v}{P(\mathcal{E}_v)(l)} - \frac{\operatorname{rk}\mathcal{E}^{\text{total}}}{P(\mathcal{E}^{\text{total}})(l)}\right)\\
&\qquad \cdot\left(\sum_{j=1}^{s_v+1}\gamma_j^v\left(P(\widehat{\mathcal{E}}_j^v)(l) - P(\widehat{\mathcal{E}}_{j-1}^v)(l)\right)\right).
\end{aligned}$$

Proof. For $l \gg 0$, we have

- $\widehat{\mathcal{E}}_j^v(l)$ is globally generated, $v \in V$, $j = 1, \ldots, s_v + 1$,
- $H^i(\widehat{\mathcal{E}}_j^v(l)) = 0$, $i > 0$, $v \in V$, and $j = 1, \ldots, s_v + 1$.

Then, we may write $\mathcal{E}_v(l)$ as a quotient $q_v\colon\ \mathcal{O}_X^{\oplus P(\mathcal{E}_v)(l)} \longrightarrow \mathcal{E}_v(l)$, such that $H^0(q_v)$ is an isomorphism, $v \in V$. Restricting this to a general point $x \in X$ yields $k_v\colon\ \mathbb{C}^{P(\mathcal{E}_v)(l)} \longrightarrow \mathbb{C}^{\operatorname{rk}\mathcal{E}_v}$, $v \in V$. Now, apply Proposition 3.2.2 to the tuple $((H^0(\widehat{\mathcal{E}}_\bullet^v(l)), \underline{\gamma}^v), v \in V)$ under the identification of $\mathbb{C}^{P(\mathcal{E}_v)(l)}$ with $H^0(\mathcal{E}_v(l))$. □

3.4 *ρ-pairs*

In this section, we will fix a dimension vector $\underline{r} = (r_v, v \in V)$ and a homogeneous representation $\rho\colon\ \mathrm{GL}(V, \underline{r}) \longrightarrow \mathrm{GL}(A)$. By slightly deviating from the conventions in the introduction (see Remark 3.4.1), the objects we would like to consider are pairs $(\mathcal{E}_v, v \in V; \Psi)$ where $(\mathcal{E}_v, v \in V)$ is a V-split vector bundle, such that $\operatorname{rk}\mathcal{E}_v = r_v$, $v \in V$, and $\Psi\colon\ X \longrightarrow \mathbb{P}(\mathcal{F}_\rho^\vee)$ is a section. Here, $\mathcal{F}_\rho$ is the vector bundle with fibre A associated to $(\mathcal{E}_v, v \in V)$ via the representation ρ. Now, the section Ψ is specified by a line bundle $\mathcal{N}$ and a surjective homomorphism $\phi\colon\ \mathcal{F}_\rho \longrightarrow \mathcal{N}$, and two such homomorphisms will yield the same section, if and only if they differ by a constant $z \in \mathbb{C}^*$. Thus, in order to find projective moduli spaces (at least over curves), we consider tuples $(\mathcal{E}_v, v \in V; \mathcal{N}; \phi)$ where $\mathcal{N}$ is a line bundle and $\phi\colon\ \mathcal{F}_\rho \longrightarrow \mathcal{N}$ is a non-trivial homomorphism. Such an object will be referred to as a *ρ-pair*, and the tuple $(\underline{P}(\mathcal{E}_v, v \in V), \mathcal{N})$ is called the *type*.

Two ρ-pairs $(\mathcal{E}_v, v \in V; \mathcal{N}; \phi)$ and $(\mathcal{E}'_v, v \in V; \mathcal{N}'; \phi')$ will be considered *equivalent*, if there are isomorphisms $\psi_v\colon\ \mathcal{E}_v \longrightarrow \mathcal{E}'_v$, $v \in V$, and $z\colon\ \mathcal{N} \longrightarrow \mathcal{N}'$, such that

$$\phi' = z \circ \phi \circ \psi_\rho^{-1},$$

with $\psi_\rho\colon\ \mathcal{F}_\rho \longrightarrow \mathcal{F}'_\rho$ the isomorphism induced by ψ_v, $v \in V$.

Given a tuple $\underline{P} = (P_v, v \in V)$ of Hilbert polynomials and a line bundle $\mathcal{N}$, a *family of ρ-pairs of type $(\underline{P}, \mathcal{N})$ parametrized by the scheme S* is the datum of a tuple $(\mathcal{E}_{S,v}, v \in V; \mathcal{L}_S, \phi_S)$ with vector bundles $\mathcal{E}_{S,v}$, $v \in V$, on $S \times X$, such that $P(\mathcal{E}_{S,v|\{s\}\times X}) = P_v$ for all $s \in S$, $v \in V$, $\mathcal{L}_S$ a line bundle on S, and $\phi_S\colon\ \mathcal{F}_{S,\rho} \longrightarrow \pi_S^*\mathcal{L}_S \otimes \pi_X^*\mathcal{N}$ a homomorphism with non-trivial restriction to every fibre $\{s\} \times X$, $s \in S$. Two such families $(\mathcal{E}_{S,v}, v \in V; \mathcal{L}_S, \phi_S)$ and $(\mathcal{E}'_{S,v}, v \in V; \mathcal{L}'_S, \phi'_S)$ will be considered *equivalent*, if there are isomorphisms $\psi_{S,v}\colon\ \mathcal{E}_{S,v} \longrightarrow \mathcal{E}'_{S,v}$, $v \in V$ and $z_S\colon\ \mathcal{L}_S \longrightarrow \mathcal{L}'_S$, such that

$$\phi'_S = (\pi_S^*(z_S) \otimes \mathrm{id}_{\pi_X^*\mathcal{N}}) \circ \phi_S \circ \psi_{S,\rho}^{-1}.$$

Remark 3.4.1. First, we note that using $\mathcal{F}_\rho^\vee$ instead of $\mathcal{F}_\rho$ is for notational convenience only. Then, the right analogue to the problems mentioned in the introduction would be the study of tuples $(\mathcal{E}_v, v \in V; \phi)$ where $\phi\colon\ \mathcal{F}_\rho \longrightarrow \mathcal{O}_X$ is a surjective homomorphism, and $(\mathcal{E}_v, v \in V; \phi)$ and $(\mathcal{E}'_v, v \in V; \phi')$ should be identified, if and only if there are isomorphisms $\psi_v\colon\ \mathcal{E}_v \longrightarrow \mathcal{E}'_v$ with $\phi' = \phi \circ \psi_\rho^{-1}$. In the case of a homogeneous representation of non-zero degree, this equivalence relation will identify $(\mathcal{E}_v, v \in V; \phi)$ and $(\mathcal{E}_v, v \in V; z \cdot \phi)$, $z \in \mathbb{C}^*$, anyway. Otherwise, one may add the trivial representation to ρ. This means that we consider tuples $(\mathcal{E}_v, v \in V; \phi, \varepsilon)$ with $(\mathcal{E}_v, v \in V; \phi)$ as before and $\varepsilon \in \mathbb{C}$, but the equivalence relation becomes $(\mathcal{E}_v, v \in V; \phi, \varepsilon) \sim (\mathcal{E}'_v, v \in V; \phi', \varepsilon')$, if there are isomorphisms $\psi_v\colon\ \mathcal{E}_v \longrightarrow \mathcal{E}'_v$, $v \in V$, and a $z \in \mathbb{C}^*$, such that

$$z \cdot \phi' = \phi \circ \psi_\rho^{-1}, \quad \text{and} \quad z \cdot \varepsilon' = \varepsilon.$$

Then, we may recover the original objects in the form $(\mathcal{E}_v, v \in V; \phi, 1)$. Thus, our concept is more flexible rather than more restrictive than the one presented in the introduction.

In order to define the semistability concept we introduce additional parameters:

- a tuple $\underline{\sigma} = (\sigma_v, v \in V)$ of positive integers.
- a tuple $\underline{\eta} = (\eta_v, v \in V)$ of rational numbers, such that $\sum_{v\in V} \eta_v r_v = 0$.
- a positive polynomial $\delta \in \mathbb{Q}[x]$. Then, $\chi_v := \eta_v \cdot \delta$, $v \in V$.

Given any torsion-free $\mathcal{O}_X$-module $\mathcal{E}$, we call a submodule $\mathcal{F} \subset \mathcal{E}$ *saturated*, if the quotient $\mathcal{E}/\mathcal{F}$ is still torsion free. The test objects for the semistability concept will be weighted filtrations $(\mathcal{E}_\bullet, \underline{\gamma})$ of the V-split vector bundle $(\mathcal{E}_v, v \in V)$ where each $(\mathcal{E}_j^v, v \in V)$ consists of saturated subsheaves $\mathcal{E}_j^v \subset \mathcal{E}_v$, $v \in V$, $j = 1, \dots, s$. For such a weighted filtration, we define $\underline{\alpha} = (\alpha_1, \dots, \alpha_{s+1})$ by $\alpha_i := (\gamma_{i+1} - \gamma_i)/R$, $R := \mathrm{rk}(\bigoplus_{v\in V} \mathcal{E}_v^{\oplus\sigma_v})$, $i = 1, \dots, s$. We now set

$$M_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_\bullet, \underline{\alpha}) := \sum_{j=1}^{s} \alpha_j \cdot \big(P_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v \in V) \cdot \mathrm{rk}_{\underline{\sigma}}(\mathcal{E}_j^v, v \in V) - P_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_j^v, v \in V) \cdot \mathrm{rk}_{\underline{\sigma}}(\mathcal{E}_v, v \in V)\big).$$

If we are also given a non-trivial homomorphism $\phi\colon\ \mathcal{F}_\rho \longrightarrow \mathcal{N}$, we have to define the quantity

$$\mu(\mathcal{E}_\bullet, \underline{\gamma}; \phi).$$

The natural, though complicated, definition for $\#V = 1$ was explained in [24]. We adapt it to our setting. Let $M := \bigoplus_{v\in V} \mathbb{C}^{\sigma_v\cdot r_v}$. Then, for appropriate a, b, c, the module A will be a submodule of $M^{\otimes a \oplus b} \otimes (\bigwedge^{\dim W} W)^{\otimes -c}$. We introduce $\mathcal{E}^{\text{total}}_j := \bigoplus_{v\in V} (\mathcal{E}^v_j)^{\oplus\sigma_v}$, $j = 1, \ldots, s$. There is the weighted filtration $(\mathcal{E}^{\text{total}}_\bullet, \underline{\alpha})$

$$0 \subsetneq \mathcal{E}^{\text{total}}_1 \subsetneq \cdots \subsetneq \mathcal{E}^{\text{total}}_s \subsetneq \mathcal{E}^{\text{total}}$$

of $\mathcal{E}^{\text{total}} = \bigoplus_{v\in V} \mathcal{E}_v^{\oplus\sigma_v}$. We choose a flag

$$0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_s \subsetneq M$$

with $\dim M_i = \operatorname{rk} \mathcal{E}^{\text{total}}_i$, $i = 1, \ldots, s$. Over a suitable open subset $\mathcal{U}$, the homomorphism ϕ will be surjective, and there will be a trivialization $\psi\colon\ M \otimes \mathcal{O}_{\mathcal{U}} \cong \mathcal{E}^{\text{total}}$ with $\psi(M_i \otimes \mathcal{O}_{\mathcal{U}}) = \mathcal{E}^{\text{total}}_i$, $i = 1, \ldots, s$. We may write

$$\mathcal{F}_{a,b,c} := \mathcal{E}^{\text{total}^{\otimes a \oplus b}} \otimes \det(\mathcal{E}^{\text{total}})^{\otimes -c} = \mathcal{F}_\rho \oplus \mathcal{F}',$$

so that ϕ and the trivialization ψ yield a morphism

$$\sigma\colon\ \mathcal{U} \longrightarrow \mathbb{P}(\mathcal{F}_\rho^\vee) \hookrightarrow \mathbb{P}(\mathcal{F}^\vee_{a,b,c}) \cong \mathbb{P}(M^{\otimes a \oplus b^\vee}) \times \mathcal{U} \longrightarrow \mathbb{P}(M^{\otimes a \oplus b^\vee}).$$

After the choice of a one-parameter subgroup $\lambda\colon\ \mathbb{C}^* \longrightarrow \mathrm{SL}(M)$ which induces the weighted filtration $(M_\bullet, \underline{\gamma}')$ with

$$\underline{\gamma}' = (\gamma'_1, \ldots, \gamma'_{s+1}) = \sum_{i=1}^{s} \alpha_i (\underbrace{R_i - R, \ldots, R_i - R}_{R_i\times}, \underbrace{R_i, \ldots, R_i}_{(R-R_i)\times}),$$

$$R_i := \operatorname{rk} \mathcal{E}^{\text{total}}_i,\ i = 1, \ldots, s,$$

we set

$$\mu(\mathcal{E}_\bullet, \underline{\gamma}; \phi) := \max\{\mu(\lambda, \sigma(x)) \mid x \in \mathcal{U}\}.$$

As in [24], one verifies that this is, in fact, well-defined.

Remark 3.4.2.

(i) One might define $\mu(\mathcal{E}_\bullet, \underline{\gamma}; \phi)$ without reference to the embedding of A into $M^{\otimes a\oplus b} \otimes (\bigwedge^{\dim W} W)^{\otimes -c}$, by working with one-parameter subgroups of $\mathrm{GL}(V, \underline{r})$.

(ii) An easier, more elegant, and equivalent definition [8] is

$$\mu(\mathcal{E}_\bullet, \underline{\gamma}; \phi) := -\min_{\substack{(i_1,\ldots,i_a)\in\\ \{1,\ldots,s+1\}^{\times a}}} \{\gamma'_{i_1} + \cdots + \gamma'_{i_a} \mid \tau \text{ restricted to}$$

$$(\mathcal{E}^{\text{total}}_{i_1} \otimes \cdots \otimes \mathcal{E}^{\text{total}}_{i_a})^{\oplus b} \text{ is non-trivial}\}.$$

Here, $\tau\colon\ \mathcal{F}_{a,b,c} \longrightarrow \mathcal{F}_\rho \xrightarrow{\phi} \mathcal{N}$. However, for the computations in examples, the above definition turns out to be more useful (see [24]).

Convention. Since the quantities introduced above depend only on $\underline{\alpha}$, we will refer to a pair $(\mathcal{E}_\bullet, \underline{\alpha})$, composed of a filtration

$$\mathcal{E}_\bullet\colon\ 0 \subsetneq (\mathcal{E}_1^v, v \in V) \subsetneq \cdots \subsetneq (\mathcal{E}_s^v, v \in V) \subsetneq (\mathcal{E}_v, v \in V)$$

of $(\mathcal{E}_v, v \in V)$ by non-trivial, proper, and saturated V-split subsheaves and a tuple $\underline{\alpha} = (\alpha_1, \dots, \alpha_s)$ of positive rational numbers, as a *weighted filtration* in the future.

A ρ-pair $(\mathcal{E}_v, v \in V; \mathcal{N}, \phi)$ will be called $(\underline{\sigma}, \underline{\eta}, \delta)$-*(semi)stable* or just *(semi)stable*, if for every weighted filtration $(\mathcal{E}_\bullet, \underline{\alpha})$ of $(\mathcal{E}_v, v \in V)$,

$$M_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_\bullet, \underline{\alpha}) + \delta \cdot \mu(\mathcal{E}_\bullet, \underline{\alpha}; \phi)(\succeq)0.$$

A few comments are in order.

Remark 3.4.3. Since $\sum_{v\in V} \chi_v \cdot r_v \equiv 0$, we may write

$$M_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_\bullet, \underline{\alpha}) = \operatorname{rk}\left(\bigoplus_{v\in V} \mathcal{E}_v^{\oplus\sigma_v}\right) \cdot \sum_{j=1}^{s} \alpha_j \left(\sum_{v\in V} \chi_v \operatorname{rk}\mathcal{E}_j^v\right)$$
$$+ \underbrace{\sum_{j=1}^{s} \alpha_j \left(P\left(\bigoplus_{v\in V} \mathcal{E}_v^{\oplus\sigma_v}\right) \cdot \operatorname{rk}\left(\bigoplus_{v\in V} \mathcal{E}_j^{v,\oplus\sigma_v}\right) - P\left(\bigoplus_{v\in V} \mathcal{E}_j^{v,\oplus\sigma_v}\right) \cdot \operatorname{rk}\left(\bigoplus_{v\in V} \mathcal{E}_v^{\oplus\sigma_v}\right)\right)}_{=:M_{\underline{\sigma}}(\mathcal{E}_\bullet,\underline{\alpha})}.$$

A bounded family of V-split vector bundles $(\mathcal{E}_v, v \in V)$ may be parametrized by a product of quot schemes $\underline{\mathfrak{Q}} = \times_{v\in V} \mathfrak{Q}_v$. Assigning to a point $([q_v], v \in V) \in \underline{\mathfrak{Q}}$ the quotient $[\bigoplus_{v\in V} q_v^{\oplus\sigma_v}]$ induces an injective and proper morphism from $\underline{\mathfrak{Q}}$ to some other quot scheme $\widetilde{\mathfrak{Q}}$. In this way, we can induce linearizations on $\underline{\mathfrak{Q}}$ by linearizations on $\widetilde{\mathfrak{Q}}$, and this shows how the quantity $M_{\underline{\sigma}}(\mathcal{E}_\bullet, \underline{\alpha})$ is obtained. The linearization of the $\mathrm{GL}(V, \underline{r})$-action on the space $\mathbb{P}(M^{\otimes a\oplus b})$ in $\mathcal{O}(1)$ induced by $\rho_{a,b,c}$ may be modified by a character, so that the determinant on $M^{\otimes a\oplus b}$ induces the trivial character on the center $\mathcal{Z}$ of $\mathrm{GL}(V, \underline{r})$, and the quantity $\mu(\mathcal{E}_\bullet, \underline{\alpha}; \phi)$ has been defined with respect to such a linearization. The parameter δ reflects the fact that the given linearization in $\mathcal{O}(1)$ may be raised to some tensor power. Finally, any linearization might be altered by a character χ of $\mathrm{GL}(V, \underline{r})$. The choice of such a character is encoded by the rational numbers $\eta_v, v \in V$. These considerations explain how the semistability concept we have introduced naturally 'mixes' the semistability concept for vector bundles and the invariant theory of the representation ρ.

We fix the Hilbert polynomials $\underline{P} = (P_v, v \in V)$ and the line bundle $\mathcal{N}$ and define the functors

$$\underline{\mathrm{M}}(\rho)^{(\underline{\sigma},\underline{\eta},\delta)-(\mathrm{s})\mathrm{s}}_{\underline{P}/\mathcal{N}}\colon\ \underline{\mathrm{Sch}_{\mathbb{C}}} \longrightarrow \underline{\mathrm{Sets}}$$
$$S \longmapsto \left\{\begin{matrix}\text{Equivalence classes of families of (semi)stable}\\ \rho\text{-pairs of type } (\underline{P}, \mathcal{N}) \text{ parametrized by } S\end{matrix}\right\}.$$

The condition $\sum_{v\in V} \eta_v r_v = 0$ is used to simplify some computations. It can, however, be assumed *without loss of generality*. For this, note that for arbitrary parameters σ_v and η_v, $v \in V$,

- $M_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_\bullet, \underline{\alpha})$ is defined the same way,
- $\mu(\mathcal{E}_\bullet, \underline{\gamma}; \phi)$ does not depend on the η_v, $v \in V$.

In particular, we may define (semi)stability with respect to these parameters. Suppose we are given arbitrary rational numbers η_v, $v \in V$, and $d \in \mathbb{Q}$. Define $\eta'_v := \eta_v - d\cdot\sigma_v$, $v \in V$. Then,

$$P_{\underline{\sigma},\underline{\chi}'}(\mathcal{F}_v, v \in V) = P_{\underline{\sigma},\underline{\chi}}(\mathcal{F}_v, v \in V) + d \cdot \delta \cdot \left(\sum_{v\in V} \sigma_v \mathrm{rk}\,(\mathcal{F}_v)\right).$$

It follows that

$$\begin{aligned} &P_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v \in V) \cdot \mathrm{rk}_{\underline{\sigma}}(\mathcal{F}_v, v \in V) - P_{\underline{\sigma},\underline{\chi}}(\mathcal{F}_v, v \in V) \cdot \mathrm{rk}_{\underline{\sigma}}(\mathcal{E}_v, v \in V) \\ &\quad = P_{\underline{\sigma},\underline{\chi}'}(\mathcal{E}_v, v\in V) \cdot \mathrm{rk}_{\underline{\sigma}}(\mathcal{F}_v, v \in V) - P_{\underline{\sigma},\underline{\chi}'}(\mathcal{F}_v, v \in V) \cdot \mathrm{rk}_{\underline{\sigma}}(\mathcal{E}_v, v\in V), \end{aligned}$$

so that always

$$M_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_\bullet, \underline{\alpha}) = M_{\underline{\sigma},\underline{\chi}'}(\mathcal{E}_\bullet, \underline{\alpha}),$$

and the concept of (semi)stability defined with respect to the parameters σ_v, η_v, $v \in V$, and δ equals the one defined with respect to the parameters σ_v, η'_v, $v \in V$, and δ. If we now set

$$d := \frac{\sum_{v\in V} \eta_v r_v}{\sum_{v\in V} \sigma_v r_v},$$

then

$$\sum_{v\in V} \eta'_v r_v = \left(\sum_{v\in V} \eta_v r_v\right) - d \cdot \left(\sum_{v\in V} \sigma_v r_v\right) = 0.$$

Theorem 3.4.4. (i) *If* $\dim(X) = 1$, *there exist a projective scheme* $\mathcal{M} := \mathcal{M}(\rho)^{(\underline{\sigma},\underline{\eta},\delta)-\mathrm{ss}}_{\underline{P}/\mathcal{N}}$ *and a natural transformation* $\theta\colon\ \underline{\mathrm{M}}(\rho)^{(\underline{\sigma},\underline{\eta},\delta)-\mathrm{ss}}_{\underline{P}/\mathcal{N}} \longrightarrow h_{\mathcal{M}}$, *such that for any other scheme* $\mathcal{M}'$ *and any other natural transformation* $\theta'\colon\ \underline{\mathrm{M}}(\rho)^{(\underline{\sigma},\underline{\eta},\delta)-\mathrm{ss}}_{\underline{P}/\mathcal{N}} \longrightarrow h_{\mathcal{M}'}$, *there is a unique morphism* $\zeta\colon\ \mathcal{M} \longrightarrow \mathcal{M}'$ *with* $\theta' = h_\zeta \circ \theta$. *The space* $\mathcal{M}$ *contains an open subscheme* $\mathcal{M}^{\mathrm{s}}$ *which is a coarse moduli scheme for the functor* $\underline{\mathrm{M}}(\rho)^{(\underline{\sigma},\underline{\eta},\delta)-\mathrm{s}}_{\underline{P}/\mathcal{N}}$. (ii) *If* $\dim(X) \geq 1$, *there exists a quasi-projective coarse moduli scheme* $\mathcal{M}(\rho)^{(\underline{\sigma},\underline{\eta},\delta)-\mathrm{s}}_{\underline{P}/\mathcal{N}}$ *for the functor* $\underline{\mathrm{M}}(\rho)^{(\underline{\sigma},\underline{\eta},\delta)-\mathrm{s}}_{\underline{P}/\mathcal{N}}$.

Example 3.4.5. The above results yield a semistability concept for coverings in the description of Casnati and Ekedahl (as reviewed in the introduction) and provides moduli spaces.

It is obvious that the theorem has to be proved only for representations of the type $\rho_{a,b,c}$. If $\dim(X) > 1$, then in order to compactify the moduli spaces, one also needs torsion free sheaves. It is however not clear how to associate a sheaf $\mathcal{F}_\rho$ to a V-split torsion free sheaf $(\mathcal{E}_v, v \in V)$ via an arbitrary representation ρ. However, for representations of the form $\rho_{a,b,c}$, this is obvious and one obtains natural compactifications. In the setting of quiver representations, we will exhibit another natural method to reduce a moduli problem to one for $\rho_{a,b,c}$-pairs. This illustrates the importance and the usefulness of the theory which we will outline in the next section.

3.5 *Decorated V-split sheaves*

We fix the following data

- a tuple of Hilbert polynomials $\underline{P} = (P_v,\ v \in V)$;
- a positive polynomial $\delta \in \mathbb{Q}[x]$ of degree at most $\dim X - 1$;
- a tuple of rational numbers $\underline{\eta} = (\eta_v,\ v \in V)$ with $\sum_{v\in V} \eta_v \cdot r_v = 0$. Here, r_v is the rank dictated by the Hilbert polynomial P_v, $v \in V$. Define $\chi_v := \eta_v \cdot \delta$, $v \in V$.
- a tuple $\underline{\sigma} = (\sigma_v,\ v \in V)$ of positive integers.

Given a V-split sheaf $(\mathcal{E}_v,\ v \in V)$ of type $\underline{P}$ and non-negative integers a, b, c, and m, a *decoration of type* (a, b, c, m) on $(\mathcal{E}_v,\ v \in V)$ is a homomorphism

$$\tau\colon\ \left(\left(\bigoplus_{v\in V} \mathcal{E}_v^{\oplus\sigma_v}\right)^{\otimes a}\right)^{\oplus b} \longrightarrow \det\left(\bigoplus_{v\in V} \mathcal{E}_v^{\oplus\sigma_v}\right)^{\otimes c} \otimes \mathcal{O}_X(m).$$

Two tuples $(\mathcal{E}_v,\ v \in V; \tau)$ and $(\mathcal{E}'_v,\ v \in V; \tau')$ are called *equivalent*, if there are $z \in \mathbb{C}^*$ and isomorphisms $\psi_v\colon\ \mathcal{E}_v \longrightarrow \mathcal{E}'_v$, such that

$$z \cdot \left(\left(\det\left(\bigoplus_{v\in V} \psi_v^{\oplus\sigma_v}\right)^{\otimes c} \otimes \mathrm{id}_{\mathcal{O}_X(m)}\right) \circ \tau \circ \left(\left(\left(\bigoplus_{v\in V} \psi_v^{\oplus\sigma_v}\right)^{\otimes a}\right)^{\oplus b}\right)^{-1}\right) = \tau'.$$

A family of V-split sheaves of type $\underline{P}$ with a decoration of type (a, b, c, m) parametrized by the scheme S consists of

- a tuple $(\mathcal{E}_{S,v},\ v \in V)$ on $S \times X$ of S-flat families $\mathcal{E}_{S,v}$ of torsion free sheaves on X with Hilbert polynomial P_v, $v \in V$;
- a line bundle $\mathcal{L}_S$ on S;
- a homomorphism

$$\tau_S\colon\ \left(\left(\bigoplus_{v\in V} \mathcal{E}_{S,v}^{\oplus\sigma_v}\right)^{\otimes a}\right)^{\oplus b} \longrightarrow \det\left(\bigoplus_{v\in V} \mathcal{E}_{S,v}^{\oplus\sigma_v}\right)^{\otimes c} \otimes \pi_S^*\mathcal{L}_S \otimes \pi_X^*\mathcal{O}_X(m).$$

Two such families will be called *equivalent*, if there are isomorphisms $\psi_{S,v}\colon\ \mathcal{E}_{S,v} \longrightarrow \mathcal{E}'_{S,v}$, $v \in V$, and $z_S\colon\ \mathcal{L}_S \longrightarrow \mathcal{L}'_S$, such that

$$\left(\det\left(\bigoplus_{v\in V} \psi_{S,v}^{\oplus\sigma_v}\right)^{\otimes c} \otimes \pi_S^*(z_S) \otimes \mathrm{id}_{\pi_X^*\mathcal{O}_X(m)}\right) \circ \tau \circ \left(\left(\left(\bigoplus_{v\in V} \psi_{S,v}^{\oplus\sigma_v}\right)^{\otimes a}\right)^{\oplus b}\right)^{-1} = \tau'.$$

The semistability condition

Let $(\mathcal{E}_v,\ v \in V)$ be a V-split sheaf. Then — as agreed upon before — a *weighted filtration of* $(\mathcal{E}_v,\ v \in V)$ is a pair $(\mathcal{E}_\bullet, \underline{\alpha})$, composed of a filtration

$$\mathcal{E}_\bullet\colon\ 0 \subsetneq (\mathcal{E}_1^v,\ v \in V) \subsetneq \cdots \subsetneq (\mathcal{E}_s^v,\ v \in V) \subsetneq (\mathcal{E}_v,\ v \in V)$$

of $(\mathcal{E}_v, v \in V)$ by non-trivial, proper, and saturated V-split subsheaves and a tuple $\underline{\alpha} = (\alpha_1, \ldots, \alpha_s)$ of positive rational numbers. Recall that

$$M_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_\bullet, \underline{\alpha}) = \sum_{j=1}^{s} \alpha_j \cdot (P_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v \in V) \cdot \mathrm{rk}_{\underline{\sigma}}(\mathcal{E}_j^v, v \in V) - P_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_j^v, v \in V) \cdot \mathrm{rk}_{\underline{\sigma}}(\mathcal{E}_v, v \in V)).$$

The number

$$\mu(\mathcal{E}_\bullet, \underline{\alpha}; \tau)$$

is defined by the formula in Remark 3.4.2(ii).

Now, we call a V-split sheaf $(\mathcal{E}_v, v \in V; \tau)$ with a decoration of type (a, b, c, m) *(semi) stable* (or more precisely $(\underline{\sigma}, \underline{\eta}, \delta)$-*(semi)stable*), if for every weighted filtration $(\mathcal{E}_\bullet, \underline{\alpha})$ of $(\mathcal{E}_v, v \in V)$,

$$M_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_\bullet, \underline{\alpha}) + \delta \cdot \mu(\mathcal{E}_\bullet, \underline{\alpha}; \tau)(\succeq)0.$$

The first main result

We define the functors

$$\underline{\mathrm{M}}^{(\underline{\sigma},\underline{\eta},\delta)-(\mathrm{s})\mathrm{s}}_{\underline{P}/a/b/c/m} \colon \underline{\mathrm{Sch}}_{\mathbb{C}} \longrightarrow \underline{\mathrm{Sets}}$$

$$S \longmapsto \left\{ \begin{matrix} \text{Equivalence classes of families of (semi)stable } V\text{-split} \\ \text{sheaves of type } \underline{P} \text{ with a decoration of type } (a, b, c, m) \end{matrix} \right\}.$$

Theorem 3.5.1. (i) *There exist a projective scheme* $\mathcal{M} := \mathcal{M}^{(\underline{\sigma},\underline{\eta},\delta)-\mathrm{ss}}_{\underline{P}/a/b/c/m}$ *and a natural transformation* $\theta \colon \underline{\mathrm{M}}^{(\underline{\sigma},\underline{\eta},\delta)-\mathrm{ss}}_{\underline{P}/a/b/c/m} \longrightarrow h_{\mathcal{M}}$, *such that for any other scheme* $\mathcal{M}'$ *and any other natural transformation* $\theta' \colon \underline{\mathrm{M}}^{(\underline{\sigma},\underline{\eta},\delta)-\mathrm{ss}}_{\underline{P}/a/b/c/m} \longrightarrow h_{\mathcal{M}'}$, *there is a unique morphism* $\zeta \colon \mathcal{M} \longrightarrow \mathcal{M}'$ *with* $\theta' = h_\zeta \circ \theta$. (ii) *The space* $\mathcal{M}$ *contains an open subscheme* $\mathcal{M}^{\mathrm{s}}$ *which is a coarse moduli scheme for the functor* $\underline{\mathrm{M}}^{(\underline{\sigma},\underline{\eta},\delta)-\mathrm{s}}_{\underline{P}/a/b/c/m}$.

3.6 *Applications to quiver problems*

Let $Q = (V, A, t, h)$ be a quiver with vertices $V = \{v_1, \ldots, v_t\}$, arrows $A = \{a_1, \ldots, a_u\}$, the tail map $t \colon A \longrightarrow V$, and the head map $h \colon A \longrightarrow V$. We assume that no multiple arrows occur. Fix a tuple of coherent sheaves $\underline{\mathcal{G}} = (\mathcal{G}_a, a \in A)$. An *(augmented) representation* of Q of type $(\underline{P}, \underline{\mathcal{G}})$ is a tuple $(\mathcal{E}_v, v \in V; f_a, a \in A; \varepsilon)$, consisting of

- a V-split sheaf $(\mathcal{E}_v, v \in V)$ of type $\underline{P}$,
- homomorphisms $f_a \colon \mathcal{G}_a \otimes \mathcal{E}_{t(a)} \longrightarrow \mathcal{E}_{h(a)}, a \in A$,
- a complex number ε,

such that either $\varepsilon \neq 0$ or one of the $f_a, a \in A$, is non-trivial. For simplicity, we will often drop the term 'augmented' in the following. Two representations $(\mathcal{E}_v, v \in V; f_a, a \in A; \varepsilon)$ and $(\mathcal{E}'_v, v \in V; f'_a, a \in A; \varepsilon)$ are called *equivalent*, if there are isomorphisms $\psi_v \colon \mathcal{E}_v \longrightarrow \mathcal{E}'_v, v \in V$, and $z \in \mathbb{C}^*$, such that

$$z \cdot (\psi_{h(a)} \circ f_a \circ (\mathrm{id}_{\mathcal{G}_a} \otimes \psi_{t(a)})^{-1}) = f'_a, \quad a \in A, \quad \text{and} \quad z \cdot \varepsilon = \varepsilon'.$$

A *family of representations of Q of type $(\underline{P}, \underline{\mathcal{G}})$ parametrized by S* is a tuple $(\mathcal{E}_{S,v}, v \in V; f_{S,a}, a \in A; \mathcal{L}_S, \varepsilon_S)$ which consists of

- S-flat families $\mathcal{E}_{S,v}$ on $S \times X$ of torsion free sheaves on X with Hilbert polynomial P_v, $v \in V$;
- a line bundle $\mathcal{L}_S$ on S;
- a section $\varepsilon_S \in H^0(S, \mathcal{L}_S)$;
- homomorphisms $f_{S,a}\colon\ \pi_X^*\mathcal{G}_a \otimes \mathcal{E}_{S,t(a)} \longrightarrow \mathcal{E}_{S,h(a)} \otimes \pi_S^*\mathcal{L}_S$.

An *equivalence of the families* $(\mathcal{E}^1_{S,v}, v \in V; f^1_{S,a}, a \in A; \mathcal{L}^1_S, \varepsilon^1_S)$ and $(\mathcal{E}^2_{S,v}, v \in V; f^2_{S,a}, a \in A; \mathcal{L}^2_S, \varepsilon^2_S)$ consists of an isomorphism $z_S\colon\ \mathcal{L}^1_S \longrightarrow \mathcal{L}^2_S$ and isomorphisms $\psi_v\colon\ \mathcal{E}^1_{S,v} \longrightarrow \mathcal{E}^2_{S,v}$, $v \in V$, such that

$$f^2_{S,a} = (\psi_{h(a)} \otimes \pi_S^*(z_S)) \circ f^1_{S,a} \circ (\mathrm{id}_{\pi_X^*\mathcal{G}_a} \otimes \psi_{t(a)})^{-1},\ a \in A,\ z_S \circ \varepsilon^1_S = \varepsilon^2_S.$$

Associated decorations

Fix the parameters $\underline{\sigma} = (\sigma_v \in \mathbb{Z}_{>0}, v \in V)$. There are $m \geq 0$ and $b > 0$, such that we have surjections $\nu_a\colon\ \mathcal{O}_C(-m)^{\oplus b} \longrightarrow \mathcal{G}_a$ for all $a \in A$, and an embedding $v_0\colon\ \mathcal{O}_X \subset \mathcal{O}_X(m)$. Set $M := \bigoplus_{v\in V} \mathbb{C}^{\sigma_v \cdot r_v}$. Next, decompose the $\mathrm{GL}(M)$-module $Z := (M^{\otimes \dim M} \otimes (\bigwedge^{\dim M} M)^{\otimes -1})^{\oplus(b+1)}$ as $\mathrm{Hom}\,(M, M)^{\oplus b} \oplus \mathbb{C} \oplus Z'$.

Let $(\mathcal{E}_{S,v}, v \in V; f_{S,a}, a \in A; \mathcal{L}_S, \varepsilon)$ be a family of representations on $S \times X$ and $\iota\colon\ U \subset S \times X$ the maximal open subset over which all the $\mathcal{E}_{S,v}$, $v \in V$, are locally free. Then, the restrictions of the $f_{S,a}^{\oplus\sigma_{t(a)}\cdot\sigma_{h(a)}}$ to U together with the pullback of ε_S to U may be interpreted as a homomorphism

$$\tau'\colon\ \underline{\mathrm{Hom}}\left(\bigoplus_{v\in V} \mathcal{E}^{\oplus\sigma_v}_{S,v|U}, \bigoplus_{v\in V} \mathcal{E}^{\oplus\sigma_v}_{S,v|U}\right)^{\oplus b} \oplus \mathcal{O}_U \longrightarrow \iota^*(\pi_S^*\mathcal{L}_S \otimes \pi_X^*\mathcal{O}_X(m)).$$

The splitting of Z yields a natural projection

$$\det\,(\mathcal{E}^{\oplus\sigma_v}_{S,v|U})^{\otimes -1} \otimes \left(\left(\bigoplus_{v\in V} \mathcal{E}^{\oplus\sigma_v}_{S,v|U}\right)^{\otimes \sum_{v\in V} \sigma_v\cdot r_v}\right)^{\oplus(b+1)}$$

$$\longrightarrow \underline{\mathrm{Hom}}\left(\bigoplus_{v\in V} \mathcal{E}^{\oplus\sigma_v}_{S,v|U}, \bigoplus_{v\in V} \mathcal{E}^{\oplus\sigma_v}_{S,v|U}\right)^{\oplus b} \oplus \mathcal{O}_U,$$

so that we get

$$\tau''\colon\ \left(\left(\bigoplus_{v\in V} \mathcal{E}^{\oplus\sigma_v}_{S,v|U}\right)^{\otimes \sum_{v\in V} \sigma_v\cdot r_v}\right)^{\oplus(b+1)} \longrightarrow \iota^*(\det\,(\mathcal{E}^{\oplus\sigma_v}_{S,v}) \otimes \pi_S^*\mathcal{L}_S \otimes \pi_X^*\mathcal{O}_X(m)).$$

We finally define

$$\begin{aligned}\tau_S\colon \left(\left(\bigoplus_{v\in V} \mathcal{E}^{\oplus\sigma_v}_{S,v}\right)^{\otimes \sum_{v\in V} \sigma_v\cdot r_v}\right)^{\oplus(b+1)} &\longrightarrow \iota_*\left(\left(\bigoplus_{v\in V} \mathcal{E}^{\oplus\sigma_v}_{S,v|U}\right)^{\otimes \sum_{v\in V} \sigma_v\cdot r_v}\right)^{\oplus(b+1)} \\ &\stackrel{\iota_*(\tau'')}{\longrightarrow} \iota_*\iota^*(\det\,(\mathcal{E}^{\oplus\sigma_v}_{S,v}) \otimes \pi_S^*\mathcal{L}_S \otimes \pi_X^*\mathcal{O}_X(m)) \\ &\stackrel{\text{Prop. A.1.1}}{=} \det\,(\mathcal{E}^{\oplus\sigma_v}_{S,v}) \otimes \pi_S^*\mathcal{L}_S \otimes \pi_X^*\mathcal{O}_X(m).\end{aligned}$$

We call $(\mathcal{E}_{S,v}, v \in V; \tau_S)$ the *associated family of V-split sheaves with a decoration of type* $(s, b+1, 1, m)$, $s := \sum_{v\in V} \sigma_v \cdot r_v$. When S is just a point, we simply speak of the *associated decoration* τ.

The semistability condition

Fix the same data as before. We call a representation $(\mathcal{E}_v, v \in V; f_a, a \in A; \varepsilon)$ $(\underline{\sigma}, \underline{\eta}, \delta)$-*(semi)stable*, if the V-split sheaf $(\mathcal{E}_v, v \in V; \tau)$ with the associated decoration is $(\underline{\sigma}, \underline{\eta}, \delta)$-(semi)stable.

The second main result

Define the functors

$$\underline{\mathrm{R}}(Q)^{(\underline{\sigma},\underline{\eta},\delta)-(\mathrm{s})\mathrm{s}}_{\underline{P}/\underline{\mathcal{G}}}\colon \quad \underline{\mathrm{Sch}}_{\mathbb{C}} \longrightarrow \underline{\mathrm{Sets}}$$
$$S \longmapsto \left\{ \begin{matrix} \text{Equivalence classes of families of } (\underline{\sigma}, \underline{\eta}, \delta)\text{-} \\ \text{(semi)stable representations of } Q \text{ of type } (\underline{P}, \underline{\mathcal{G}}) \end{matrix} \right\}.$$

Theorem 3.6.1. (i) *There exist a projective scheme* $\mathcal{R} := \mathcal{R}(Q)^{(\underline{\sigma},\underline{\eta},\delta)-\mathrm{ss}}_{\underline{P}/\underline{\mathcal{G}}}$ *and a natural transformation* $\theta\colon \underline{\mathrm{R}}(Q)^{(\underline{\sigma},\underline{\eta},\delta)-\mathrm{ss}}_{\underline{P}/\underline{\mathcal{G}}} \longrightarrow h_{\mathcal{R}}$, *such that for any other scheme* $\mathcal{R}'$ *and any other natural transformation* $\theta'\colon \underline{\mathrm{R}}(Q)^{(\underline{\sigma},\underline{\eta},\delta)-\mathrm{ss}}_{\underline{P}/\underline{\mathcal{G}}} \longrightarrow h_{\mathcal{R}'}$, *there exists a unique morphism* $\zeta\colon \mathcal{R} \longrightarrow \mathcal{R}'$ *with* $\theta' = h_\zeta \circ \theta$. (ii) *The scheme* $\mathcal{R}$ *contains an open subscheme* $\mathcal{R}^{\mathrm{s}}$ *which is a coarse moduli scheme for the functor* $\underline{\mathrm{R}}(Q)^{(\underline{\sigma},\underline{\eta},\delta)-\mathrm{s}}_{\underline{P}/\underline{\mathcal{G}}}$.

3.7 *Behaviour for large* δ

Intuitively, one would like to have a semistability concept for representations of quivers which poses conditions on subrepresentations only. However, as illustrated in [24] for the example of the quiver consisting of one vertex and an arrow, connecting the vertex to itself, this property cannot be expected for general δ. However, as in the case of the aforementioned quiver, for large δ, the semistability concept will stabilize to one which is a condition on subrepresentations only. Another nice feature is that, for large δ, one has a generalized Hitchin map. Our first result is the following:

Theorem 3.7.1. *Fix the data* $\underline{\sigma}$, $\underline{\eta}$, $\underline{P}$, *and* $\underline{\mathcal{G}}$. *Let* δ *be a positive polynomial of degree exactly* $\dim X - 1$. *Then, there exists a natural number* n_∞, *such that for all* $n \geq n_\infty$, *the following two conditions on a representation* $(\mathcal{E}_v, v \in V; f_a, a \in A; \varepsilon)$ *of* Q *of type* $(\underline{P}, \underline{\mathcal{G}})$ *are equivalent*

1. *The representation* $(\mathcal{E}_v, v \in V; f_a, a \in A; \varepsilon)$ *is* $(\underline{\sigma}, \underline{\eta}/n, n\cdot\delta)$-*(semi)stable*, $\underline{\eta}/n := (\eta_v/n, v \in V)$.
2. (a) *For any non-trivial, proper subrepresentation* $(\mathcal{F}_v, v \in V)$ (*i.e., V-split subsheaf, such that* $f_a(\mathcal{G}_a \otimes \mathcal{F}_{t(a)}) \subset \mathcal{F}_{h(a)}$ *for all* $a \in A$) *one has*

$$\frac{P_{\underline{\sigma},\underline{\chi}}(\mathcal{F}_v, v \in V)}{\mathrm{rk}_{\underline{\sigma}}(\mathcal{F}_v, v \in V)} \quad (\preceq) \quad \frac{P_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v \in V)}{\mathrm{rk}_{\underline{\sigma}}(\mathcal{E}_v, v \in V)},$$

and

(b) *either* $\varepsilon \neq 0$, *or the restriction of* $(\mathcal{E}_v, v \in V; f_a, a \in A; \varepsilon)$ *to a general point* $x \in X$ *in the open subset where all the* $\mathcal{E}_v$, $v \in V$, *are locally free is semistable.*

We call a representation $(\mathcal{E}_v, v \in V; f_a, a \in A; \varepsilon)$ of Q of type $(\underline{P}, \underline{\mathcal{G}})$ $(\underline{\sigma}, \underline{\chi})$-*(semi)stable*, if it satisfies Condition 2 in Theorem 3.7.1 with $(\preceq)$. Note that, for representations of the form $(\mathcal{E}_v, v \in V; f_a, a \in A; \varepsilon = 1)$, this is exactly the 'Gieseker-analogue' of the semistability definition given by Álvarez-Cónsul and García-Prada [1]. Observe that for $(\underline{\sigma}, \underline{\chi})$-semistable representations, one has the concepts of a *Jordan–Hölder filtration*, *the associated graded object*, and *S-equivalence*. Therefore, one can also speak of $(\underline{\sigma}, \underline{\chi})$-*polystable representations*.

Invariants of quivers and the generalized Hitchin map

Recall that we may find $b > 0$ and $m \geq 0$, such that there are surjections $\nu_a\colon \mathcal{O}_X(-m)^{\oplus b} \longrightarrow \mathcal{G}_a$ for all $a \in A$ as well as an embedding $\nu_0\colon \mathcal{O}_X \longrightarrow \mathcal{O}_X(m)$. Therefore, we now look at the quiver $Q^b := (V, A^b := A \times \{1, \ldots, b\}, t, h)$ where the tail and head maps are given by the projection onto the first factor followed by the tail and head map of Q. In other words, any arrow in Q is replaced by b copies of the same arrow. We choose $\underline{r} = (r_v, v \in V)$ as the dimension vector. The variety of representations with this dimension vector is thus

$$\operatorname{Rep}_{\underline{r}}(Q^b) := \bigoplus_{a \in A^b} \operatorname{Hom}\left(\mathbb{C}^{r_{t(a)}}, \mathbb{C}^{r_{h(a)}}\right).$$

The variety $\operatorname{Rep}_{\underline{r}}(Q^b)$ comes with an action of the group $\mathsf{X}_{v \in V} \operatorname{GL}(r_v, \mathbb{C})$. By the work of LeBruyn and Procesi [13], one knows explicit generators for the ring of $(\mathsf{X}_{v \in V} \operatorname{GL}(r_v, \mathbb{C}))$-invariant regular functions on $\operatorname{Rep}_{\underline{r}}(Q^b)$. To state the result, let $\mathfrak{o} = (a_1, \ldots, a_l)$ be an *oriented cycle*, i.e., a sequence of arrows $a_1, \ldots, a_l$, such that $h(a_i) = t(a_{i-1})$, $i = 2, \ldots, l$, and $h(a_1) = t(a_l) =: v(\mathfrak{o})$. We call l the *length of the cycle*. For any such cycle and any point $x \in \operatorname{Rep}_{\underline{r}}(Q^b)$, we get an endomorphism $f_{x,\mathfrak{o}}\colon \mathbb{C}^{r_{v(\mathfrak{o})}} \longrightarrow \mathbb{C}^{r_{v(\mathfrak{o})}}$. We then define

$$\begin{aligned} t_{\mathfrak{o}}\colon \operatorname{Rep}_{\underline{r}}(Q^b) &\longrightarrow \mathbb{C} \\ x &\longmapsto \operatorname{Trace}(f_{x,\mathfrak{o}}). \end{aligned}$$

The function $t_{\mathfrak{o}}$ is obviously invariant under the $(\mathsf{X}_{v \in V} \operatorname{GL}(r_v, \mathbb{C}))$-action. The result of [13] states that the invariants of the form $t_{\mathfrak{o}}$, $\mathfrak{o}$ an oriented cycle, generate the ring of invariants $\mathcal{R}_{\underline{r}}(Q^b) := \mathbb{C}[\operatorname{Rep}_{\underline{r}}(Q^b)]^{\mathsf{X}_{v \in V} \operatorname{GL}(r_v, \mathbb{C})}$. Moreover, one may restrict to oriented cycles of length at most $\left(\sum_{v \in V} r_v\right)^2 + 1$. We also look at the affine variety

$$\operatorname{Rep}^{\varepsilon}_{\underline{r}}(Q^b) := \operatorname{Rep}_{\underline{r}}(Q^b) \oplus \mathbb{C}.$$

This is also a $(\mathsf{X}_{v \in V} \operatorname{GL}(r_v, \mathbb{C}))$-variety, the action on $\mathbb{C}$ being trivial. Denote by t_0 the projection onto the second factor. This is a $(\mathsf{X}_{v \in V} \operatorname{GL}(r_v, \mathbb{C}))$-invariant function, and the above result implies

$$\begin{aligned} \mathcal{R}^{\varepsilon}_{\underline{r}}(Q^b) &:= \mathbb{C}[\operatorname{Rep}^{\varepsilon}_{\underline{r}}(Q^b)]^{\mathsf{X}_{v \in V} \operatorname{GL}(r_v, \mathbb{C})} \\ &= \mathbb{C}\left[t_0;\ t_{\mathfrak{o}}, \mathfrak{o} \text{ an oriented cycle of length } \leq \left(\sum_{v \in V} r_v\right)^2 + 1 \right]. \end{aligned}$$

Next, assign degree one to t_0 and degree length of $\mathfrak{o}$ to $t_{\mathfrak{o}}$, $\mathfrak{o}$ an oriented cycle. Then, $\mathcal{R}^{\varepsilon}_{\underline{r}}(Q^b)$ is a graded ring, and $\text{Proj}(\mathcal{R}^{\varepsilon}_{\underline{r}}(Q^b))$ identifies with the $(\mathbb{C}^* \times \mathsf{X}_{v\in V}\, \text{GL}(r_v, \mathbb{C}))$-quotient of $\text{Rep}^{\varepsilon}_{\underline{r}}(Q^b)$ where $\mathbb{C}^*$ acts by scalar multiplication. We may choose a degree d such that the subring $\mathcal{R}^{(d)} \subset \mathcal{R}^{\varepsilon}_{\underline{r}}(Q^b)$ of elements the degree of which is a multiple of d is generated by elements of degree d, say, $\mathfrak{i}_0, \ldots, \mathfrak{i}_q$ (see [18], § III.8, Lemma). This yields an embedding

$$\text{Proj}(\mathcal{R}^{\varepsilon}_{\underline{r}}(Q^b)) \hookrightarrow \mathbb{P}_q.$$

Now, we return to the setting of representations of Q of type $(\underline{P}, \underline{\mathcal{G}})$ where we fix b, m, ν_a, $a \in A$, and ν_0 as before. Set $\mathbb{H}(Q, \underline{P}, \underline{\mathcal{G}}) := \mathbb{P}(H^0(\mathcal{O}_X(d \cdot m))^{\oplus q})$.

Let S be a scheme, and $(\mathcal{E}_{S,v}, v \in V; f_{S,a}, a \in A; \mathcal{L}_S, \varepsilon)$ a family of $(\underline{\sigma}, \underline{\chi})$-semistable representations of Q of type $(\underline{P}, \underline{\mathcal{G}})$ parametrized by S. Denote by $\iota\colon\ U \subset S \times X$ the maximal open subset where all the $\mathcal{E}_{S,v}$, $v \in V$, are locally free. To the invariant t_0 corresponds the homomorphism

$$t_0''\colon\ \mathcal{O}_{S\times X} \xrightarrow{\pi_S^*\varepsilon_S} \pi_S^*\mathcal{L}_S \xrightarrow{\text{id}_{\pi_S^*\mathcal{L}_S}\otimes \pi_X^*\nu_0} \pi_S^*\mathcal{L}_S \otimes \pi_X^*\mathcal{O}_X(m).$$

Furthermore, using the quotients ν_a, $a \in A$, for any oriented cycle $\mathfrak{o}$ of length l, we get a homomorphism

$$f_{\mathfrak{o}}'\colon\ \mathcal{E}_{S,v(\mathfrak{o})} \longrightarrow \mathcal{E}_{S,v(\mathfrak{o})} \otimes (\pi_S^*\mathcal{L}_S \otimes \pi_X^*\mathcal{O}_X(m))^{\otimes l}.$$

If we restrict $f_{\mathfrak{o}}'$ to U and take traces, we obtain a section

$$t_{\mathfrak{o}}'\colon\ \mathcal{O}_U \longrightarrow \iota^*(\pi_S^*\mathcal{L}_S \otimes \pi_X^*\mathcal{O}_X(m))^{\otimes l}.$$

By Proposition A.1.1, this extends to

$$t_{\mathfrak{o}}''\colon\ \mathcal{O}_{S\times X} \longrightarrow (\pi_S^*\mathcal{L}_S \otimes \pi_X^*\mathcal{O}_X(m))^{\otimes l}.$$

Therefore, any invariant $\mathfrak{i}_j$, $j = 0, \ldots, q$, provides a section

$$\begin{aligned} i_j := i_j(\mathcal{E}_{S,v}, v \in V; f_{S,a}, a \in A; \mathcal{L}_S, \varepsilon)\colon\ &\mathcal{O}_{S\times X} \\ &\longrightarrow (\pi_S^*\mathcal{L}_S \otimes \pi_X^*\mathcal{O}_X(m))^{\otimes d}, \quad j = 0, \ldots, q. \end{aligned}$$

Condition 2(b) now grants that one of the homomorphisms i_j, $j = 0, \ldots, q$, will be non-zero. Hence, we get a morphism

$$\text{Hit}(\mathcal{E}_{S,v}, v \in V; f_{S,a}, a \in A; \mathcal{L}_S, \varepsilon)\colon\ S \longrightarrow \mathbb{H} := \mathbb{H}(Q, \underline{P}, \underline{\mathcal{G}})$$

with $\text{Hit}^*\mathcal{O}_{\mathbb{H}}(1) = \mathcal{L}_S^{\otimes d}$.

The third main result

This time, we look at the functors

$$\underline{\text{R}}(Q)^{(\underline{\sigma},\underline{\chi})-(\text{s})\text{s}}_{\underline{P}/\underline{\mathcal{G}}}\colon\ \underline{\text{Sch}}_{\mathbb{C}} \longrightarrow \underline{\text{Sets}}$$

$$S \longmapsto \left\{ \begin{matrix} \text{Equivalence classes of families of } (\underline{\sigma}, \underline{\chi})\text{-} \\ \text{(semi)stable representations of } Q \text{ of type } (\underline{P}, \underline{\mathcal{G}}) \end{matrix} \right\}.$$

Theorem 3.7.2. (i) *There exist a projective scheme* $\mathcal{R} := \mathcal{R}(Q)^{(\underline{\sigma},\underline{\chi})-\mathrm{ss}}_{\underline{P}/\underline{\mathcal{G}}}$ *and a natural transformation* $\theta\colon\ \underline{\mathrm{R}}(Q)^{(\underline{\sigma},\underline{\chi})-\mathrm{ss}}_{\underline{P}/\underline{\mathcal{G}}} \longrightarrow h_{\mathcal{R}}$, *such that for any other scheme* $\mathcal{R}'$ *and any other natural transformation* $\theta'\colon\ \underline{\mathrm{R}}(Q)^{(\underline{\sigma},\underline{\chi})-\mathrm{ss}}_{\underline{P}/\underline{\mathcal{G}}} \longrightarrow h_{\mathcal{R}'}$, *there exists a unique morphism* $\zeta\colon\ \mathcal{R} \longrightarrow \mathcal{R}'$ *with* $\theta' = h_\zeta \circ \theta$. (ii) *The scheme* $\mathcal{R}$ *contains an open subscheme* $\mathcal{R}^{\mathrm{s}}$ *which is a coarse moduli scheme for the functor* $\underline{\mathrm{R}}(Q)^{(\underline{\sigma},\underline{\chi})-\mathrm{s}}_{\underline{P}/\underline{\mathcal{G}}}$. (iii) *The closed points of* $\mathcal{R}$ *are in bijection to the set of S-equivalence classes of* $(\underline{\sigma}, \underline{\chi})$*-semistable representations of* Q *of type* $(\underline{P}, \underline{Q})$*, or, equivalently, to the set of isomorphism classes of* $(\underline{\sigma}, \underline{\chi})$*-polystable representations of* Q *of type* $(\underline{P}, \underline{Q})$. (iv) *There is a generalized Hitchin morphism*

$$\mathrm{Hit}(Q, \underline{P}, \underline{\mathcal{G}})\colon\ \mathcal{R} \longrightarrow \mathbb{H}(Q, \underline{P}, \underline{\mathcal{G}}).$$

Note that the theorem in the introduction now follows by taking $\mathcal{D}$ and $\mathbb{D}$ as the open subscheme $\varepsilon = 1$ in $\mathcal{R}$ and $\mathbb{H}$, respectively (cf. Remark 3.4.2).

4. The proofs

4.1 *Proof of Theorem* 3.5.1

In order to prove Theorem 3.5.1, one can copy almost word by word the proofs in [24], §2.3.6, or [8]. The only point which has to be given special attention and which is indeed rather tricky is the correct choice of a linearization on the parameter space. We will, therefore, construct the parameter space, give the linearization of the respective group action, and show in a sample computation that it is the correct one.

The parameter space

We denote by $\mathfrak{A}_v$, $v \in V$, the union of those components of $\mathrm{Pic}(X)$ which contain line bundles of the form $\det(\mathcal{E}_v)$ where $(\mathcal{E}_v, v \in V; \tau)$ is a semistable V-split sheaf of type $\underline{P}$ with a decoration of type (a, b, c, m). We also set $\mathfrak{A} := \mathsf{X}_{v\in V}\mathfrak{A}_v$. By the usual boundedness arguments, we can find an l_0, such that for all $l \geq l_0$, all semistable V-split sheaves $(\mathcal{E}_v, v \in V; \tau)$ of type $\underline{P}$ with a decoration of type (a, b, c, m), all $v \in V$, all $[\mathcal{L}] \in \mathfrak{A}_v$, and all $\mathcal{N} = \bigotimes_{v\in V} \mathcal{L}_v^{\otimes\sigma_v}$ with $[\mathcal{L}_v] \in \mathfrak{A}_v$, $v \in V$,

- $\mathcal{E}_v(l)$ is globally generated and $H^i(\mathcal{E}_v(l)) = 0$ for all $i > 0$;
- $\mathcal{L}(r_v \cdot l)$ is globally generated and $H^i(\mathcal{L}(r_v \cdot l)) = 0$ for all $i > 0$;
- $\mathcal{N}^{\otimes c}(a \cdot l)$ is globally generated and $H^i(\mathcal{N}^{\otimes c}(a \cdot l)) = 0$ for all $i > 0$.

We fix such an l, and set $p_v := P_v(l)$, $v \in V$, and $p := \sum_{v\in V} \sigma_v \cdot p_v$. Moreover, we choose vector spaces W_v of dimension p_v and let $\mathfrak{Q}_v^0$ be the quasi-projective quot scheme parametrizing quotients $q\colon\ W_v \otimes \mathcal{O}_X(-l) \longrightarrow \mathcal{F}$ with $\mathcal{F}$ a torsion free coherent $\mathcal{O}_X$-module with Hilbert polynomial P_v and $H^0(q(l))$ an isomorphism, $v \in V$. Let $\mathfrak{E}_v$ be the universal quotient on $\mathfrak{Q}_v^0 \times X$, $v \in V$, and

$$\mathfrak{E}^{\mathrm{total}} := \bigoplus_{v\in V} \pi^*_{\mathfrak{Q}_v^0}\mathfrak{E}_v^{\oplus\sigma_v}$$

be the resulting sheaf on $(\mathsf{X}_{v\in V}\ \mathfrak{Q}_v^0) \times X$. Define $M := \bigoplus_{v\in V} W_v^{\oplus\sigma_v}$ and

$$\mathfrak{P} := \mathbb{P}(((M^{\otimes a})^{\oplus b})^\vee \otimes \pi_{(\mathsf{X}_{v\in V}\ \mathfrak{Q}_v^0)*}(\det(\mathfrak{E}^{\mathrm{total}})^{\otimes c} \otimes \pi_X^*\mathcal{O}_X(a \cdot l))).$$

This is a projective bundle over $\bigtimes_{v\in V} \mathfrak{Q}_v^0$, and the parameter space $\mathfrak{M}$ is constructed in the usual way as a closed subscheme of $\mathfrak{P}$. In particular, it is projective over $\bigtimes_{v\in V} \mathfrak{Q}_v^0$. Furthermore, $\mathfrak{M}$ comes with an action of the group $(\bigtimes_{v\in V} \mathrm{GL}(W_v))/\mathbb{C}^*$, $\mathbb{C}^*$ being diagonally embedded. We define

$$\begin{aligned}\widetilde{G} &:= \left(\bigtimes_{v\in V} \mathrm{GL}(W_v)\right) \cap \mathrm{SL}(M)\\ &= \left\{(h_1, \dots, h_t) \in \bigtimes_{v\in V} \mathrm{GL}(W_v) \mid \det(h_1)^{\sigma_{v_1}} \cdot \ldots \cdot \det(h_t)^{\sigma_{v_t}} = 1\right\}.\end{aligned}$$

The group $\widetilde{G}$ maps with finite kernel onto $(\bigtimes_{v\in V} \mathrm{GL}(W_v))/\mathbb{C}^*$, whence we may restrict our attention to the action of $\widetilde{G}$.

The linearization of the above group action will be induced via a Gieseker morphism to some other scheme. For this, we fix Poincaré line bundles $\mathcal{P}_v$ over $\mathfrak{A}_v \times X$, $v \in V$, and set

$$\mathfrak{G}_v := \mathbb{P}\left(\left(\bigwedge^{r_v} W_v\right)^{\vee} \otimes \pi_{\mathfrak{A}_v *}(\mathcal{P}_v \otimes \pi_X^* \mathcal{O}_X(r_v \cdot l))\right).$$

Choosing $\mathcal{P}_v$ appropriately, we may assume that $\mathcal{O}_{\mathfrak{G}_v}(1)$ is very ample for all $v \in V$. On $\mathfrak{A} \times X$, we get the line bundle

$$\mathcal{P} := \bigotimes_{v\in V} \pi^*_{\mathfrak{A}_v \times X} \mathcal{P}_v^{\otimes \sigma_v}.$$

Then, we define

$$\mathfrak{P}' := \mathbb{P}(((M^{\otimes a})^{\oplus b})^{\vee} \otimes \pi_{\mathfrak{A}*}(\mathcal{P}^{\otimes c} \otimes \pi_X^* \mathcal{O}_X(a \cdot l)))$$

as a projective bundle over $\mathfrak{A}$. Again, $\mathcal{O}_{\mathfrak{P}'}(1)$ can be assumed to be ample. We now have a $\widetilde{G}$-equivariant and injective morphism

$$\Gamma\colon\ \mathfrak{M} \longrightarrow \mathfrak{P}' \times \bigtimes_{v\in V} \mathfrak{G}_v.$$

For a given $\beta \in \mathbb{Z}_{>0}$, and $\kappa_v \in \mathbb{Z}_{>0}$, $v \in V$, there is a natural linearization of the $\widetilde{G}$-action on $\mathfrak{P}' \times \bigtimes_{v\in V} \mathfrak{G}_v$ in the ample line bundle $\mathcal{O}(\beta; \kappa_v, v \in V)$. This may be altered by any character of $\bigtimes_{v\in V} \mathrm{GL}(W_v)$. Let $d := \delta(l)$, $x_v := -\chi_v(l)/d$, $x'_v := r_v x_v / p_v$, $v \in V$,

$$\varepsilon := \frac{p - a \cdot d}{r \cdot d}, \quad \varepsilon_v := \sigma_v - \frac{x_v}{\varepsilon} = \sigma_v + \frac{r \cdot \chi_v(l)}{p - a \cdot d}, \quad v \in V,$$

and

$$x''_v := \varepsilon \cdot \sigma_v \cdot \left(\frac{r}{p} - \frac{r_v}{p_v}\right), \quad v \in V.$$

Remark 4.1.1. To be very precise, the quantities ε and ε_v, $v \in V$, are functions in l. Since $p = P(l)$ is a positive polynomial of degree $\dim X$ and both δ and χ_v are polynomials of degree at most $\dim X - 1$, it is clear that ε and ε_v, $v \in V$, will be positive for $l \gg 0$, i.e., the line bundle in which the action is linearized is really ample.

Now, we choose $\beta \in \mathbb{Z}_{>0}$ and $\kappa_v \in \mathbb{Z}_{>0}$ such that

$$\frac{\kappa_v}{\beta} = \varepsilon \cdot \varepsilon_v, \quad v \in V.$$

We modify the linearization of the $\widetilde{G}$-action on $\mathsf{X}_{v\in V} \mathfrak{G}_v$ in $\mathcal{O}(\kappa_v, v \in V)$ by a character, such that $\mathbb{C}^{*l} = C^* \cdot \mathrm{id}_{W_{v_1}} \times \cdots \times \mathbb{C}^* \cdot \mathrm{id}_{W_{v_l}}$ acts via a $(z_v, v \in V) \longmapsto \Pi_{v\in V} z_v^{p_v \cdot e_v}$ with

$$e_v := \beta \cdot (x'_v + x''_v), \quad v \in V.$$

Note that this character is just the restriction of the character

$$(m_1, \ldots, m_l) \longmapsto \det(m_1)^{e_{v_1}} \cdots \det(m_l)^{e_{v_l}}$$

of $\mathsf{X}_{v\in V}\mathrm{GL}(W_v)$ to the center $\mathcal{Z}$. We work with the resulting linearization of the $\widetilde{G}$-action on $\mathfrak{P}' \times \mathsf{X}_{v\in V} \mathfrak{G}_v$ in $\mathcal{O}(\beta; \kappa_v, v \in V)$.

A sample computation

In order to illustrate that our choice of the linearization is accurate, we go through part of the calculations which are analogous to those in §2.3 of [24]. More precisely, we show the following: Let $m = (q_v\colon\ W_v \otimes \mathcal{O}_X(-l) \longrightarrow \mathcal{E}_v, v \in V; \tau)$ be a point in the parameter space $\mathfrak{M}$, such that $\Gamma(m)$ is (semi)stable with respect to the chosen linearization in $\mathcal{O}(\beta; \kappa_v, v \in V)$, then $(\mathcal{E}_v, v \in V; \tau)$ is a (semi)stable V-split sheaf with a decoration of type (a, b, c, m). First, as in [24], one verifies that the (semi)stability condition has to be checked only for those weighted filtrations $(\mathcal{E}_\bullet, \underline{\alpha})$ which satisfy

$$\begin{aligned}&\mathcal{E}^v_j(l) \text{ is globally generated and } H^i(\mathcal{E}^v_j(l)) = 0,\\ &\quad i > 0,\ j = 1, \ldots, s, \quad v \in V.\end{aligned}$$

For weighted filtrations of that type, we have to prove that

$$M_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_\bullet, \underline{\alpha})(l) + \delta(l) \cdot \mu(\mathcal{E}_\bullet, \underline{\alpha}; \tau)(\geq)0. \tag{3}$$

Define $\underline{\gamma} = (\gamma_1, \ldots, \gamma_{s+1})$ by the conditions

$$\frac{\gamma_{j+1} - \gamma_j}{p} = \alpha_j, \quad j = 1, \ldots, s,$$

and, setting $\mathcal{E}^{\mathrm{total}}_j := \bigoplus_{v\in V} \mathcal{E}^{v,\oplus\sigma_v}_j$, $j = 1, \ldots, s+1$,

$$\sum_{j=1}^{s+1} \gamma_j \cdot (h^0(\mathcal{E}^{\mathrm{total}}_j(l)) - h^0(\mathcal{E}^{\mathrm{total}}_{j-1}(l))) = 0.$$

Then, we obtain a weighted filtration $(\mathcal{E}_\bullet, \underline{\gamma})$ and, thus, weighted filtrations $(\widehat{\mathcal{E}}^v_\bullet, \underline{\gamma}^v)$ of $\mathcal{E}_v$, $v \in V$. Next, we choose bases $\underline{w}^v = (w^v_1, \ldots, w^v_{p_v})$ of W_v with

$$\langle w^v_1, \ldots, w^v_{h^0(\widehat{\mathcal{E}}^v_j(l))}\rangle = H^0(\widehat{\mathcal{E}}^v_j(l)), \quad j = 1, \ldots, s_v,\ v \in V,$$

and set

$$\underline{\widetilde{\gamma}}^v := (\underbrace{\gamma_1^v, \dots, \gamma_1^v}_{h^0(\widehat{\mathcal{E}}_j^v(l))\times}, \dots, \underbrace{\gamma_{s_v+1}^v, \dots, \gamma_{s_v+1}^v}_{(p_v - h^0(\widehat{\mathcal{E}}_{s_v}^v(l)))\times}).$$

This yields the one-parameter subgroup

$$\lambda := (\lambda(\underline{w}^{v_1}, \underline{\widetilde{\gamma}}^{v_1}), \dots, \lambda(\underline{w}^{v_t}, \underline{\widetilde{\gamma}}^{v_t}))$$

of $\widetilde{G}$. Now, with $\Gamma(m) = ([m']; [m_v], v \in V)$,

$$\begin{aligned}
\frac{\mu(\lambda, \Gamma(m))}{\beta} &= \mu(\lambda, [m']) \\
&+ \varepsilon \cdot \underbrace{\left(\sum_{v \in V} \varepsilon_v \cdot \mu(\lambda, [m_v]) - \sum_{v \in V} \sigma_v \left(\frac{r_v}{p_v} - \frac{r}{p} \right) \cdot \left(\sum_{j=1}^{s_v+1} \gamma_j^v (h^0(\widehat{\mathcal{E}}_j^v(l)) - h^0(\widehat{\mathcal{E}}_{j-1}^v(l))) \right) \right)}_{=:A} \\
&+ \underbrace{\sum_{v \in V} \left(x'_v \cdot \sum_{j=1}^{s_v+1} \gamma_j^v (h^0(\widehat{\mathcal{E}}_j^v(l)) - h^0(\widehat{\mathcal{E}}_{j-1}^v(l))) \right)}_{=:B}.
\end{aligned}$$

Observe

$$\begin{aligned}
&\sum_{j=1}^{s_v+1} \gamma_j^v (h^0(\widehat{\mathcal{E}}_j^v(l)) - h^0(\widehat{\mathcal{E}}_{j-1}^v(l))) \\
&\quad = \gamma_{s_v+1}^v \cdot p_v - \sum_{j=1}^{s_v} (\gamma_{j+1}^v - \gamma_j^v) \cdot h^0(\widehat{\mathcal{E}}_j^v(l)) \\
&\quad = \gamma_{s+1} \cdot p_v - \sum_{j=1}^{s_v} (\gamma_{j+1}^v - \gamma_j^v) \cdot P(\widehat{\mathcal{E}}_j^v)(l).
\end{aligned}$$

As $\sum_{v \in V} x'_v \cdot p_v = 0$, we find

$$B = -\sum_{v \in V} \left(x'_v \cdot \sum_{j=1}^{s_v} (\gamma_{j+1}^v - \gamma_j^v) \cdot P(\widehat{\mathcal{E}}_j^v)(l) \right).$$

Next,

$$\begin{aligned}
\mu(\lambda, [m_v]) &= \sum_{j=1}^{s_v} \frac{\gamma_{j+1}^v - \gamma_j^v}{p_v} \cdot (p_v \cdot \operatorname{rk} \widehat{\mathcal{E}}_j^v - h^0(\widehat{\mathcal{E}}_j^v(l)) \cdot r_v) \\
&= \sum_{j=1}^{s_v} \frac{\gamma_{j+1}^v - \gamma_j^v}{p_v} \cdot (p_v \cdot \operatorname{rk} \widehat{\mathcal{E}}_j^v - P(\widehat{\mathcal{E}}_j^v)(l) \cdot r_v).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\varepsilon \cdot \varepsilon_v \cdot \mu(\lambda, [m_v]) - x'_v \cdot \sum_{j=1}^{s_v} (\gamma^v_{j+1} - \gamma^v_j) \cdot h^0(\widehat{\mathcal{E}}^v_j(l)) \\
&\quad = \sum_{j=1}^{s_v} \frac{\gamma^v_{j+1} - \gamma^v_j}{p_v} \cdot (\varepsilon \cdot \varepsilon_v \cdot (p_v \cdot \operatorname{rk} \widehat{\mathcal{E}}^v_j - P(\widehat{\mathcal{E}}^v_j)(l) \cdot r_v) - x'_v \cdot p_v \cdot P(\widehat{\mathcal{E}}^v_j)(l)) \\
&\quad = \sum_{j=1}^{s_v} \frac{\gamma^v_{j+1} - \gamma^v_j}{p_v} \cdot (\varepsilon \cdot \varepsilon_v \cdot (p_v \cdot \operatorname{rk} \widehat{\mathcal{E}}^v_j - P(\widehat{\mathcal{E}}^v_j)(l) \cdot r_v) - x_v \cdot r_v \cdot P(\widehat{\mathcal{E}}^v_j)(l)) \\
&\quad = \sum_{j=1}^{s_v} \frac{\gamma^v_{j+1} - \gamma^v_j}{p_v} \cdot (\varepsilon \cdot \sigma_v \cdot (p_v \cdot \operatorname{rk} \widehat{\mathcal{E}}^v_j - P(\widehat{\mathcal{E}}^v_j)(l) \cdot r_v) - x_v \cdot p_v \cdot \operatorname{rk} \widehat{\mathcal{E}}^v_j) \\
&\quad = \varepsilon \cdot \sigma_v \cdot \sum_{j=1}^{s_v} \left(\frac{\gamma^v_{j+1} - \gamma^v_j}{p_v} \cdot (p_v \cdot \operatorname{rk} \widehat{\mathcal{E}}^v_j - P(\widehat{\mathcal{E}}^v_j)(l) \cdot r_v) \right) \\
&\qquad - \sum_{j=1}^{s_v} (x_v \cdot (\gamma^v_{j+1} - \gamma^v_j) \cdot \operatorname{rk} \widehat{\mathcal{E}}^v_j).
\end{aligned}$$

For a given vertex $v_0 \in V$ and a given index $j_0 \in \{1, \ldots, s_v\}$, let $s(v_0, j_0) \le s'(v_0, j_0) \in \{1, \ldots, s\}$ be the minimal and the maximal index among those indices j with $\mathcal{E}^{v_0}_j = \widehat{\mathcal{E}}^{v_0}_{j_0}$. Then, by definition,

$$p \cdot \sum_{j=s(v_0,j_0)}^{s'(v_0,j_0)} \alpha_j = \gamma^{v_0}_{j_0+1} - \gamma^{v_0}_{j_0}.$$

Hence,

$$\sum_{v \in V} \left(\sum_{j=1}^{s_v} (x_v \cdot (\gamma^v_{j+1} - \gamma^v_j) \cdot \operatorname{rk} \widehat{\mathcal{E}}^v_j) \right) = p \cdot \sum_{j=1}^{s} \alpha_j \left(\sum_{v \in V} (x_v \cdot \operatorname{rk} \mathcal{E}^v_j) \right).$$

Using Proposition 3.3.5, we discover that $\varepsilon \cdot A + B$ equals

$$\begin{aligned}
&\varepsilon \cdot \sum_{j=1}^{s} \alpha_j (p \cdot \operatorname{rk} \mathcal{E}^{\text{total}}_j - P(\mathcal{E}^{\text{total}}_j)(l) \cdot r) - p \cdot \sum_{j=1}^{s} \alpha_j \left(\sum_{v \in V} (x_v \cdot \operatorname{rk} \mathcal{E}^v_j) \right) \\
&\quad = \sum_{j=1}^{s} \alpha_j \cdot \left(\frac{p^2 \operatorname{rk} \mathcal{E}^{\text{total}}_j}{r \cdot d} - \frac{p \cdot a \cdot \operatorname{rk} \mathcal{E}^{\text{total}}_j}{r} - \frac{p \cdot P(\mathcal{E}^{\text{total}}_j)(l)}{d} + a \cdot P(\mathcal{E}^{\text{total}}_j)(l) \right) \\
&\qquad - p \cdot \sum_{j=1}^{s} \alpha_j \left(\sum_{v \in V} (x_v \cdot \operatorname{rk} \mathcal{E}^v_j) \right).
\end{aligned}$$

In order to conclude, we have to compute $\mu(\lambda, [m'])$. Under the identification of M with the space $H^0(\mathcal{E}^{\text{total}}(l))$, we define

$$\operatorname{gr}_j(M) = H^0((\mathcal{E}^{\text{total}}_j / \mathcal{E}^{\text{total}}_{j-1})(l)), \quad j = 1, \ldots, s+1.$$

The basis $\underline{m}$ of M induced by the bases $\underline{w}^v$ for W_v, $v \in V$, yields a natural isomorphism

$$M \cong \bigoplus_{j=1}^{s+1} \mathrm{gr}_j(M).$$

For an index tuple $\underline{\iota} \in J^a := \{1, \ldots, s+1\}^{\times a}$, we define $M_{\underline{\iota}} := \mathrm{gr}_{\iota_1}(M) \otimes \cdots \otimes \mathrm{gr}_{\iota_a}(M)$, and for $k \in \{1, \ldots, b\}$, we let $M^k_{\underline{\iota}}$ be $M_{\underline{\iota}}$ embedded into the k-th copy of $M^{\otimes a}$ in $M^{\otimes a \oplus b}$. If we denote $P(\mathcal{E}^{\mathrm{total}}_j(l)) = h^0(\mathcal{E}^{\mathrm{total}}_j(l))$ by m_j, $j = 1, \ldots, s$, then $\lambda = \sum_{j=1}^{s} \alpha_j \lambda(\underline{m}, \underline{\gamma}_p^{(m_j)})$ as a one-parameter subgroup of $\mathrm{SL}(M)$. Therefore,

$$\mu(\lambda, [m'])$$
$$= -\min\left\{ \sum_{j=1}^{s} \alpha_j (a \cdot m_j - \nu_j(\underline{\iota}) \cdot p) \,|\, k \in \{1, \ldots, b\},\ \underline{\iota} \in J^a:\ M^k_{\underline{\iota}} \not\subset \ker(m') \right\}.$$

Here,

$$\nu_j(\underline{\iota}) = \#\{\iota_i \le j \,|\, \underline{\iota} = (\iota_1, \ldots, \iota_a),\ i = 1, \ldots, a\}.$$

Let $\iota_0 \in J^a$ be an index which realizes the precise value of $\mu(\lambda, [m'])$. Then, altogether, we find

$$\sum_{j=1}^{s} \alpha_j \left(\frac{p^2 \mathrm{rk}\, \mathcal{E}^{\mathrm{total}}_j}{r \cdot d} - \frac{p \cdot a \cdot \mathrm{rk}\, \mathcal{E}^{\mathrm{total}}_j}{r} - \frac{p \cdot P(\mathcal{E}^{\mathrm{total}}_j(l))}{d} \right)$$
$$+ p \cdot \sum_{j=1}^{s} \alpha_j \cdot \nu_j(\underline{\iota_0}) - p \cdot \sum_{j=1}^{s} \alpha_j \left(\sum_{v \in V} (x_v \cdot \mathrm{rk}\, \mathcal{E}^v_j) \right)$$

as the value for $\mu(\lambda, \Gamma(m))/\beta$. We multiply this by $r \cdot d/p$ and get

$$\sum_{j=1}^{s} \alpha_j \cdot (p \cdot \mathrm{rk}\, \mathcal{E}^{\mathrm{total}}_j - r \cdot P(\mathcal{E}^{\mathrm{total}}_j(l))) + d \cdot \left(\sum_{j=1}^{s} \alpha_j \cdot (\nu_j(\underline{\iota_0}) \cdot r - a \cdot \mathrm{rk}\, \mathcal{E}^{\mathrm{total}}_j) \right)$$
$$+ r \cdot \sum_{j=1}^{s} \alpha_j \left(\sum_{v \in V} (\chi_v(l) \cdot \mathrm{rk}\, \mathcal{E}^v_j) \right).$$

As in [24], one verifies that

$$\mu(\mathcal{E}_\bullet, \underline{\alpha}; \tau) = \sum_{j=1}^{s} \alpha_j \cdot (\nu_j(\underline{\iota_0}) \cdot r - a \cdot \mathrm{rk}\, \mathcal{E}^{\mathrm{total}}_j),$$

so, by Remark 3.4.3, $\mu(\lambda, \Gamma(m))$ $(\ge)$ 0 implies inequality (3). □

4.2 *Proof of Theorem* 3.6.1

We use the same set up and the same notation as in §3.6 on the associated decorations and, with respect to the corresponding parameters, at the beginning of §4.1. This time, we set $N := \bigoplus_{v \in V} W_v$. The space

$$\mathfrak{P}'' := \mathbb{P}\left(N^\vee \otimes \pi_{(\times_{v\in V}\mathfrak{Q}_v^0)*}\left(\bigoplus_{v\in V}\pi^*_{\mathfrak{Q}_v^0}\mathfrak{E}_v \otimes \pi_X^*\mathcal{O}_X(l+m)\right)^{\oplus b}\right.$$
$$\left.\oplus H^0(X, \mathcal{O}_X(m)) \otimes \mathcal{O}_{\times_{v\in V}\mathfrak{Q}_v^0}\right)$$

is a projective bundle over $\times_{v\in V}\mathfrak{Q}_v^0$. Denote by $\mathfrak{E}_{\mathfrak{P}'',v}$ the pullback of $\mathfrak{E}_v$ to $\mathfrak{P}''\times X$. On $\mathfrak{P}''\times X$, there are the tautological homomorphisms

$$\phi'' \colon\ N \otimes \mathcal{O}_{\mathfrak{P}''\times X} \longrightarrow \left(\bigoplus_{v\in V}\mathfrak{E}_{\mathfrak{P}'',v}\otimes \pi_X^*\mathcal{O}_X(l+m)\right)^{\oplus b} \otimes \pi^*_{\mathfrak{P}''}\mathcal{O}_{\mathfrak{P}''}(1)$$

and

$$\varepsilon'' \colon\ \mathcal{O}_{\mathfrak{P}''\times X} \longrightarrow \pi^*_{\mathfrak{P}''}\mathcal{O}_{\mathfrak{P}''}(1) \otimes \pi_X^*\mathcal{O}_X(m).$$

First, we define $\mathfrak{R}'$ as the closed subscheme where ϕ'' factorizes over the quotient $\bigoplus_{v\in V}\mathfrak{E}_{\mathfrak{P}'',v}\otimes\pi_X^*\mathcal{O}_X(l)$. Then, $\phi''_{|\mathfrak{R}'\times X}$ may be considered as a collection of homomorphisms

$$f'_{v,v'} \colon\ \mathfrak{E}_{\mathfrak{R}',v} \longrightarrow \mathfrak{E}_{\mathfrak{R}',v'}\otimes \pi_X^*\mathcal{O}_X(m)^{\oplus b} \otimes \pi^*_{\mathfrak{R}'}\mathcal{O}_{\mathfrak{R}'}(1), \quad v, v' \in V.$$

Moreover, we have

$$\varepsilon' \colon\ \mathcal{O}_{\mathfrak{R}'\times X} \longrightarrow \pi^*_{\mathfrak{R}'}\mathcal{O}_{\mathfrak{R}'}(1) \otimes \pi_X^*\mathcal{O}_X(m).$$

Now, we can define $\mathfrak{R}$ as a closed subscheme of $\mathfrak{R}'$ by the following conditions:

- The restriction of $f'_{v,v'}$ to $\mathfrak{R}\times X$ is trivial, if $(v, v') \notin A$;
- If $(v, v') = a \in A$, then the corresponding homomorphism

  $$f''_{v,v'} \colon\ \pi_X^*\mathcal{O}_X(-m)^{\oplus b} \otimes \mathfrak{E}_{\mathfrak{R}',v} \longrightarrow \mathfrak{E}_{\mathfrak{R}',v'} \otimes \pi^*_{\mathfrak{R}'}\mathcal{O}_{\mathfrak{R}'}(1)$$

  vanishes on

  $$\pi_X^*\ker(\mathcal{O}_X(-m)^{\oplus b} \longrightarrow \mathcal{G}_a) \otimes \mathfrak{E}_{\mathfrak{R}',v}.$$

  Note that this is a closed condition, by Proposition A.2.1.
- The restriction of

  $$\mathcal{O}_{\mathfrak{R}'\times X} \xrightarrow{\varepsilon'} \pi_X^*\mathcal{O}_X(m) \otimes \pi^*_{\mathfrak{R}'}\mathcal{O}_{\mathfrak{R}'}(1) \longrightarrow \pi_X^*(\mathcal{O}_X(m)/\mathcal{O}_X) \otimes \pi^*_{\mathfrak{R}'}\mathcal{O}_{\mathfrak{R}'}(1)$$

  to $\mathfrak{R}'\times X$ is trivial, too.

The space $\mathfrak{R}$ is the correct parameter space and parametrizes a universal family $(\mathcal{E}_{\mathfrak{R},v}, v \in V;\ f_{\mathfrak{R},a}, a \in A;\ \mathcal{L}_{\mathfrak{R}}, \varepsilon_{\mathfrak{R}})$. It comes with an action of $\times_{v\in V}\mathrm{GL}(W_v)$, and the universal family is linearized with respect to that group action. The parameter space is also projective over $\times_{v\in V}\mathfrak{Q}_v^0$.

The associated family of V-split sheaves $(\mathcal{E}_{\mathfrak{R},v}, v \in V; \mathcal{L}_{\mathfrak{R}}, \tau_{\mathfrak{R}})$ of type $\underline{P}$ with a decoration of type $(s, b+1, 1, m)$ defines a $(\mathsf{X}_{v\in V}\, \mathrm{GL}(W_v))$-equivariant morphism

$$I\colon\ \mathfrak{R} \longrightarrow \mathfrak{M}$$

over the base scheme $\mathsf{X}_{v\in V}\, \mathfrak{Q}^0_v$. Since $\mathfrak{R}$ is proper over $\mathsf{X}_{v\in V}\, \mathfrak{Q}^0_v$, the morphism I is automatically proper ([9], ch. II, Cor. 4.8(e)). It is also injective. To see this, let $r \in \mathfrak{R}$ be a point which corresponds to the representation $(\mathcal{E}_v, v \in V; f_a, a \in A; \varepsilon)$ of Q of type $(\underline{P}, \underline{\mathcal{G}})$. For any $a \in A$, the surjection $\mathcal{O}_X(-m)^{\oplus b} \otimes W_{t(a)} \otimes \mathcal{O}_X(-l) \longrightarrow \mathcal{G}_a \otimes \mathcal{E}_{t(a)}$ yields an injective homomorphism

$$\begin{aligned}\mathrm{Hom}\,(\mathcal{G}_a \otimes \mathcal{E}_{t(a)}, \mathcal{E}_{h(a)}) &\subset \mathrm{Hom}\,(\mathcal{O}_X(-m)^{\oplus b} \otimes W_{t(a)} \otimes \mathcal{O}_X(-l), \mathcal{E}_{h(a)})\\ &= H^0({W^{\vee}_{t(a)}}^{\oplus b} \otimes \mathcal{E}_{h(a)}(l+m)).\end{aligned}$$

Since $\mathcal{E}_{h(a)}$ is torsion free, the restriction map

$$H^0(X, {W^{\vee}_{t(a)}}^{\oplus b} \otimes \mathcal{E}_{h(a)}(l+m)) \longrightarrow H^0(U, {W^{\vee}_{t(a)}}^{\oplus b} \otimes \mathcal{E}_{h(a)}(l+m)_{|U})$$

is injective for any open subset $U = X \setminus Z$ with $\mathrm{codim}_X(Z) \geq 2$. If $\mathfrak{U} \subset \mathfrak{R} \times X$ is the maximal open subset where all the $\mathcal{E}_{\mathfrak{R},v}$, $v \in V$ are locally free, then $U := \mathfrak{U} \cap (\{r\} \times X)$ is the maximal open subset where all the $\mathcal{E}_v$, $v \in V$, are locally free ([10], Lemma 2.1.7). In particular, the complement of U in X has codimension at least two. Since $\tau_{\mathfrak{R}|\{q\}\times X}$ determines all the f_a, $a \in A$, and ε over U, we are done. Because I is injective and proper and, thus finite, Theorem 3.6.1 follows immediately from Theorem 3.5.1 and its proof. □

4.3 *Proof of Theorem* 3.7.1

The proof of Theorem 3.7.1 is basically a formal adaptation of the corresponding result for Hitchin pairs. If, in the following, a representation $(\mathcal{E}_v, v \in V; f_a, a \in A; \varepsilon)$ is given, τ will always stand for the associated decoration. We first observe the following.

Lemma 4.3.1. *Suppose we are given* $\underline{\sigma}$, $\underline{\eta}$, *and* δ, *as well as a* $(\underline{\sigma}, \underline{\eta}, \delta)$*-semistable representation* $(\mathcal{E}_v, v \in V; f_a, a \in A)$ *of type* $(\underline{P}, \underline{\mathcal{G}})$. *Then, it satisfies Condition* 2(a) *of Theorem* 3.7.1.

Proof. Let $(\mathcal{F}_v, v \in V)$ be a non-trivial, proper subrepresentation of $(\mathcal{E}_v, v \in V; f_a, a \in A)$. Set

$$\mathcal{E}_\bullet\colon\ 0 \subsetneq (\mathcal{F}_v, v \in V) \subsetneq (\mathcal{E}_v, v \in V).$$

Then, one verifies $\mu(\mathcal{E}_\bullet, (1); \tau) \leq 0$, from which the assertion follows. □

PROPOSITION 4.3.2

Fix $\underline{\sigma}$ *and* $\underline{\chi}$. *Then, the set of torsion free sheaves occurring in representations of type* $(\underline{P}, \underline{\mathcal{G}})$ *which satisfy Condition* 2(a) *of Theorem* 3.7.1 *is bounded.*

Proof. We fix surjections $\nu_a\colon\ \mathcal{O}_X(-m)^{\oplus b} \longrightarrow \mathcal{G}_a$, $a \in A$. We may now adapt Nitsure's argument ([20], Proposition 3.2). Let

$$\begin{aligned}0 = (\mathcal{F}^v_0, v \in V) &\subsetneq (\mathcal{F}^v_1, v \in V) \subsetneq \cdots \subsetneq (\mathcal{F}^v_s, v \in V) \subsetneq (\mathcal{F}^v_{s+1}, v \in V)\\ &= (\mathcal{E}_v, v \in V)\end{aligned}$$

be the Harder–Narasimhan filtration of $(\mathcal{E}_v, v \in V)$ defined with respect to the parameters $\underline{\sigma}$ and $\underline{\chi}$. It will suffice to bound $\mu_{\underline{\sigma},\underline{\chi}}(\mathcal{F}_1^v, v \in V)$. Define

$$D := \deg \mathcal{O}_X(m) + \max_{v\in V}\left\{\frac{\overline{\chi}_v}{\sigma_v}\right\} - \min_{v\in V}\left\{\frac{\overline{\chi}_v}{\sigma_v}\right\}.$$

We claim that

$$\begin{aligned}&\mu_{\underline{\sigma},\underline{\chi}}(\mathcal{F}_1^v, v \in V)\\ &\quad\le \min\left\{\mu_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v \in V),\ \mu_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v \in V) + \frac{((\sum_{v\in V} r_v)-1)^2}{\sum_{v\in V} r_v}\cdot D\right\}.\end{aligned}$$

We can view the collection f_a, $a \in A$, together with the zero homomorphisms $\mathcal{E}_{v_1} \longrightarrow \mathcal{E}_{v_2} \otimes \mathcal{O}_X(m)^{\oplus b}$ for $(v_1, v_2) \notin A$ as a homomorphism

$$f\colon \bigoplus_{v\in V}\mathcal{E}_v \longrightarrow \left(\bigoplus_{v\in V}\mathcal{E}_v\right)\otimes \mathcal{O}_X(m)^{\oplus b}.$$

For any V-split subsheaf $(\mathcal{F}_v, v \in V)$, the condition of being a subrepresentation is, thus, equivalent to $f(\bigoplus_{v\in V}\mathcal{F}_v) \subseteq (\bigoplus_{v\in V}\mathcal{F}_v)\otimes\mathcal{O}_X(m)^{\oplus b}$. We simply say that $\bigoplus_{v\in V}\mathcal{F}_v$ is *f-invariant*. If the condition

$$\mu_{\underline{\sigma},\underline{\chi}}(\mathcal{F}_1^v, v \in V) \le \mu_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v \in V)$$

is violated, then, by definition, none of the sheaves $\mathcal{F}_j^{\text{total}} := \bigoplus_{v\in V}\mathcal{F}_j^v$ can be f-invariant, $j = 1, \ldots, s$, i.e., the homomorphisms $\phi_j\colon\ \mathcal{F}_j^{\text{total}} \longrightarrow (\mathcal{E}^{\text{total}}/\mathcal{F}_j^{\text{total}})\otimes\mathcal{O}_X(m)^{\oplus b}$ are non-trivial, $\mathcal{E}^{\text{total}} := \bigoplus_{v\in V}\mathcal{E}_v$, $j = 1, \ldots, s$. For any $j = 1, \ldots, s$, there exist $\iota \le j-1$ and $\kappa \ge j+1$, such that ϕ_j induces a non-trivial homomorphism $\overline{\phi}_j\colon\ \mathcal{F}_{\iota+1}^{\text{total}}/\mathcal{F}_\iota^{\text{total}} \longrightarrow (\mathcal{F}_\kappa^{\text{total}}/\mathcal{F}_{\kappa-1}^{\text{total}})\otimes\mathcal{O}_X(m)^{\oplus b}$. Now, Propositions 3.3.3 and 3.3.4 imply

$$\begin{aligned}\mu_{\underline{\sigma},\underline{\chi}}(\mathcal{F}_j^v/\mathcal{F}_{j-1}^v, v \in V) &\le \mu_{\underline{\sigma},\underline{\chi}}(\mathcal{F}_{\iota+1}^v/\mathcal{F}_\iota^v, v \in V)\\ &\le \mu_{\underline{\sigma},\underline{\chi}}(\mathcal{F}_\kappa^v/\mathcal{F}_{\kappa-1}^v, v \in V) + D\\ &\le \mu_{\underline{\sigma},\underline{\chi}}(\mathcal{F}_{j+1}^v/\mathcal{F}_j^v, v \in V) + D.\end{aligned}$$

Thus,

$$\begin{aligned}\mu_{\underline{\sigma},\underline{\chi}}(\mathcal{F}_1^v, v \in V) &\le \mu_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v/\mathcal{F}_s^v, v \in V) + s\cdot D\\ &\le \mu_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v/\mathcal{F}_s^v, v \in V) + \left(\left(\sum_{v\in V} r_v\right)-1\right)\cdot D.\end{aligned}$$

Finally, one finds that

$$\mu_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v/\mathcal{F}_s^v, v \in V) \le \frac{\deg_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v \in V) - \mu_{\underline{\sigma},\underline{\chi}}(\mathcal{F}_1^v, v \in V)}{(\sum_{v\in V} r_v) - 1}$$

from which the assertion follows. □

We first show that (1) implies (2). We have already checked that 2(a) holds. Before we check Condition 2(b), we review the linear algebra setting. The space $\bigoplus_{a\in A}\operatorname{Hom}(\mathbb{C}^{r_{t(a)}},\mathbb{C}^{r_{h(a)}})^{\oplus b}$ can be $(\mathsf{X}_{v\in V}\operatorname{GL}(\mathbb{C}^{r_v}))$-equivariantly embedded into

$$\mathbb{E} := \operatorname{End}(M)^{\oplus b}, \quad M := \bigoplus_{v\in V}\mathbb{C}^{\sigma_v\cdot r_v}.$$

Let $[f_a\colon \mathbb{C}^{r_{t(a)}} \longrightarrow (\mathbb{C}^{r_{h(a)}})^{\oplus b}, a\in A]$ in $\mathbb{P}(\bigoplus_{a\in A}\operatorname{Hom}(\mathbb{C}^{r_{t(a)}},\mathbb{C}^{r_{h(a)}})^{\oplus b})$ be an element which is unstable with respect to the $\widetilde{G}$-action. Let $[f]\in\mathbb{P}(\mathbb{E})$ be the associated element which is equally unstable. As explained before, a one-parameter subgroup $\lambda\colon \mathbb{C}^*\longrightarrow\widetilde{G}$ yields a weighted flag $(M_\bullet,\underline{\alpha})$ with

$$M_\bullet\colon\ 0\subsetneq U_1\subsetneq\cdots\subsetneq U_s\subsetneq\mathbb{C}^{\sum_{v\in V}\sigma_v\cdot r_v}.$$

Here, $U_j=\bigoplus_{v\in V}U_j^{v,\oplus\sigma_v}$ for suitable subspaces $U_j^v\subset\mathbb{C}^{r_v}$, $j=1,\ldots,s$. Then, $\mu(\lambda,[f])<0$ will occur if and only if

$$f(U_j)\subset U_{j-1}^{\oplus b}, \quad j=1,\ldots,s+1,$$

i.e.,

$$f\left(\bigoplus_{v\in V}U_j^{v,\oplus\sigma_v}\right)\subset\left(\bigoplus_{v\in V}U_{j-1}^{v,\oplus\sigma_v}\right)^{\oplus b}, \quad j=1,\ldots,s+1,$$

or equivalently

$$U_j^{v,\oplus\sigma_v}\subset\ker\left(M\xrightarrow{f}\bigoplus_{v\in V}(\mathbb{C}^{\sigma_v\cdot r_v}/U_{j-1}^{v,\oplus\sigma_v})^{\oplus b}\right), \quad \text{for all } v\in V.$$

Conversely, we may define

$$Y_1^v := \ker(\mathbb{C}^{r_v}\overset{\text{diag}}{\hookrightarrow}\mathbb{C}^{\sigma_v\cdot r_v}\xrightarrow{f}M^{\oplus b}), \quad v\in V,$$

and

$$Y_j^v := \ker\left(\mathbb{C}^{r_v}\overset{\text{diag}}{\hookrightarrow}\mathbb{C}^{\sigma_v\cdot r_v}\xrightarrow{f}\bigoplus_{v\in V}(\mathbb{C}^{\sigma_v\cdot r_v}/Y_{j-1}^{v,\oplus\sigma_v})^{\oplus b}\right), \quad v\in V.$$

By our previous observations, this process will stop after at most s steps, i.e., we get a flag

$$M'_\bullet\colon\ 0\subsetneq\bigoplus_{v\in V}Y_1^{v,\oplus\sigma_v}\subsetneq\cdots\subsetneq\bigoplus_{v\in V}Y_{s'}^{v,\oplus\sigma_v}\subsetneq\mathbb{C}^{\sum_{v\in V}\sigma_v\cdot r_v},$$

and $(M'_\bullet,(1,\ldots,1))$ comes from a suitable one-parameter subgroup $\lambda\colon\mathbb{C}^*\longrightarrow\widetilde{G}$ with $\mu(\lambda,[f])=-\sum_{v\in V}\sigma_v\cdot r_v$.

The latter construction can be extended to the setting of sheaves, i.e., given a representation $(\mathcal{E}_v, v\in V; f_a, a\in A; \varepsilon=0)$ as in (1) for which Condition 2(b) fails, we define

$$\mathcal{F}_1^v := \ker\left(\mathcal{E}_v\overset{\text{diag}}{\hookrightarrow}\mathcal{E}_v^{\oplus\sigma_v}\xrightarrow{f}\bigoplus_{v\in V}\mathcal{E}_v^{\oplus\sigma_v}\otimes\mathcal{O}_X(m)^{\oplus b}\right), \quad v\in V,$$

and

$$\mathcal{F}_j^v := \ker\left(\mathcal{E}_v \stackrel{\text{diag}}{\hookrightarrow} \mathcal{E}_v^{\oplus\sigma_v} \stackrel{f}{\longrightarrow} \bigoplus_{v\in V}(\mathcal{E}_v^{\oplus\sigma_v}/\mathcal{F}_{j-1}^{v,\oplus\sigma_v}) \otimes \mathcal{O}_X(m)^{\oplus b}\right),$$

$$v \in V,\ j > 1.$$

Then, we find the weighted filtration $(\mathcal{E}^\bullet, (1, \dots, 1))$ with

$$\mathcal{E}_\bullet\colon\ 0 \subsetneq (\mathcal{F}_1^v, v \in V) \subsetneq \cdots \subsetneq (\mathcal{F}_{s'}^v, v \in V) \subsetneq (\mathcal{E}_v, v \in V)$$

and

$$\mu(\mathcal{E}^\bullet, (1, \dots, 1); \tau) = -\sum_{v\in V} \sigma_v \cdot r_v.$$

By the boundedness result 4.3.1, it is clear that the sheaves of the form $\mathcal{F}_j^v$ as just-defined live in bounded families, too. In particular, there is a constant $C > 0$, such that

$$\begin{aligned}&\deg_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_v, v \in V) \cdot \mathrm{rk}_{\underline{\sigma}}(\mathcal{F}_j^v, v \in V)\\ &\quad - \deg_{\underline{\sigma},\underline{\chi}}(\mathcal{F}_j^v, v \in V) \cdot \mathrm{rk}_{\underline{\sigma}}(\mathcal{E}_v, v \in V) < C\end{aligned}$$

for any filtration as above. But then, with $\overline{\delta} > 0$, the coefficient of $x^{\dim X-1}$ in δ, the condition of $(\underline{\sigma}, \underline{\eta}/n, n \cdot \delta)$-semistability requires

$$\begin{aligned}0 &\preceq M_{\underline{\sigma},\underline{\chi}}(\mathcal{E}_\bullet, (1, \dots, 1)) + n \cdot \delta \cdot \mu(\mathcal{E}_\bullet, (1, \dots, 1); \tau)\\ &\preceq \left(\left(\left(\sum_{v\in V} r_v\right) - 1\right) \cdot C - n \cdot \overline{\delta} \cdot \sum_{v\in V}(\sigma_v \cdot r_v)\right) \cdot x^{\dim X-1},\end{aligned}$$

but for large n, this is impossible.

The converse is an easy adaptation of the argument given in ([24], Example 3.6), and is left as an exercise to the reader. □

4.4 *Proof of Theorem* 3.7.2

The points (i) and (ii) are just a reformulation of Theorem 3.6.1. Point (iii) is proved by standard arguments and will be omitted here. Finally, the constructions carried out in Section 3.7 show that the universal family on the parameter space $\mathfrak{R}$ defines a morphism $\mathfrak{R} \longrightarrow \mathbb{H}(Q, \underline{P}, \underline{\mathcal{G}})$. This morphism is invariant under the $\widetilde{G}$-action and, thus, descends to the moduli space $\mathcal{R}(Q)^{(\underline{\sigma},\underline{\chi})-\mathrm{ss}}_{\underline{P}/\underline{\mathcal{G}}}$. □

Appendix A: Two auxiliary results

A.1 *Restrictions of families of locally free sheaves to open subsets*

Let X be a smooth projective manifold and S a noetherian scheme. Let $\iota\colon\ U \subset S \times X$ be an open subset, such that

$$\mathrm{codim}(X \setminus (U \cap \{s\} \times X), X) \geq 2, \quad \text{for all } s \in S.$$

PROPOSITION A.1.1

In the above situation, the natural homomorphism $\mathcal{O}_{S\times X} \longrightarrow \iota_*\mathcal{O}_U$ *is an isomorphism. In particular, for any locally free sheaf* $\mathcal{V}$ *on* $S \times X$, *we have*

$$\mathcal{V} = \iota_*\iota^*\mathcal{V}.$$

Proof. We refer to ([16], p. 111f). □

A.2 *Zero loci of sheaf homomorphisms*

The following result may be found in ([8], Lemma 3.1).

PROPOSITION A.2.1

Let S be a noetherian scheme, $\mathcal{A}^1_S$ *and* $\mathcal{A}^2_S$ *coherent sheaves on* $S\times X$, *and* $\phi_S\colon\ \mathcal{A}^1_S \longrightarrow \mathcal{A}^2_S$ *a homomorphism. Assume that* $\mathcal{A}^2_S$ *is S-flat. Then, there is a closed subscheme* $\mathfrak{Y} \subset S$ *the closed points of which are those* $s \in S$ *for which* $\phi_{S|\{s\}\times X} \equiv 0$. *More precisely, it has the property that any morphism* $f\colon\ T \longrightarrow S$ *factors through* $\mathfrak{Y}$, *if and only if* $(f \times \mathrm{id}_X)^*\phi_S$ *is the zero homomorphism.*

Appendix B: A concluding remark

It was pointed out to me by Balázs Szendröi that the case in which X is a point is formally not covered by our formalism. It would be formally included, if one allowed δ to have degree $\dim(X)$ (or higher). Then, for X a point, $\delta = 1$, η_v, $v \in V$, with $\sum_{v\in V}\eta_v r_v = 0$, and a representation $(f_a, a \in A) \in \oplus_{a\in A}\mathrm{Hom}(\mathbb{C}^{r_{t(a)}}, \mathbb{C}^{r_{h(a)}})$, the condition of (semi)stability would read

$$\frac{\sum_{v\in V}\sigma_v\dim(W_v) - \sum_{v\in V}\eta_v\dim(W_v)}{\sum_{v\in V}\sigma_v\dim(W_v)} = 1 - \frac{\sum_{v\in V}\eta_v\dim(W_v)}{\sum_{v\in V}\sigma_v\dim(W_v)}$$
$$(\leq)\ \frac{\sum_{v\in V}\sigma_v r_v - \sum_{v\in V}\eta_v r_v}{\sum_{v\in V}\sigma_v r_v} = 1,$$

i.e.,

$$-\sum_{v\in V}\eta_v\dim(W_v)\ (\leq)\ 0$$

for any non-trivial subrepresentation $(W_v, v \in V)$. This is precisely King's definition with respect to the character

$$(U_v, v \in V;\ f_a, a \in A) \longmapsto -\sum_{v\in V}\eta_v\cdot\dim(U_v).$$

Note that for $\dim(X) \geq 0$, a positive polynomial δ of degree $\dim(X)$, and η_v, $v \in V$, with $\sum_{v\in V}\eta_v r_v = 0$, the condition of (semi)stability can be restated as follows:

- For any non-trivial subrepresentation $(\mathcal{F}_v, v \in V)$ one has

$$-\sum_{v\in V}\eta_v\mathrm{rk}\,(\mathcal{F}_v)\ (\leq)\ 0,$$

and,

- if '=' occurs, then

$$\frac{\sum_{v\in V}\sigma_v P(\mathcal{F}_v)}{\sum_{v\in V}\sigma_v \mathrm{rk}\,(\mathcal{F}_v)} \quad (\preceq) \quad \frac{\sum_{v\in V}\sigma_v P(\mathcal{E}_v)}{\sum_{v\in V}\sigma_v \mathrm{rk}\,(\mathcal{E}_v)}.$$

Let us call representations which satisfy this condition *asymptotically* (*semi*)*stable*. The moduli spaces for asymptotically (semi)stable objects might be obtained as follows:

- Fix the data $\sigma_v, \eta_v, v \in V$.
- There is a positive polynomial δ_0 (depending on the Hilbert polynomials P_v, σ_v, and $\eta_v, v \in V$) of degree $\dim(X) - 1$, such that, for any $\delta \succ \delta_0$, a representation $(\mathcal{E}_v, v \in V;\ f_a, a \in A)$ will be (semi)stable with respect to the parameters $\sigma_v, \eta_v, v \in V$, and δ, if and only if it is asymptotically (semi)stable.

The techniques to prove this should be adapted from my recent paper 'Global boundedness for decorated sheaves' *Int. Math. Res. Not.* **68** (2004) 3637–71.

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