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Derivations into duals of ideals of Banach algebras

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Abstract. We introduce two notions of amenability for a Banach algebra $\mathcal{A}$. Let I be a closed two-sided ideal in $\mathcal{A}$, we say $\mathcal{A}$ is I-weakly amenable if the first cohomology group of $\mathcal{A}$ with coefficients in the dual space I^* is zero; i.e., $H^1(\mathcal{A}, I^*) = \{0\}$, and, $\mathcal{A}$ is ideally amenable if $\mathcal{A}$ is I-weakly amenable for every closed two-sided ideal I in $\mathcal{A}$. We relate these concepts to weak amenability of Banach algebras. We also show that ideal amenability is different from amenability and weak amenability. We study the I-weak amenability of a Banach algebra $\mathcal{A}$ for some special closed two-sided ideal I.

Keywords. Amenability; weak-amenability; ideal weak-amenability.

1. Introduction

Let $\mathcal{A}$ be a Banach algebra and X be a Banach $\mathcal{A}$-bimodule, that is X is a Banach space and an $\mathcal{A}$-bimodule such that the module operations $(a, x) \longmapsto ax$ and $(a, x) \longmapsto xa$ from $\mathcal{A} \times X$ into X are jointly continuous. Then X^* is also a Banach $\mathcal{A}$-bimodule if we define

$$\langle x, ax^* \rangle = \langle xa, x^* \rangle; \quad \langle x, x^*a \rangle = \langle ax, x^* \rangle \quad (a \in \mathcal{A},\ x \in X,\ x^* \in X^*).$$

We say that Y is a dual $\mathcal{A}$-bimodule if there is a Banach $\mathcal{A}$-bimodule X such that Y is isometrically module isomorphic with X^*. Thus in particular I is a Banach $\mathcal{A}$-bimodule and I^* is a dual $\mathcal{A}$-bimodule for every closed two-sided ideal I in $\mathcal{A}$.

If X is a Banach $\mathcal{A}$-bimodule, then a derivation from $\mathcal{A}$ into X is a continuous linear operator D with

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in \mathcal{A}).$$

If $x \in X$ and we define δ_x by

$$\delta_x(a) = a \cdot x - x \cdot a, \quad (a \in \mathcal{A})$$

then δ_x is a derivation, and such derivations are called inner. The space of characters on an algebra $\mathcal{A}$ is denoted by $\Phi_{\mathcal{A}}$. Let $\varphi \in \Phi_{\mathcal{A}} \cup \{0\}$. Then $\mathbb{C}$ is a symmetric $\mathcal{A}$-bimodule for the products $a \cdot z = z \cdot a = \varphi(a)z$, $(a \in \mathcal{A}, z \in \mathbb{C})$. In this case the bimodule is denoted by $\mathbb{C}_\varphi$. A derivation from $\mathcal{A}$ into $\mathbb{C}_\varphi$ is a linear functional d on $\mathcal{A}$ such that

$$d(ab) = \varphi(a) \cdot d(b) + d(a) \cdot \varphi(b) \quad (a, b \in \mathcal{A}).$$

Such a linear functional is called a point derivation at φ. Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is amenable if, whenever D is a derivation from $\mathcal{A}$ to a dual $\mathcal{A}$-bimodule, then D is inner; this definition was introduced by Johnson [Jo1]. $\mathcal{A}$ is weakly amenable if, whenever D is a derivation from $\mathcal{A}$ to $\mathcal{A}^*$, then D is inner. Bade, Curtis and Dales [B-C-D] have introduced the concept of weak amenability for commutative Banach algebras.

For example, it was shown in [Jo1] that the group algebra $L^1(G)$ is amenable if and only if G is an amenable group and in [Jo2] (or [D-Gh]) that $L^1(G)$ is weakly amenable for each locally compact group G.

The following definition describes the main new property that we shall study.

DEFINITION 1.1

Let $\mathcal{A}$ be a Banach algebra and I be a closed two-sided ideal in $\mathcal{A}$. Then $\mathcal{A}$ is I-weakly amenable if $H^1(\mathcal{A}, I^*) = \{0\}$; $\mathcal{A}$ is ideally amenable if $\mathcal{A}$ is I-weakly amenable for every closed two-sided ideal I in $\mathcal{A}$.

We begin with the following trivial observations:

(i) An amenable Banach algebra is ideally amenable.
(ii) An ideally amenable Banach algebra is weakly amenable.

Let $\mathcal{A}^\#$ be the unitization of the commutative Banach algebra $\mathcal{A}$. Then for each closed two-sided ideal I of $\mathcal{A}$ consider the following short exact sequence.

$$(\Sigma)\colon 0 \longrightarrow K \xrightarrow{\iota} \mathcal{A}^\#\hat{\otimes} I \xrightarrow{\pi} I \longrightarrow 0,$$

where π is given by $\pi(a \otimes i) = ai$ for all $a \in \mathcal{A}^\#$, $i \in I$, ι is the embedding map and $K = \ker \pi$. It is well-known that $B(\mathcal{A}^\#, I^*)$ is isometrically isomorphic to $(\mathcal{A}^\#\hat{\otimes} I)^*$, so we get the following short exact sequence of linear maps:

$$(\Sigma^*)\colon 0 \longrightarrow I^* \xrightarrow{\pi^*} B(\mathcal{A}^\#, I^*) \xrightarrow{\iota^*} K^* \longrightarrow 0.$$

Let $[K; \mathcal{A}] = \mathrm{Span}\,\{u \cdot a - a \cdot u\colon u \in K,\ a \in \mathcal{A}\}$, then by ([Gr1], Proposition 3.1) there is no non-zero bounded derivation into I^* if and only if $\overline{[K; \mathcal{A}]} = K$.

Let $\mathcal{A}$ be a Banach algebra, and let I be a closed two-sided ideal in $\mathcal{A}$. We consider the following complex

$$\cdots \longrightarrow \mathcal{A}\hat{\otimes}\mathcal{A}\hat{\otimes} I \xrightarrow{d_2} \mathcal{A}\hat{\otimes} I \xrightarrow{d_1} I \longrightarrow 0,$$

where the maps d_1 and d_2 are specified by the formulae:

$$d_1(a \otimes i) = ai - ia \quad (a \in \mathcal{A},\ i \in I);$$
$$d_2(a \otimes b \otimes i) = b \otimes ia - ab \otimes i + a \otimes bi \quad (a, b \in \mathcal{A},\ i \in I).$$

By ([Jo1], Corollary 1.3) and ([Kh], II.5.29), $H^1(\mathcal{A}, I^*) = \{0\}$ if and only if im d_1 is closed in I and im d_2 is dense in ker d_1.

Theorem 1.2. *Let $\mathcal{A}$ be a commutative Banach algebra and I be a closed two-sided ideal in $\mathcal{A}$. Then the following assertions are equivalent*:

(i) *$\mathcal{A}$ is I-weakly amenable.*
(ii) Span $\{(a \otimes b - b \otimes a) \cdot i - ab \otimes i : a, b \in \mathcal{A},\ i \in I\}$ *is dense in* $\mathcal{A} \otimes I$.
(iii) $[K; \mathcal{A}]^{-} = K$.

As in ([B-C-D], Theorem 1.5), we have the following theorem.

Theorem 1.3. *A commutative Banach algebra is weakly amenable if and only if every derivation from $\mathcal{A}$ into a commutative Banach $\mathcal{A}$-bimodule is zero.*

By Theorem 1.3 we conclude that a commutative Banach algebra $\mathcal{A}$ is weakly amenable if and only if it is ideally amenable. As in ([B-C-D], Theorem 3.14), let K be an infinite, compact metric space and let $\alpha \in (0, 1/2)$ then $\mathrm{lip}_\alpha K$ is weakly amenable but it is not amenable, therefore this is an example of ideally amenable Banach algebra, that is not amenable.

We now consider an example of non-commutative Banach algebra that is ideally amenable but it is not amenable.

Example. Let $\mathcal{A} = \ell^1(\mathbb{N})$. We define the product on $\mathcal{A}$ by $f \cdot g = f(1)g$ for all f and g in $\mathcal{A}$. It is obvious that $\mathcal{A}$ is a Banach algebra with this product and norm $\| \cdot \|_1$. It is straightforward that $\mathcal{A}$ has no approximate identity, so by ([Jo1], Proposition 1.6) $\mathcal{A}$ is not amenable. Let I be a closed two-sided ideal of $\mathcal{A}$, it is easy to see that if $I \neq \mathcal{A}$, then $I \subseteq \{f \in \mathcal{A};\ f(1) = 0\}$.

Let $I \neq \mathcal{A}$ and $D\colon \mathcal{A} \longrightarrow I^*$ be a derivation. For $f \in \mathcal{A}$ we consider

$$\tilde{f}\colon \mathbb{N} \longrightarrow \mathbb{C},$$

$$\tilde{f}(n) = \begin{cases} f(n), & n \geq 2 \\ 0, & n = 1 \end{cases},$$

then $f = f \cdot e + \tilde{f}$ such that

$$e(n) = \begin{cases} 1, & n = 1 \\ 0, & n \neq 1 \end{cases}.$$

Therefore $D(f) = f(1)D(e) + D(\tilde{f})$ but we have $\tilde{D}(\tilde{f} \cdot e) = \tilde{f} \cdot D(e) + D(\tilde{f}) \cdot e$, $\tilde{f} \cdot e = \tilde{f} \cdot D(e) = 0$ and $D(\tilde{f}) \cdot e = D(\tilde{f})$. Consequently $D(\tilde{f}) = 0$. Let $x^* = -D(e)$.

$$\begin{aligned} f \cdot x^* - x^* \cdot f &= 0 - f(1)x^* \\ &= -f(1)(-D(e)) \\ &= f(1)D(e) \\ &= D(f). \end{aligned}$$

So $D = \delta_{x^*}$ and $H'(\mathcal{A}, I^*) = \{0\}$. If $I = \mathcal{A}$, then by Assertion 1 of [Zh], $H^1(\mathcal{A}, I^*) = \{0\}$. ■

Let $\mathcal{A}$ be a Banach algebra with a bounded right(left) approximate identity and let X be a Banach $\mathcal{A}$-bimodule on which $\mathcal{A}$ acts trivially on the left(right). Then by ([Jo1], Proposition 1.5) $H^1(\mathcal{A}, X^*) = \{0\}$, so if I is a closed two-sided ideal in $\mathcal{A}$ and $\mathcal{A}I = \{0\}(I\mathcal{A} = \{0\})$, then $\mathcal{A}$ is I-weakly amenable. If G is a commutative non-discrete group, then by [Gh-L-W], $L^1(G)^{**}$ is not weakly amenable. Let $J = \{F \in L^1(G)^{**}\colon L^1(G)^{**}F = 0\}$. Then J is a closed two-sided ideal in $L^1(G)^{**}$ and $L^1(G)^{**}$ is J-weakly amenable.

Theorem 1.4. *Let $\mathcal{A}$ be a Banach algebra and I be a closed two-sided ideal in $\mathcal{A}$ and $\mathcal{A}$ be I-weakly amenable, let $\varphi \in \Phi_{\mathcal{A}}$, such that $I \nsubseteq \ker \varphi$. Then there is no non-zero point derivation at φ.*

Proof. Let $d\colon \mathcal{A} \longrightarrow \mathbb{C}_\varphi$ be a non-zero point derivation, and let $\pi\colon \mathcal{A}^* \longrightarrow I^*$ be the adjoint of $\imath\colon I \longrightarrow \mathcal{A}$. Consider the map $D\colon \mathcal{A} \longrightarrow I^*$ defined by $D(a) = d(a)\pi(\varphi)$. It is easy to see that D is a derivation. Since $\mathcal{A}$ is I-weakly amenable, there exists $\lambda \in I^*$ with $D(a) = a\cdot\lambda - \lambda\cdot a$ $(a \in \mathcal{A})$. Take $i \in I$ with $\varphi(i) = 1$. If $\ker\varphi \subseteq \ker d$, then there exists $\alpha \in \mathbb{C}$ such that $d = \alpha\varphi$. So $2\alpha = 2\alpha\varphi(i) = 2d(i) = 2d(i)\varphi(i) = d(i^2) = \alpha\varphi(i^2) = \alpha$. Therefore, $\alpha = 0$ and this is a contradiction. Consequently $\ker\varphi \nsubseteq \ker d$ and there exists $a \in \ker\varphi$ with $d(a) = 1$. Set $i' = i + (1 - d(i))ia = i + ia - d(i)ia$. Then $\varphi(i') = d(i') = 1$ and so

$$1 = (D(i'))(i') = (i' \cdot \lambda - \lambda \cdot i')(i') = \lambda(i'^2) - \lambda(i'^2) = 0,$$

is a contradiction. ■

Theorem 1.5. *Let $\mathcal{A}$ be a weakly amenable Banach algebra and for each closed two-sided ideal I such that $I = \overline{\mathcal{A}I \bigcup I\mathcal{A}}$, $\mathcal{A}$ is I-weakly amenable. Then $\mathcal{A}$ is ideally amenable.*

Proof. Let I be a closed two-sided ideal in $\mathcal{A}$. Put $J = \overline{\mathcal{A}I \bigcup I\mathcal{A}}$. It is easy to see that J is a closed two-sided ideal in $\mathcal{A}$ and $J = \overline{J\mathcal{A} \bigcup \mathcal{A}J}$. Let $\imath\colon J \longrightarrow I$ be the natural embedding and $D\colon \mathcal{A} \longrightarrow I^*$ be a derivation. Then $\imath^* \circ D\colon \mathcal{A} \longrightarrow J^*$ is a derivation. So there is a $m \in J^*$ such that $\imath^* \circ D = \delta_m$. Let x^* be the extension of m into I by Hahn–Banach theorem. For every $a, b \in \mathcal{A}$ we have

$$\begin{aligned}
\langle i, D(ab)\rangle &= \langle ia, D(b)\rangle + \langle bi, D(a)\rangle \\
&= \langle \imath(ia), D(b)\rangle + \langle \imath(bi), D(a)\rangle \\
&= \langle ia, bm - mb\rangle + \langle bi, am - ma\rangle \\
&= \langle iab - bia + bia - abi, m\rangle \\
&= \langle i, abx^* - x^*ab\rangle = \langle i, \delta_{x^*}(ab)\rangle \quad (i \in I).
\end{aligned}$$

Hence $D(ab) = \delta_{x^*}(ab)$. Since $\mathcal{A}$ is weakly amenable, so $\mathcal{A}^2 = \mathcal{A}$ and therefore $D = \delta_{x^*}$ and D is inner. ■

Theorem 1.6. *Let $\mathcal{A}$ be a Banach algebra, X be a Banach $\mathcal{A}$-bimodule and Y be a closed submodule of X. If $H^1(\mathcal{A}, Y^*) = \{0\}$ and $H^1(\mathcal{A}, (X/Y)^*) = \{0\}$, then $H^1(\mathcal{A}, X^*) = \{0\}$.*

Proof. Let $D\colon \mathcal{A} \longrightarrow X^*$ be a derivation, and $\pi\colon X^* \longrightarrow Y^*$ be adjoint of $\imath\colon Y \longrightarrow X$. Then π is a $\mathcal{A}$-bimodule homomorphism and $\pi \circ D\colon \mathcal{A} \longrightarrow Y^*$ is a derivation. Therefore there exists $y^* \in Y^*$ such that $\pi \circ D = \delta_{y^*}$. Let x^* be an extension of y^* by Hahn–Banach theorem. Then $d = D - \delta_{x^*}\colon \mathcal{A} \longrightarrow Y^\perp$ is a derivation, but $Y^\perp = (X/Y)^*$. Therefore there exists $x_1^* \in Y^\perp \subseteq X^*$ such that $d = \delta_{x_1^*}$, so $D = \delta_{(x^* + x_1^*)}$. ■

COROLLARY 1.7

Let I and J be closed two-sided ideals in Banach algebra $\mathcal{A}$, and $J \subseteq I$. If $\mathcal{A}$ is J-weakly amenable and $H^1(\mathcal{A}, (I/J)^*) = \{0\}$, then $\mathcal{A}$ is I-weakly amenable.

COROLLARY 1.8

Let I be a closed two-sided ideal in Banach algebra $\mathcal{A}$. If $H^1(\mathcal{A}, (\mathcal{A}/I)^*) = \{0\}$ and $\mathcal{A}$ is I-weakly amenable, then $\mathcal{A}$ is weakly amenable.

Theorem 1.9. *Let $\mathcal{A}$ be a Banach algebra and I be a closed two-sided ideal in $\mathcal{A}$ with a bounded approximate identity. If $\mathcal{A}$ is ideally amenable, then I is ideally amenable.*

Proof. Let J be a closed two-sided ideal in I. It is easy to see that J is an ideal in $\mathcal{A}$. Let $D: I \longrightarrow J^*$ be a derivation. By ([Ru], Proposition 2.1.6), D can be extended to a derivation $\tilde{D}: \mathcal{A} \longrightarrow J^*$. So there is a $m \in J^*$ such that $\tilde{D} = \delta_m$. Then $D(i) = \tilde{D}(i) = \delta_m$ for each $i \in I$. So D is inner. ■

COROLLARY 1.10

Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity and $\mathcal{M}(\mathcal{A})$ be the multiplier algebra of $\mathcal{A}$. If $\mathcal{M}(\mathcal{A})$ is ideally amenable, then $\mathcal{A}$ is ideally amenable.

DEFINITION 1.11 [Gr3]

Let $\mathcal{A}$ be a Banach algebra and I be a closed two-sided ideal in $\mathcal{A}$. We say that I has the trace extension property, if every $m \in I^*$ such that $am = ma$ for each $a \in \mathcal{A}$, can be extended to $a^* \in A^*$ such that $aa^* = a^*a$ for each $a \in \mathcal{A}$.

Let I be a closed two-sided ideal in Banach algebra $\mathcal{A}$. Johnson in [Jo1] has shown that, if $\mathcal{A}$ is amenable, then I is amenable if and only if I has a bounded approximate identity, and $\mathcal{A}$ is amenable if I and $\mathcal{A}/I$ are amenable. Also if $\mathcal{A}/I$ and I are weakly amenable, then $\mathcal{A}$ is weakly amenable [Gr2]. Also if I has the trace extension property and $\mathcal{A}$ is weakly amenable, then $\mathcal{A}/I$ is weakly amenable [Gr2]. We prove a similar proposition for ideal amenability.

Theorem 1.12. *Let $\mathcal{A}$ be an ideally amenable Banach algebra and I be a closed two-sided ideal in $\mathcal{A}$ that has the trace extension property. Then, $\mathcal{A}/I$ is ideally amenable.*

Proof. Let J/I be a closed two-sided ideal in $\mathcal{A}/I$. Then J is a closed two-sided ideal in $\mathcal{A}$. We write $\pi: J \longrightarrow J/I$, $q: \mathcal{A} \longrightarrow \mathcal{A}/I$ for the natural quotient maps and π^* for the adjoint of π. Let $D: \mathcal{A}/I \longrightarrow (J/I)^*$ be a derivation. Then $d = \pi^* \circ D \circ q: \mathcal{A} \longrightarrow J^*$ is a derivation, so there exists $x^* \in J^*$ such that $d = \delta_{x^*}$. Let m be the restriction of x^* to I. Then $m \in I^*$ and for all $i \in I$ we have

$$\begin{aligned}\langle i, am - ma\rangle &= \langle ia - ai, m\rangle = \langle ia - ai, x^*\rangle \\ &= \langle i, \delta_{x^*}(a)\rangle = \langle i, \pi^* \circ D \circ q(a)\rangle \\ &= \langle \pi(i), D \circ q(a)\rangle = \langle I, D(a + I)\rangle \\ &= 0.\end{aligned}$$

Therefore, $am = ma (a \in \mathcal{A})$, and so $a^* \in \mathcal{A}^*$ such that $aa^* = a^*a$ and a^* is an extension of m. Let y^* be the restriction of a^* to J. Then $y^* \in J^*$ and $x^* - y^* = 0$ on I. Therefore, $x^* - y^* \in (J/I)^*$ and we have

$$\begin{aligned}\langle j + I, D(a + I)\rangle &= \langle \pi(j), D(q(a))\rangle \\ &= \langle j, \pi^* \circ D \circ q(a)\rangle \\ &= \langle j, \delta_{x^*}(a)\rangle = \langle j, \delta_{x^*-y^*}(a)\rangle \\ &= \langle j + I, \delta_{x^*-y^*}(a + I).\end{aligned}$$

Hence $D = \delta_{x^*-y^*}$ and therefore $\mathcal{A}/I$ is ideally amenable. ∎

Let I be a closed two-sided ideal in Banach algebra $\mathcal{A}$. We consider some easy remarks about the relations between I-weak amenability of $\mathcal{A}$ and weak amenability of I and $\mathcal{A}$.

As in [Jo2] we know that the group algebra $L^1(G)$ is weakly amenable for every locally compact group G, but we do not know whether or not $L^1(G)$ is ideally amenable. By the following theorem we can show that $M(G)$ is I-weakly amenable if and only if $L^1(G)$ is I-weakly amenable for every closed two-sided ideal I in $L^1(G)$.

Theorem 1.13. *Let $\mathcal{A}$ be a Banach algebra and let J be a closed two-sided ideal in $\mathcal{A}$ with a bounded approximate identity. Then for every closed two-sided ideal I in J, J is I-weakly amenable if and only if $\mathcal{A}$ is I-weakly amenable.*

Proof. Let $(j_\alpha)_{\alpha\in\Lambda}$ be a bounded approximate identity for J and D: $J \longrightarrow I^*$ be a derivation. Consider the map $\tilde{D}$: $\mathcal{A} \longrightarrow I^*$ defined by

$$\tilde{D}(a) = w^* - \lim_\alpha (D(aj_\alpha) - a \cdot D(j_\alpha)) \quad (a \in \mathcal{A}).$$

By ([Ru], Proposition 2.1.16) $\tilde{D}$ is a continuous derivation. If $D = 0$, then $\tilde{D} = 0$, since $JI = IJ = I$. Therefore $H^1(J, I^*) = H^1(\mathcal{A}, I^*)$, and this implies that $\mathcal{A}$ is I-weakly amenable if and only if J is I-weakly amenable. ∎

Let $\mathcal{A}^\#$ be the unitization of $\mathcal{A}$. We know that $\mathcal{A}$ is amenable if and only if $\mathcal{A}^\#$ is amenable. If $\mathcal{A}$ is weakly amenable then $\mathcal{A}^\#$ is weakly amenable (see ([D-Gh-G], Proposition 1.4, (ii))). For ideal amenability we have the following:

PROPOSITION 1.14

Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is ideally amenable if and only if $\mathcal{A}^\#$ is ideally amenable.

Proof. Let $\mathcal{A}^\#$ be ideally amenable, I be a closed two-sided ideal of $\mathcal{A}$ and D: $\mathcal{A} \longrightarrow I^*$ be a derivation. It is easy to see that I is a closed two-sided ideal of $\mathcal{A}^\#$, and $\tilde{D}$: $\mathcal{A}^\# \longrightarrow I^*$ such that $\tilde{D}(a+\alpha) = D(a)$, $(a \in \mathcal{A},\ \alpha \in \mathbb{C})$ is a derivation. Since $\mathcal{A}^\#$ is ideally amenable, $\tilde{D}$ is inner and hence D is inner.

Conversely, let $\mathcal{A}$ be ideally amenable and I be a closed two-sided ideal in $\mathcal{A}^\#$. Since $\mathcal{A}$ is ideally amenable, therefore $\mathcal{A}$ is weakly amenable, so $\mathcal{A}^\#$ is weakly amenable. Therefore, we can suppose that $1 \notin I$ and I is an ideal of $\mathcal{A}$. Let D: $\mathcal{A}^\# \longrightarrow I^*$ be a derivation, then $D(1) = 0$ and we can consider D as a derivation from $\mathcal{A}$ into I^* and therefore D is inner. ∎

Let $\mathcal{A}$ be a non-unital Banach algebra. Then similar to the Proposition 1.14 we can show that $\mathcal{A}^{\#}$ is not $\mathcal{A}$-weakly amenable. Let $\mathcal{A}$ be the augmentation ideal of $L^1(PLS_2(R))$. Michael White has shown that $\mathcal{A}$ is not weakly amenable and $\mathcal{A}^{\#}$ is an example of weakly amenable Banach algebra that is not ideally amenable.

2. $\mathcal{C}^*$-algebras

Recall first that, a $\mathcal{C}^*$-algebra is amenable if and only if it is nuclear [Ha]. However, every $\mathcal{C}^*$-algebra is weakly amenable [Ha]. We show that every $\mathcal{C}^*$-algebra is ideally amenable.

Lemma 2.1. *Let $\mathcal{A}$ be a Banach algebra and I be a closed two-sided ideal in $\mathcal{A}$. If I is weakly amenable, then $\mathcal{A}$ is I-weakly amenable.*

Proof. Let $D\colon \mathcal{A} \longrightarrow I^*$ be a derivation and $\iota\colon I \longrightarrow \mathcal{A}$ be the embedding map. Then $D \circ \iota\colon I \longrightarrow I^*$ is a derivation, and so there exists $m \in I^*$ such that $D \circ \iota = \delta_m$. Since I is weakly amenable, $\overline{I^2} = I$, and for $i, j \in I$ we have

$$\begin{aligned} \langle ij, D(a)\rangle &= \langle i, jD(a)\rangle = \langle i, D(ja) - D(j)a\rangle \\ &= \langle i, jam - mja\rangle - \langle ai, jm - mj\rangle \\ &= \langle ija, m\rangle - \langle aij, m\rangle = \langle ij, am - ma\rangle \\ &= \langle ij, \delta_m(a)\rangle \quad (a \in \mathcal{A}). \end{aligned}$$

Therefore $D = \delta_m$, and so D is inner. ■

COROLLARY 2.2

Every C^*-algebra is ideally amenable.

Proof. Let $\mathcal{A}$ be a C^*-algebra and I be a two-sided closed ideal in $\mathcal{A}$, then I is a C^*-algebra by ([B-D], Theorem 38.18), so I is weakly amenable. By Lemma 2.1, $\mathcal{A}$ is I-weakly amenable. ■

COROLLARY 2.3

Let H be a Hilbert space, $K(H)$ be the space of compact operators in $B(H)$. Then $H^1(B(H), K(H)^*) = \{0\}$.

COROLLARY 2.4

Let H be an infinite dimensional Hilbert space. Then $B(H)$ is ideally amenable, but not amenable.

Let G be a locally compact topological group. Then $L^1(G)$ is an ideal in $L^1(G)^{**}$ if and only if G is compact, then by Lemma 2.1 we have the following result.

COROLLARY 2.5

Let G be a compact topological group. Then $L^1(G)^{**}$ is $L^1(G)$-weakly amenable.

3. Maximal ideals

Let M be a maximal ideal in $\mathcal{A}$. We study conditions when $\mathcal{A}$ is M-weakly amenable.

Theorem 3.1. *Let $\mathcal{A}$ be a unital weakly amenable Banach algebra. Then for every maximal ideal M in $\mathcal{A}$, $\mathcal{A}$ is M-weakly amenable.*

Proof. Let $D\colon \mathcal{A} \longrightarrow M^*$ be a derivation. For each a in $\mathcal{A}$, $D(a)$ has an extension $\tilde{D}(a) \in \mathcal{A}^*$ such that $\tilde{D}(1) = 0$. For $a, b \in \mathcal{A}$ there exists $\lambda_1, \lambda_2 \in \mathbb{C}$ and $m, n \in M$ such that

$$\begin{aligned}\tilde{D}(ab) &= \tilde{D}((\lambda_1 \cdot 1 + m)(\lambda_2 \cdot 1 + n))\\ &= \lambda_1 D(n) + \lambda_2 D(m) + \tilde{D}(mn)\\ &= \lambda_1 D(n) + \lambda_2 D(m) + m \cdot D(n) + D(m) \cdot n\\ &= (\lambda_1 \cdot 1 + m) \cdot D(n) + D(m) \cdot (\lambda_2 + n)\\ &= a \cdot \tilde{D}(b) + \tilde{D}(a) \cdot b.\end{aligned}$$

Therefore $\tilde{D} \in \mathcal{Z}^1(\mathcal{A}, \mathcal{A}^*)$ and there exists $a^* \in \mathcal{A}^*$ such that $\tilde{D} = \delta_{a^*}$. Obviously $D = \delta_{a^*|M}$. ■

Theorem 3.2. *Let $\mathcal{A}$ be a Banach algebra and M be a closed two-sided ideal in $\mathcal{A}$ of codimension one. If $\mathcal{A}$ is M-weakly amenable, then M is weakly amenable.*

Proof. Let $D\colon M \longrightarrow M^*$ be a derivation. Since $\mathcal{A} = M \oplus \mathbb{C}$, the map $D_1\colon \mathcal{A} \longrightarrow M^*$ defined by $D_1(m + c) = D(m)$ for $m \in \mathcal{M}, \alpha \in \mathbb{C}$ is an inner derivation. Consequently D is inner. ■

Let F_2 be the free group on two generators. Let $\ell^0(F_2) = \{\mu \in \ell^1(F_2)\colon \mu(F_2) = 0\}$. Then by Theorem 3.1, $\ell^1(F_2)$ is $\ell^0(F_2)$-weakly amenable, and by Theorem 3.2, $\ell^0(F_2)$ is weakly amenable.

Lemma 3.3. *Let $\mathcal{A}$ be a unital commutative Banach algebra and M be a closed two-sided ideal in $\mathcal{A}$ of codimension one. If $\mathcal{A}$ is M-weakly amenable, then $\mathcal{A}$ is weakly amenable.*

Proof. By the above theorem, we know that M is weakly amenable. On the other hand, we have $\mathcal{A} = M \oplus \mathbb{C}$. Therefore by ([Gr1], Proposition 2.3), $\mathcal{A}$ is weakly amenable. ■

Now we have the following theorem:

Theorem 3.4. *Let $\mathcal{A}$ be a commutative unital Banach algebra, the following assertions are equivalent*:

(i) *$\mathcal{A}$ is weakly amenable*,
(ii) *$\mathcal{A}$ is ideal weakly amenable*,
(iii) *$\mathcal{A}$ is M-weakly amenable for some maximal ideal M in $\mathcal{A}$*,
(iv) $H^1(\mathcal{A}, X^*) = \{0\}$ *for every commutative Banach $\mathcal{A}$-bimodule X.*

PROPOSITION 3.5

Let $\mathcal{A}$ be a commutative Banach algebra with a bounded approximate identity and M be a maximal modular ideal in $\mathcal{A}$ also $\mathcal{A}_z \cap M = 0$ (where $\mathcal{A}_z$ is the set of topological divisor of zero elements in $\mathcal{A}$). Let D: $\mathcal{A} \longrightarrow M$ be a derivation such that $D(M) = \{0\}$, then $D = 0$.

Proof. Let $(e_\alpha)_{\alpha\in I}$ be a bounded approximate identity for $\mathcal{A}$. We may suppose that $e_\alpha \in \mathcal{A} \setminus M$ for every $\alpha \in I$. Let $a \in \mathcal{A}$ and $D(a) \neq 0$. Then for every $\alpha \in I$ there exists $a_\alpha \in \mathcal{A} \setminus M$ and $m_\alpha \in M$ such that

$$e_\alpha = m_\alpha + a_\alpha a. \tag{1}$$

Therefore $0 = D^2(e_\alpha) = 2D(a_\alpha)D(a)$. Since $D(a) \in M$ and $\mathcal{A}_z \cap M = 0$, for every $\alpha \in I$, $D(a) = 0$. On the other hand, for every $a' \in \mathcal{A}$, we have

$$\begin{aligned} D(a') &= \lim_\alpha D(e_\alpha a') \\ &= \lim_\alpha D(e_\alpha)a' + \lim_\alpha e_\alpha D(a') \\ &= \lim_\alpha D(e_\alpha)a' + D(a'). \end{aligned}$$

Consequently $\lim_\alpha D(e_\alpha)a' = 0$ for every $a' \in A$. Let a' be a non-zero element of M, since $\mathcal{A}_z \cap M = 0$, $\lim_\alpha D(e_\alpha) = 0$ and by (1), we have $\lim_\alpha a_\alpha D(a) = 0$. Therefore $\lim_\alpha a_\alpha = 0$. Now for each $b \in \mathcal{A}$, $b = \lim_\alpha be_\alpha = \lim_\alpha b(m_\alpha + a_\alpha a) = \lim_\alpha [b(m_\alpha) + ba_\alpha a] = \lim_\alpha bm_\alpha$. Consequently $(m_\alpha)_{\alpha\in I}$ is a approximate identity for $\mathcal{A}$ but by ([Pa], Theorem 5.2.7) M is closed and so $M = \mathcal{A}$ and this is a contradiction. ■

4. Problems

We are interested in the problems listed below.

Johnson [Jo1] has shown that $\mathcal{A}\widehat{\otimes}\mathcal{B}$ is amenable whenever $\mathcal{A}$ and $\mathcal{B}$ are amenable Banach algebras. So we can raise the following question.

Question 4.1. If $\mathcal{A}$ and $\mathcal{B}$ are ideally amenable Banach algebras, then is $\mathcal{A}\widehat{\otimes}\mathcal{B}$ ideally amenable?

We know that $L^1(G)$ is amenable if and only if G is an amenable group [Jo1], and also $L^1(G)$ is weakly amenable for every locally compact group ([Jo1] or [D-Gh]).

Question 4.2. Under what conditions the group algebra $L^1(G)$ is ideally amenable?

Question 4.3. If $L^1(G)^{**}$ is ideally amenable, then so are $M(G)$ and $L^1(G)$ ideally amenable?

Question 4.4. Is $\ell^1(F_2)$ ideally amenable?

For a Banach algebra $\mathcal{A}$, the amenability of $\mathcal{A}^{**}$ necessitates the amenability of $\mathcal{A}$ ([Da], Proposition 2.8.59) and similarly for weak amenability provided $\mathcal{A}$ is a left ideal in $\mathcal{A}^{**}$ [Gh-L-W]. So we can raise the following question.

Question 4.5. If $\mathcal{A}^{**}$ is ideally amenable, then is $\mathcal{A}$ ideally amenable?

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