Proc. Indian Acad. Sci. (Math. Sci.) Vol. 114, No. 4, November 2004, pp. 375–387.

The solutions of the n-dimensional Bessel diamond operator and the Fourier–Bessel transform of their convolution

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MS received 21 April 2004; revised 8 September 2004

Abstract. In this article, the operator $\Diamond_B^k$ is introduced and named as the Bessel diamond operator iterated k times and is defined by

$$\Diamond_B^k = [(B_{x_1} + B_{x_2} + \cdots + B_{x_p})^2 - (B_{x_{p+1}} + \cdots + B_{x_{p+q}})^2]^k,$$

where $p + q = n$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, where $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$ [8], $x_i > 0$, $i = 1, 2, \ldots, n$, k is a non-negative integer and n is the dimension of $\mathbb{R}_n^+$. In this work we study the elementary solution of the Bessel diamond operator and the elementary solution of the operator $\Diamond_B^k$ is called the Bessel diamond kernel of Riesz. Then, we study the Fourier–Bessel transform of the elementary solution and also the Fourier–Bessel transform of their convolution.

Keywords. Diamond operator; tempered distribution; Fourier–Bessel transform.

1. Introduction

Gelfand and Shilov [2] have first introduced the elementary solution of the n-dimensional classical diamond operator. Later, Kananthai [3–5] has proved the distribution related to the n-dimensional ultra-hyperbolic equation, the solutions of n-dimensional classical diamond operator and Fourier transformation of the diamond kernel of Marcel Riesz. Furthermore, Kananthai [4] has showed that the solution of the convolution form $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is a unique elementary solution of the $\Diamond^k u(x) = \delta$.

In this article, we will define the Bessel ultra-hyperbolic type operator iterated k times with $x \in \mathbb{R}_n^+ = \{x: x = (x_1, \ldots, x_n), x_1 > 0, \ldots, x_n > 0\}$,

$$\Box_B^k = (B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - \cdots - B_{x_{p+q}})^k, \quad p + q = n.$$

We will show that the generalized function $R_{2k}(x)$ as defined by (10) is the unique elementary solution of the operator $\Box_B^k$, that is $\Box_B^k R_{2k}(x) = \delta$ where $x \in \mathbb{R}_n^+$ and δ is the Dirac-delta distribution. S is the Shwartz space of any testing functions and S' is a space of tempered distribution.

Furthermore, we will show that the function $E(x)$ as defined by (8) is an elementary solution of the Laplace–Bessel operator

$$\Delta_B = \sum_{i=1}^{n} B_{x_i} = \sum_{i=1}^{n} \left(\frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial}{\partial x_i} \right), \tag{1}$$

that is, $\Delta_B E(x) = \delta$ where $x \in \mathbb{R}_n^+$.

The operator $\diamond_B^k$ can be expressed as the product of the operators $\Box_B$ and Δ_B, that is,

$$\begin{aligned}
\diamond_B^k &= \left[\left(\sum_{i=1}^{p} B_{x_i} \right)^2 - \left(\sum_{i=p+1}^{p+q} B_{x_i} \right)^2 \right]^k \\
&= \left[\sum_{i=1}^{p} B_{x_i} - \sum_{i=p+1}^{p+q} B_{x_i} \right]^k \left[\sum_{i=1}^{p} B_{x_i} + \sum_{i=p+1}^{p+q} B_{x_i} \right]^k \\
&= \Box_B^k \Delta_B^k.
\end{aligned} \tag{2}$$

Denoted by T^y the generalized shift operator acting according to the law [8]

$$T_x^y \varphi(x) = C_v^* \int_0^{\pi} \dots \int_0^{\pi} \varphi \left(\sqrt{x_1^2 + y_1^2 - 2x_1 y_1 \cos\theta_1}, \dots, \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\theta_n} \right)$$
$$\times \left(\prod_{i=1}^{n} \sin^{2v_i - 1} \theta_i \right) \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n,$$

where $x, y \in \mathbb{R}_n^+$, $C_v^* = \prod_{i=1}^{n} \frac{\Gamma(v_i+1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$. We remark that this shift operator is closely connected with the Bessel differential operator [8].

$$\begin{aligned}
\frac{\mathrm{d}^2U}{\mathrm{d}x^2} + \frac{2v}{x}\frac{\mathrm{d}U}{\mathrm{d}x} &= \frac{\mathrm{d}^2U}{\mathrm{d}y^2} + \frac{2v}{y}\frac{\mathrm{d}U}{\mathrm{d}y}, \\
U(x,0) &= f(x), \\
U_y(x,0) &= 0.
\end{aligned}$$

The convolution operator determined by T^y is as follows:

$$(f * \varphi)(y) = \int_{\mathbb{R}_n^+} f(y) T_x^y \varphi(x) \left(\prod_{i=1}^{n} y_i^{2v_i} \right) \mathrm{d}y. \tag{3}$$

Convolution (3) is known as a B-convolution. We note the following properties for the B-convolution and the generalized shift operator:

(a) $T_x^y \cdot 1 = 1$.
(b) $T_x^0 \cdot f(x) = f(x)$.
(c) If $f(x), g(x) \in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function, $x > 0$ and

$$\int_0^{\infty} |f(x)| \left(\prod_{i=1}^{n} x_i^{2v_i} \right) \mathrm{d}x < \infty,$$

then

$$\int_{\mathbb{R}_n^+} T_x^y f(x) g(y) \left(\prod_{i=1}^{n} y_i^{2v_i} \right) \mathrm{d}y = \int_{\mathbb{R}_n^+} f(y)\, T_x^y g(x) \left(\prod_{i=1}^{n} y_i^{2v_i} \right) \mathrm{d}y.$$

(d) From (c), we have the following equality for $g(x) = 1$.

$$\int_{\mathbb{R}_n^+} T_x^y f(x) \left(\prod_{i=1}^{n} y_i^{2v_i} \right) \mathrm{d}y = \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^{n} y_i^{2v_i} \right) \mathrm{d}y.$$

(e) $(f * g)(x) = (g * f)(x)$.

The Fourier–Bessel transformation and its inverse transformation are defined as follows [9]:

$$(F_B f)(x) = C_v \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^{n} j_{v_i - \frac{1}{2}} (x_i, y_i)\, y_i^{2v_i} \right) \mathrm{d}y,$$

$$(F_B^{-1} f)(x) = (F_B f)(-x), \; C_v = \left(\prod_{i=1}^{n} 2^{v_i - \frac{1}{2}} \left(v_i + \frac{1}{2} \right) \right)^{-1},$$

where $j_{v_i-(1/2)}(x_i, y_i)$ is the normalized Bessel function which is the eigenfunction of the Bessel differential operator. The following are the equalities for Fourier–Bessel transformation [6,7]:

$$\begin{aligned} &F_B \delta(x) = 1 \\ &F_B(f * g)(x) = F_B f(x) \cdot F_B g(x). \end{aligned} \tag{4}$$

Now we are finding the solution of the equation

$$\diamond_B^k u(x) = \sum_{r=0}^{m} \diamond_B^r \delta, \quad \diamond_B^0 \delta = \delta$$

or

$$\square_B^k \Delta_B^k u(x) = \sum_{r=0}^{m} \square_B^r \Delta_B^r \delta. \tag{5}$$

In finding the solutions of (5), we use the properties of B-convolutions for the generalized functions.

Lemma 1. *There is the following equality for Fourier–Bessel transformation*

$$F_B(|x|^{-\alpha}) = 2^{n+2|v|-2\alpha} \Gamma \left(\frac{n + 2|v| - \alpha}{2} \right) \left[\Gamma \left(\frac{\alpha}{2} \right) \right]^{-1} |x|^{\alpha - n - 2|v|},$$

where $|v| = v_1 + \cdots + v_n$.

The proof of this lemma is given in [9].

Lemma 2. *Given the equation* $\Delta_B E(x) = \delta$ *for* $x \in \mathbb{R}_n^+$, *where* Δ_B *is the Laplace–Bessel operator defined by* (1),

$$E(x) = -S_2(x) \tag{6}$$

is an elementary solution of the operator Δ_B *where*

$$S_2(x) = \frac{2^{n+2|v|-4}\Gamma\left(\frac{n+2|v|-2}{2}\right)}{\prod_{i=1}^{n} 2^{v_i-\frac{1}{2}}\Gamma\left(v_i+\frac{1}{2}\right)} |x|^{2-n-2|v|}.$$

Proof. For the equation $\Delta_B E(x) = \delta$, we have

$$F_B \Delta_B E = F_B \delta. \tag{7}$$

First, we consider the left side of (7)

$$\begin{aligned} F_B \Delta_B E(x) &= C_v \int_{\mathbb{R}_n^+} (\Delta_B E(y)) \left(\prod_{i=1}^{n} J_{v_i-\frac{1}{2}}(x_i, y_i)\, y_i^{2v_i}\right) \mathrm{d}y \\ &= C_v \int_{\mathbb{R}_n^+} \left(\sum_{i=1}^{n} \frac{\partial^2 E(y)}{\partial y_i^2}\right) \left(\prod_{i=1}^{n} J_{v_i-\frac{1}{2}}(x_i, y_i)\, y_i^{2v_i}\right) \mathrm{d}y \\ &\quad + C_v \int_{\mathbb{R}_n^+} \left(\sum_{i=1}^{n} \frac{2v_i}{y_i} \frac{\partial E(y)}{\partial y_i}\right) \left(\prod_{i=1}^{n} J_{v_i-\frac{1}{2}}(x_i, y_i)\, y_i^{2v_i}\right) \mathrm{d}y. \end{aligned}$$

If we apply partial integration twice in the first integral and once in the second integral, then we have

$$\begin{aligned} F_B \Delta_B E(x) &= C_v \int_{\mathbb{R}_n^+} E(y) \left(\sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2} \prod_{i=1}^{n} J_{v_i-\frac{1}{2}}(x_i, y_i)\right) y_i^{2v_i} \mathrm{d}y \\ &\quad + C_v \int_{\mathbb{R}_n^+} E(y) \left(\sum_{i=1}^{n} \frac{2v_i}{y_i} \frac{\partial}{\partial y_i} \prod_{i=1}^{n} J_{v_i-\frac{1}{2}}(x_i, y_i)\right) y_i^{2v_i} \mathrm{d}y \\ &= C_v \int_{\mathbb{R}_n^+} E(y) \left(\sum_{i=1}^{n} B_{y_i} \prod_{i=1}^{n} J_{v_i-\frac{1}{2}}(x_i, y_i)\right) y_i^{2v_i} \mathrm{d}y. \end{aligned}$$

Here, if we use the following equality [8],

$$\int_0^\infty E(y) B_{y_i} J_{v_i-\frac{1}{2}}(x_i, y_i)\, y_i^{2v_i} \mathrm{d}y_i = -x_i^2 \int_0^\infty E(y) J_{v_i-\frac{1}{2}}(x_i, y_i)\, y_i^{2v_i} \mathrm{d}y_i,$$

then we have

$$\begin{aligned} F_B \Delta_B E(x) &= -(x_1^2 + x_2^2 + \cdots + x_n^2) C_v \\ &\quad \times \int_{\mathbb{R}_n^+} E(y) \left(\prod_{i=1}^{n} J_{v_i-\frac{1}{2}}(x_i, y_i)\, y_i^{2v_i}\right) \mathrm{d}y \\ &= -|x|^2 F_B E(x). \end{aligned}$$

The right side of equality (7) is $F_B\delta = 1$. Then

$$F_B \Delta_B E = -|x|^2 F_B E = 1.$$

From Lemma 1 and inverse Fourier–Bessel transformation we obtain

$$E(x) = -\frac{2^{n+2|v|-4}\Gamma\left(\frac{n+2|v|-2}{2}\right)}{\prod_{i=1}^{n} 2^{v_i-\frac{1}{2}}\Gamma\left(v_i+\frac{1}{2}\right)}|x|^{2-n-2|v|}. \tag{8}$$

That completes the proof. □

Lemma 3. *Given the equation* $\Delta_B^k u(x) = \delta$ *for* $x \in \mathbb{R}_n^+$, *where* Δ_B^k *is the Laplace–Bessel operator iterated k times defined by*

$$\Delta_B^k = (B_{x_1} + B_{x_2} + \cdots + B_{x_n})^k,$$

then $u(x) = (-1)^k S_{2k}(x)$ *is an elementary solution of the operator* Δ_B^k *where*

$$S_{2k}(x) = \frac{2^{n+2|v|-4k}\Gamma\left(\frac{n+2|v|-2k}{2}\right)}{\prod_{i=1}^{n} 2^{v_i-\frac{1}{2}}\Gamma\left(v_i+\frac{1}{2}\right)}|x|^{2k-n-2|v|}. \tag{9}$$

The proof of Lemma 3 is similar to the proof of Lemma 2.

Lemma 4. *If* $\Box_B^k u(x) = \delta$ *for* $x \in \Gamma_+ = \{x \in \mathbb{R}_n^+ \colon x_1 > 0, x_2 > 0, \ldots, x_n > 0$ *and* $V > 0\}$, *where* $\Box_B^k$ *is the Bessel-ultra hyperbolic operator iterated k times defined by*

$$\Box_B^k = (B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - \cdots - B_{x_{p+q}})^k, \quad p+q=n,$$

then $u(x) = R_{2k}(x)$ *is the unique elementary solution of the operator* $\Box_B^k$ *where*

$$\begin{aligned} R_{2k}(x) &= \frac{V^{\left(\frac{2k-n-2|v|}{2}\right)}}{K_n(2k)} \\ &= \frac{(x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2)^{\left(\frac{2k-n-2|v|}{2}\right)}}{K_n(2k)} \end{aligned} \tag{10}$$

for

$$K_n(2k) = \frac{\pi^{\frac{n+2|v|-1}{2}}\Gamma\left(\frac{2+2k-n-2|v|}{2}\right)\Gamma\left(\frac{1-2k}{2}\right)\Gamma(2k)}{\Gamma\left(\frac{2+2k-p-2|v|}{2}\right)\Gamma\left(\frac{p-2k}{2}\right)}.$$

The proof of this lemma can be from Lemmas 1–3.

Lemma 5. $R_{2k}(x)$ *and* $S_{2k}(x)$ *are homogeneous distributions of order* $(2k - n - 2|v|)$.

Proof. We need to show that $R_{2k}(x)$ satisfies the Euler equation

$$(2k-n-2|v|)R_{2k}(x)=\sum_{i=1}^{n}x_i\frac{\partial R_{2k}(x)}{\partial x_i}.$$

Now

$$\begin{aligned}
&\sum_{i=1}^{n}x_i\frac{\partial R_{2k}(x)}{\partial x_i}\\
&=\frac{1}{K_n(2k)}\sum_{i=1}^{n}x_i\frac{\partial}{\partial x_i}(x_1^2+x_2^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2)^{\left(\frac{2k-n-2|v|}{2}\right)}\\
&=\frac{(2k-n-2|v|)}{K_n(2k)}(x_1^2+x_2^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2)^{\left(\frac{2k-n-2|v|}{2}-1\right)}\\
&\quad\times(x_1^2+x_2^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2)\\
&=\frac{(2k-n-2|v|)}{K_n(2k)}(x_1^2+x_2^2+\cdots+x_p^2-x_{p+1}^2-\cdots-x_{p+q}^2)^{\left(\frac{2k-n-2|v|}{2}\right)}\\
&=(2k-n-2|v|)R_{2k}(x).
\end{aligned}$$

Hence $R_{2k}(x)$ is a homogeneous distribution of order $(2k-n-2|v|)$ as required and similarly $S_{2k}(x)$ is also a homogeneous distribution of order $(2k-n-2|v|)$. □

Lemma 6. *$R_{2k}(x)$ and $S_{2k}(x)$ are the tempered distributions.*

Proof. Choose supp $R_{2k}=K\subset\bar{\Gamma}_+$, where K is a compact set. Then R_{2k} is a tempered distribution with compact support and by [1], pp. 156–159, $S_{2k}(x)*R_{2k}(x)$ exists and is a tempered distribution. □

Lemma 7. (*The B-convolutions of tempered distributions*)

$$S_{2k}(x)*S_{2m}(x)=S_{2k+2m}(x). \tag{11}$$

Proof. From Lemmas 1 and 2 we have

$$F_BS_{2k}(x)=-|x|^{-2k}\quad\text{and}\quad F_BS_{2m}(x)=-|x|^{-2m}.$$

Thus from (4) we obtain

$$\begin{aligned}
F_B\,(S_{2k}(x)*S_{2m}(x))&=F_BS_{2k}(x)F_BS_{2m}(x)\\
&=|x|^{-2k-2m}\\
S_{2k}(x)*S_{2m}(x)&=F_B^{-1}|x|^{-2k-2m}\\
&=C(v,m,k,n)\,|x|^{-2k-2m-n-2|v|}\\
&=S_{2k+2m}(x),
\end{aligned}$$

where

$$C(v, m, k, n) = \frac{2^{n+2|v|-4(m+k)}\Gamma\left(\frac{n+2|v|-2(m+k)}{2}\right)}{\prod_{i=1}^{n} 2^{v_i-\frac{1}{2}}\Gamma\left(v_i+\frac{1}{2}\right)}.$$

Now from (6) and (11) with $m = k = 1$, we have

$$\begin{aligned} E(x) * E(x) &= (-S_2(x)) * (-S_2(x)) \\ &= (-1)^2 S_{2+2}(x) \\ &= S_4(x). \end{aligned}$$

By induction, we obtain

$$\underbrace{E(x) * E(x) * \cdots * E(x)}_{k \text{ times}} = (-1)^k S_{2k}(x). \tag{12}$$

□

Lemma 8. *Given the equation* $\Delta_B^k u(x) = \delta$, *then* $u(x) = (-1)^k S_{2k}(x)$ *is an elementary solution of the operator* Δ_B^k *where,* $(-1)^k S_{2k}(x)$ *is defined by* (12).

Proof. Now $\Delta_B^k u(x) = \delta$ can be written in the form

$$\Delta_B^k \delta * u(x) = \delta.$$

B-convolving both sides by the function $E(x)$ defined by (8), we obtain

$$(E(x) * \Delta_B^k \delta) * u(x) = E(x) * \delta = E(x)$$

and

$$(\Delta_B E(x) * \Delta_B^{k-1} \delta) * u(x) = E(x).$$

Since $\Delta_B^k E(x) = \delta$ we have

$$(\delta * \Delta_B^{k-1} \delta) * u(x) = E(x).$$

Hence

$$(\Delta_B^{k-1} \delta) * u(x) = E(x).$$

By keeping on B-convolving $E(x)$, $k - 1$ times, we obtain

$$\delta * u(x) = \underbrace{E(x) * E(x) * \cdots * E(x)}_{k \text{ times}}.$$

It follows that

$$u(x) = (-1)^k S_{2k}(x)$$

by (12) as required. □

Before proving the theorems, we need to define the B-convolution of $(-1)^k S_{2k}(x)$ with $R_{2k}(x)$ defined by (10) with $k = 0, 1, 2, \ldots$. Now for the case $2k \geq n + 2\,|v|$, we obtain $(-1)^k S_{2k}(x)$ and $R_{2k}(x)$ as analytic functions that are ordinary functions. Thus the B-convolution

$$(-1)^k S_{2k}(x) * R_{2k}(x) \tag{13}$$

exists. Now for the case $2k < n + 2\,|v|$, by Lemma 6 we obtain $(-1)^k S_{2k}(x)$ and $R_{2k}(x)$ as tempered distributions.

Let K be a compact set and $K \subset \bar{\Gamma}_+$ where $\bar{\Gamma}_+$ is defined closer to Γ_+. Choose the support of $R_{2k}(x)$ equal to K, then $\mathrm{supp} R_{2k}(x)$ is compact (closed and bounded). So the B-convolution

$$(-1)^k S_{2k}(x) * R_{2k}(x) \tag{14}$$

exists and is a tempered distribution from Lemma 6.

Theorem 1. *Given the equation* $\Diamond_B^k u(x) = \delta$ *for* $x \in \mathbb{R}_n^+$, *where* $\Diamond_B^k$ *is a diamond Bessel operator iterated k times defined by* (2) *then* $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$, *defined by* (13) *and* (14), *is a unique elementary solution of the operator* $\Diamond_B^k$.

Proof. Now $\Diamond_B^k u(x) = \delta$ can be written as $\Diamond_B^k u(x) = \Box_B^k \Delta_B^k u(x)$ by (2). $\Delta_B^k u(x) = R_{2k}(x)$ is a unique elementary solution of the operator $\Box_B^k$ for n odd with p odd and q even, or for n even with p odd and q odd. By the method of B-convolution, we have

$$\Delta_B^k \delta * u(x) = R_{2k}(x).$$

B-convolving both sides by $(-1)^k S_{2k}(x)$, we obtain

$$((-1)^k S_{2k}(x) * \Delta_B^k \delta) * u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$$

or

$$\Delta_B^k((-1)^k S_{2k}(x)) * u(x) = (-1)^k S_{2k}(x) * R_{2k}(x).$$

It follows that $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ by Lemma 8. That completes the proof. □

Theorem 2. *For* $0 < r < k$,

$$\Diamond_B^r((-1)^k S_{2k}(x) * R_{2k}(x)) = (-1)^{k-r} S_{2k-2r}(x) * R_{2k-2r}(x)$$

and for $k \leq m$

$$\Diamond_B^m((-1)^k S_{2k}(x) * R_{2k}(x)) = \Diamond_B^{m-k}\delta.$$

Proof. From Theorem 1,

$$\Diamond_B^k((-1)^k S_{2k}(x) * R_{2k}(x)) = \delta.$$

Thus

$$\Diamond_B^{k-r}\Diamond_B^r((-1)^k S_{2k}(x) * R_{2k}(x)) = \delta$$

or

$$\diamond_B^{k-r}\delta * \diamond_B^r((-1)^k S_{2k}(x) * R_{2k}(x)) = \delta.$$

B-convolving both sides by $(-1)^{k-r}S_{2k-2r}(x) * R_{2k-2r}(x)$, we obtain

$$\begin{aligned}&\diamond_B^{k-r}[(-1)^{k-r}S_{2k-2r}(x) * R_{2k-2r}(x)] * \diamond_B^r[(-1)^k S_{2k}(x) * R_{2k}(x)]\\&\quad = [(-1)^{k-r}S_{2k-2r}(x) * R_{2k-2r}(x)] * \delta,\end{aligned}$$

or by Theorem 1

$$\delta * \diamond_B^r[(-1)^k S_{2k}(x) * R_{2k}(x)] = [(-1)^{k-r}S_{2k-2r}(x) * R_{2k-2r}(x)].$$

It follows that for $0 < r < k$,

$$\diamond_B^r[(-1)^k S_{2k}(x) * R_{2k}(x)] = (-1)^{k-r}S_{2k-2r}(x) * R_{2k-2r}(x)$$

as required. For $k \leq m$,

$$\begin{aligned}\diamond_B^m((-1)^k S_{2k}(x) * R_{2k}(x)) &= \diamond_B^{m-k}\diamond_B^k[(-1)^k S_{2k}(x) * R_{2k}(x)]\\&= \diamond_B^{m-k}\delta\end{aligned}$$

by Theorem 1. That completes the proofs. □

Theorem 3. *Given the linear differential equation*

$$\diamond_B^k u(x) = \sum_{r=0}^{m} c_r \diamond_B^r \delta, \tag{15}$$

where the operator $\diamond_B^k$ is defined by (2), *n is odd with p odd and q even, or n is even with p odd and q odd, c_r is a constant, δ is the Dirac-delta distribution and $\diamond_B^0\delta = \delta$.*

Then the solutions of (15) that depend on the relationship between the values of k and m are as follows:

(1) If $m < k$ and $m = 0$, then (15) has the solution $u(x) = c_0(-1)^k S_{2k}(x) * R_{2k}(x)$, which is an elementary solution of the operator $\diamond_B^k$ in Theorem 1 and is the ordinary function for $2k < n + 2\,|v|$.

(2) If $0 < m < k$, then the solution of (15) is

$$u(x) = \sum_{r=1}^{m}[(-1)^{k-r}S_{2k-2r}(x) * R_{2k-2r}(x)]$$

which is an ordinary function for $2k - 2r \geq n + 2\,|v|$, and a tempered distribution for $2k - 2r < n + 2\,|v|$.

(3) If $m \geq k$, and suppose $k \leq n + 2\,|v| \leq M$, then (15) has the solution

$$u(x) = \sum_{r=k}^{M} c_r \diamond_B^{r-k}$$

which is only the singular distribution.

Proof of this theorem can be easily seen from Theorems 1, 2 and [4].

Lemma 9 *(The Fourier–Bessel transformation $\diamond_B^k \delta$). Let $\|x\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ for $x \in \mathbb{R}_n^+$. Then*

$$|F_B \diamond_B^k \delta| \leq C_\upsilon \|x\|^{2k}. \tag{16}$$

That is, $F_B \Delta_B^k \delta$ is bounded and continuous on the space S' of the tempered distribution. Moreover, by the inverse Fourier–Bessel transformation

$$\diamond_B^k \delta = C_\upsilon F_B^{-1}[(x_1^2 + x_2^2 + \cdots + x_p^2)^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2]^k.$$

Proof. From the Fourier–Bessel transform we have

$$\begin{aligned} F_B \diamond_B^k \delta(x) &= C_v \int_{\mathbb{R}_n^+} \diamond_B^k \delta(y) \left(\prod_{i=1}^n j_{v_i - \frac{1}{2}}(x_i, y_i)\, y_i^{2v_i} \right) \mathrm{d}y \\ &= C_v \int_{\mathbb{R}_n^+} \Delta_B^k \square_B^k \delta(y) \left(\prod_{i=1}^n j_{v_i - \frac{1}{2}}(x_i, y_i)\, y_i^{2v_i} \right) \mathrm{d}y \\ &= C_v \int_{\mathbb{R}_n^+} \Delta_B^k g(y) \left(\prod_{i=1}^n j_{v_i - \frac{1}{2}}(x_i, y_i)\, y_i^{2v_i} \right) \mathrm{d}y, \end{aligned}$$

where $g(y) = \square_B^k \delta(y)$. For $k \in \mathbb{N}$ we have [10]

$$F_B(\Delta_B^k) f = (-1)^k |x|^{2k} F_B f.$$

So we have

$$\begin{aligned} F_B \diamond_B^k \delta(x) &= C_v (-1)^k |x|^{2k} F_B g(x) \\ &= C_v (-1)^k (x_1^2 + \cdots + x_n^2)^k F_B \square_B^k \delta(x). \end{aligned}$$

The same way we have following equality:

$$F_B \square_B^k \delta(x) = C_v (-1)^k (x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2)^k F_B \delta(x).$$

Since $F_B \delta(x) = 1$, we can write

$$\begin{aligned} F_B \diamond_B^k \delta(x) &= C_v (-1)^{2k} (x_1^2 + \cdots + x_n^2)^k \\ &\quad \times (x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2)^k \\ &= C_v [(x_1^2 + \cdots + x_p^2)^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2]^k. \end{aligned}$$

Then there is the following inequality:

$$\begin{aligned} |F_B \diamond_B^k \delta| &= C_v [|x_1^2 + \cdots + x_n^2||x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2|]^k \\ &\leq C_v [|x_1^2 + \cdots + x_n^2|]^{2k} \\ &= C_v \|x\|^{4k}. \end{aligned}$$

Therefore $F_B \diamond_B^k$ is bounded and continuous on the space S' of the tempered distribution.

Since F_B is 1-1 transformation from S' to $\mathbb{R}_n^+$, there is the following equation:

$$\diamond_B^k \delta = C_\upsilon F_B^{-1}[(x_1^2 + x_2^2 + \cdots + x_p^2)^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2]^k.$$

That completes the proof. □

Theorem 4.

$$F_B[(-1)^k S_{2k}(x) * R_{2k}(x)] \leq \frac{C_\upsilon}{[(x_1^2 + x_2^2 + \cdots + x_p^2)^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2]^k}$$

and

$$|F_B[(-1)^k S_{2k}(x) * R_{2k}(x)]| = C_\upsilon M \quad \textit{for a large } x_i \in \mathbb{R}^+,$$

where M is a constant. That is, F_B is bounded and continuous on the space S' of the tempered distribution.

Proof. By Lemma 8,

$$\diamond_B^k[(-1)^k S_{2k}(x) * R_{2k}(x)] = \delta$$

or

$$(\diamond_B^k \delta) * [(-1)^k S_{2k}(x) * R_{2k}(x)] = \delta. \tag{17}$$

If we applied the Fourier–Bessel transform on both sides of (17), then we obtain

$$F_B[(\diamond_B^k \delta) * [(-1)^k S_{2k}(x) * R_{2k}(x)]] = F_B \delta(x)$$

$$C_v \left\langle (\diamond_B^k \delta) * [(-1)^k S_{2k}(x) * R_{2k}(x)], \prod_{i=1}^n j_{v_i - \frac{1}{2}}(x_i, y_i) y_i^{2v_i} \right\rangle = C_v.$$

By the properties of B-convolution

$$C_v \left\langle (\diamond_B^k \delta), \left\langle [(-1)^k S_{2k}(x) * R_{2k}(x)], \prod_{i=1}^n j_{v_i - \frac{1}{2}}(z_i, y_i)\, y_i^{2v_i} \prod_{i=1}^n j_{v_i - \frac{1}{2}}(x_i, y_i)\, y_i^{2v_i} \right\rangle \right\rangle = C_v,$$

$$C_v \left\langle [(-1)^k S_{2k}(x) * R_{2k}(x)], \prod_{i=1}^n j_{v_i - \frac{1}{2}}(z_i, y_i)\, y_i^{2v_i} \right\rangle \left\langle (\diamond_B^k \delta), \prod_{i=1}^n j_{v_i - \frac{1}{2}}(x_i, y_i)\, y_i^{2v_i} \right\rangle = C_v,$$

$$F_B[(-1)^k S_{2k}(x) * R_{2k}(x)] \frac{1}{C_v} F_B(\diamond_B^k \delta) = C_v.$$

By Lemma 9,

$$F_B[(\diamond_B^k \delta) * [(-1)^k S_{2k}(x) * R_{2k}(x)]]$$
$$\times [(x_1^2 + x_2^2 + \cdots + x_p^2)^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2]^k = C_v.$$

It follows that

$$F_B[(-1)^k S_{2k}(x) * R_{2k}(x)]$$
$$= \frac{C_v}{[(x_1^2 + x_2^2 + \cdots + x_p^2)^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2]^k}.$$

Now

$$|F_B[(-1)^k S_{2k}(x) * R_{2k}(x)]|$$
$$= \frac{C_v}{|x_1^2 + x_2^2 + \cdots + x_n^2|^k |x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2|^k}, \quad (18)$$

where $x = (x_1, \ldots, x_n) \in \Gamma_+$ with Γ_+ defined by Lemma 4. Then $(x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2) > 0$ and for a large x_i and a large k, the right-hand side of (18) tends to zero. It follows that it is bounded by a positive constant say M, that is, we obtain (17) as required and also by (17). F_B is continuous on the space S' of the tempered distribution. □

Theorem 5.

$$F_B[[(-1)^k S_{2k}(x) * R_{2k}(x)] * [(-1)^k S_{2m}(x) * R_{2m}(x)]]$$
$$= F_B[(-1)^k S_{2k}(x) * R_{2k}(x)] F_B[(-1)^m S_{2m}(x) * R_{2m}(x)]$$
$$= \frac{C_v}{((x_1^2 + x_2^2 + \cdots + x_p^2)^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2)^{k+m}},$$

where k and m are non-negative integers and F_B is bounded and continuous on the space S' of the tempered distribution.

Proof. Since $S_{2k}(x)$ and $R_{2k}(x)$ are tempered distribution with compact support, from Lemmas 6 and 7 we have

$$[(-1)^k S_{2k}(x) * R_{2k}(x)] * [(-1)^m S_{2m}(x) * R_{2m}(x)]$$
$$= (-1)^{k+m} [S_{2k}(x) * S_{2m}(x)] * [R_{2k}(x) * R_{2m}(x)]$$
$$= (-1)^{k+m} S_{2(k+m)}(x) * R_{2(k+m)}(x).$$

Taking the Fourier transform on both sides and using Theorem 4 we obtain

$$F_B[(-1)^k S_{2k}(x) * R_{2k}(x)] * [(-1)^m S_{2m}(x) * R_{2m}(x)]$$
$$= \frac{C_v}{[x_1^2 + x_2^2 + \cdots + x_p^2)^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2]^{k+m}}$$

$$= \frac{1}{C_v} \frac{C_v}{[(x_1^2 + x_2^2 + \cdots + x_p^2)^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2]^k}$$

$$\times \frac{C_v}{[(x_1^2 + x_2^2 + \cdots + x_p^2)^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2]^m}$$

$$= \frac{1}{C_v} F_B[(-1)^k S_{2k}(x) * R_{2k}(x)] F_B[(-1)^m S_{2m}(x) * R_{2m}(x)].$$

Since $(-1)^{k+m} S_{2(k+m)}(x) * R_{2(k+m)}(x) \in S'$, the space of tempered distribution, and Theorem 4 we obtain that F_B is bounded and continuous on S'. □

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