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Superstability of the generalized orthogonality equation on restricted domains

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Abstract. Chmieliński has proved in the paper [4] the superstability of the generalized orthogonality equation $|\langle f(x), f(y)\rangle| = |\langle x, y\rangle|$. In this paper, we will extend the result of Chmieliński by proving a theorem: Let D_n be a suitable subset of $\mathbb{R}^n$. If a function $f: D_n \to \mathbb{R}^n$ satisfies the inequality $||\langle f(x), f(y)\rangle| - |\langle x, y\rangle|| \le \varphi(x, y)$ for an appropriate control function $\varphi(x, y)$ and for all $x, y \in D_n$, then f satisfies the generalized orthogonality equation for any $x, y \in D_n$.

Keywords. Superstability; generalized orthogonality equation.

1. Introduction

In 1931, Wigner introduced in his book [13] the generalized orthogonality equation

$$|\langle f(x), f(y)\rangle| = |\langle x, y\rangle| \tag{1}$$

for all $x, y \in E$, where E is an inner product space and $\langle\cdot\,,\cdot\rangle$ denotes the inner product on E. This functional equation was solved in [1,2,7,9,10] by many mathematicians.

Recently, Chmieliński [4] proved that the generalized orthogonality equation is superstable when the relevant functions belong to the class of functions $f: \mathbb{R}^n \to \mathbb{R}^n$. If a function $f: \mathbb{R}^n \to \mathbb{R}^n$ $(n \ge 2)$ satisfies the functional inequality

$$||\langle f(x), f(y)\rangle| - |\langle x, y\rangle|| \le \varepsilon$$

for some $\varepsilon \ge 0$ and for all $x, y \in \mathbb{R}^n$, then f is a solution of the generalized orthogonality equation (1).

We will refer the reader to [3,6,8,12] for detailed definitions of stability and superstability of functional equations.

By using ideas of Skof and Rassias [8,11], and by following the methods of Chmieliński [4,5] mainly, we will extend the result of Chmieliński by considering the case when the domain of f is restricted and by substituting an appropriate control function $\varphi(x, y)$ for ε in the relevant inequality as well.

Throughout this paper, let $c > 0$ $(c \ne 1)$ and $d > 0$ be constants and let $n \ge 2$ be a fixed natural number. By $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}$ we denote the set of positive integers, of non-negative integers and of real numbers, respectively. We will also use the notation $\text{lin}\{x_1, \ldots, x_k\}$ to denote the subspace of $\mathbb{R}^n$ spanned by $x_1, \ldots, x_k \in \mathbb{R}^n$. Let us define a subset D_n of $\mathbb{R}^n$ by

$$D_n := \begin{cases} \{x \in \mathbb{R}^n : \|x\| \geq d\}, & \text{for } 0 < c < 1, \\ \{x \in \mathbb{R}^n : \|x\| < d\}, & \text{for } c > 1, \end{cases}$$

where we denote by $\|\cdot\|$ the usual norm on $\mathbb{R}^n$ defined by

$$\|x\| := \sqrt{\langle x, x\rangle}$$

with the usual inner product $\langle \cdot, \cdot \rangle$ defined by

$$\langle x, y\rangle := x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

for all points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ of $\mathbb{R}^n$.

Suppose φ: $\mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ is a symmetric function which satisfies the following conditions:

(i) There exists a function ϕ: $[0, \infty) \times [0, \infty) \to [0, \infty)$ such that $\varphi(x, y) = \phi(\|x\|, \|y\|)$ for all $x, y \in \mathbb{R}^n$.

(ii) For all $x, y \in \mathbb{R}^n$,

$$\frac{1}{|\lambda|}\, \varphi(\lambda x, y) = O\left(-\frac{\ln c}{\ln |\lambda|}\right)$$

either as $|\lambda| \to \infty$ (for $0 < c < 1$) or as $|\lambda| \to 0$ (for $c > 1$).

(iii) If both $|\lambda|$ and $|\mu|$ are different from 1, then for all $x, y \in \mathbb{R}^n$,

$$\frac{1}{|\lambda\mu|}\, \varphi(\lambda x, \mu y) = O\left(\left|\frac{\ln c}{\ln|\lambda|}\,\frac{\ln c}{\ln|\mu|}\right|\right)$$

either as $|\lambda\mu| \to \infty$ (for $0 < c < 1$) or as $|\lambda\mu| \to 0$ (for $c > 1$).

2. Preliminaries

We begin by introducing a lemma of [4] which turns out to be very useful to prove Lemma 4 below.

Theorem 1. *Let $\varepsilon \geq 0$ be given. For each $\eta > 0$ there exists $k_0 \in \mathbb{N}$ such that if $a, u_1, u_2, \ldots, u_{n-1} \in \mathbb{R}^n \setminus \{0\}$ satisfy the conditions*

$$1 - \frac{\varepsilon}{k^2} \leq \|u_i\|^2 \leq 1 + \frac{\varepsilon}{k^2} \quad (i = 1, 2, \ldots, n-1),$$

$$|\langle u_i, u_j\rangle| \leq \frac{\varepsilon}{k^2} \quad (i, j = 1, 2, \ldots, n-1;\ i \neq j),$$

$$|\langle a, u_i\rangle| \leq \frac{\varepsilon}{k} \quad (i = 1, 2, \ldots, n-1),$$

for any $k \geq k_0$; then

(a) $u_1, \ldots, u_{n-1}$ *are linearly independent*;

(b) $|\cos A(a, \ell)| \geq 1 - \eta$, *where ℓ denotes the line in $\mathbb{R}^n$ which is the orthogonal complement of* $\text{lin}\{u_1, \ldots, u_{n-1}\}$ *and $A(\cdot, \cdot)$ stands for the angle.*

In the following five lemmas, we will modify the statements of Proposition 1 in [4] and later apply them to the proof of our main result.

In the following lemmas and theorems of this section, we assume that the function $f\colon D_n \to \mathbb{R}^n$ satisfies the inequality

$$||\langle f(x), f(y)\rangle| - |\langle x, y\rangle|| \le \varphi(x, y) \tag{2}$$

for all $x, y \in D_n$ if there is no specification for f.

It is enough to put $y = x$ in the inequality (2) to prove the following lemma.

Lemma 2. *The following inequality*

$$\|x\|^2 - \varphi(x, x) \;\le\; \|f(x)\|^2 \;\le\; \|x\|^2 + \varphi(x, x)$$

holds for any $x \in D_n$.

Lemma 3. *If* $f(x) = 0$, *then* $x = 0$.

Proof. If $f(x) = 0$, then (2) implies for each $y \in D_n$ that $|\langle x, y\rangle| \le \varphi(x, y)$. By putting $y = \lambda x \in D_n$ in the last inequality, we obtain

$$\|x\|^2 \;\le\; \frac{1}{|\lambda|}\,\varphi(x, \lambda x). \tag{3}$$

If $0 < c < 1$ and if we take the limit in (3) as $|\lambda| \to \infty$, then (ii) implies $x = 0$ which is impossible because $\|x\| \ge d > 0$. Thus, if $0 < c < 1$, then $f(x) \ne 0$ for every $x \in D_n$. When $c > 1$, we can take the limit in (3) as $|\lambda| \to 0$. Then, (ii) and (3) yield $x = 0$. □

Lemma 4. *For each* $x \in D_n\setminus\{0\}$ *there exists a function* $\mu_x\colon \mathbb{R} \to \mathbb{R}$ *such that* $f(\lambda x) = \mu_x(\lambda) f(x)$ *for all* $\lambda \in \mathbb{R}\setminus\{0\}$ *with* $\lambda x \in D_n$ *and also such that*

$$\frac{|\mu_x(\lambda)|}{|\lambda|} \not\to 0 \quad \begin{cases} \text{as } |\lambda| \to \infty & (\text{for} \quad 0 < c < 1) \\ \text{as } |\lambda| \to 0 & (\text{for} \quad c > 1) \end{cases}.$$

Proof. Assume that $x \in D_n\setminus\{0\}$ and $\lambda \ne 0$ are given with $\lambda x \in D_n$. If $f(x)$ and $f(\lambda x)$ were linearly independent, then we could select some $\omega > 0$ such that

$$|\cos A(f(x), f(\lambda x))| \;=\; 1 - \omega, \tag{4}$$

where $A(\cdot\,, \cdot)$ stands for the angle.

Since $x \ne 0$ is assumed, we can choose an orthogonal basis $\{x, v_1, \ldots, v_{n-1}\}$ for $\mathbb{R}^n$ with $\|v_1\| = \cdots = \|v_{n-1}\| = 1$. For any $i, j \in \{1, \ldots, n-1\}$ with $i \ne j$ and for any $k \in \mathbb{N}$ we have

$$\langle c^{-k} v_i, x\rangle = \langle c^{-k} v_i, \lambda x\rangle = \langle c^{-k} v_i, c^{-k} v_j\rangle = 0. \tag{5}$$

By using simple notations given by

$$a := f(x), \quad a' := f(\lambda x) \quad \text{and} \quad u_i := c^k f(c^{-k} v_i),$$

we get from (2), (5), (ii) and (iii) that

$$|\langle u_i, a\rangle| \le c^k \varphi(c^{-k} v_i, x) = O\left(\frac{1}{k}\right),$$

$$|\langle u_i, a'\rangle| \le c^k \varphi(c^{-k} v_i, \lambda x) = O\left(\frac{1}{k}\right),$$

$$|\langle u_i, u_j\rangle| \le c^{2k} \varphi(c^{-k} v_i, c^{-k} v_j) = O\left(\frac{1}{k^2}\right)$$

for all $i, j \in \{1, \dots, n-1\}$ with $i \neq j$ and for any sufficiently large $k \in \mathbb{N}$ (such that $c^{-k}v_i \in D_n$ for $i = 1, \dots, n-1$). Moreover, it follows from Lemma 2 that

$$1 - c^{2k}\varphi(c^{-k}v_i, c^{-k}v_i) \leq \|u_i\|^2 \leq 1 + c^{2k}\varphi(c^{-k}v_i, c^{-k}v_i).$$

At this point, we apply Theorem 1. First, denote by ℓ the one-dimensional orthogonal complement of the subspace $\mathrm{lin}\{u_1, \dots, u_{n-1}\}$. According to Theorem 1, (ii) and (iii), we can choose a sufficiently large integer k in order that $|\cos A(a, \ell)|$ and $|\cos A(a', \ell)|$ are arbitrarily close to 1. This fact means that $|\cos A(a, a')| = |\cos A(f(x), f(\lambda x))|$ is really 1, which is contrary to our assumption (4). Therefore, $f(x)$ and $f(\lambda x)$ have to be linearly dependent.

According to Lemma 3, $f(x) \neq 0$ and $f(\lambda x) \neq 0$ because $x \in D_n \backslash \{0\}$ and $\lambda \neq 0$ with $\lambda x \in D_n$. Thus, we can choose $\mu_x(\lambda) \in \mathbb{R}$ such that $f(\lambda x) = \mu_x(\lambda) f(x)$ for all $\lambda \neq 0$ with $\lambda x \in D_n$.

Hence, with $y = \lambda x$, (2) yields

$$\|x\|^2 - \frac{1}{|\lambda|}\varphi(x, \lambda x) \leq \frac{|\mu_x(\lambda)|}{|\lambda|}\|f(x)\|^2 \leq \|x\|^2 + \frac{1}{|\lambda|}\varphi(x, \lambda x) \tag{6}$$

for any $x \in D_n \backslash \{0\}$ and $\lambda \neq 0$ with $\lambda x \in D_n$.

When $0 < c < 1$, by taking the limit in (6) as $|\lambda| \to \infty$, (6) and (ii) imply that

$$\frac{|\mu_x(\lambda)|}{|\lambda|} \not\to 0 \quad \text{as } |\lambda| \to \infty.$$

Similarly, when $c > 1$, we can take the limit in (6) by letting $|\lambda| \to 0$ and use (6) and (ii) to obtain

$$\frac{|\mu_x(\lambda)|}{|\lambda|} \not\to 0 \quad \text{as } |\lambda| \to 0,$$

which completes the proof. □

Lemma 5. *For any $x \in D_n \backslash \{0\}$ and $y \in D_n$, it holds that $\langle x, y \rangle = 0$ if and only if $\langle f(x), f(y) \rangle = 0$.*

Proof. Let $x \in D_n \backslash \{0\}$, $y \in D_n$ and $\lambda \neq 0$ be given with $\lambda x \in D_n$. If $\langle x, y \rangle = 0$, then $\langle \lambda x, y \rangle = 0$. In this case, it follows from (2) that

$$|\langle f(\lambda x), f(y) \rangle| \leq \varphi(\lambda x, y). \tag{7}$$

On account of Lemma 4, there exists a function $\mu_x\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(\lambda x) = \mu_x(\lambda) f(x) \tag{8}$$

for all $x \in D_n \backslash \{0\}$ and $\lambda \neq 0$ with $\lambda x \in D_n$. Moreover, there is a constant $\alpha > 0$ and a strictly increasing (or decreasing) positive sequence (λ_k) with

$$\begin{cases} \lambda_k \to \infty, & \text{for} \quad 0 < c < 1, \\ \lambda_k \to 0, & \text{for} \quad c > 1, \end{cases}$$

such that

$$\frac{|\mu_x(\lambda_k)|}{\lambda_k} \geq \alpha \tag{9}$$

for every $k \in \mathbb{N}$. By (7)–(9) and (ii), we have

$$\begin{aligned}|\langle f(x), f(y)\rangle| &\leq \frac{1}{|\mu_x(\lambda_k)|}\varphi(\lambda_k x, y)\\ &\leq \frac{1}{\alpha\lambda_k}\varphi(\lambda_k x, y)\\ &\to 0, \quad \text{as} \quad k \to \infty. \end{aligned}\tag{10}$$

Suppose $x \in D_n \setminus \{0\}$ and $y \in D_n$ are given with $\langle f(x), f(y)\rangle = 0$. For each $\lambda > 0$ with $\lambda x \in D_n$, Lemma 4 gives

$$|\langle f(\lambda x), f(y)\rangle| = |\mu_x(\lambda)|\,|\langle f(x), f(y)\rangle| = 0.$$

Hence, it follows from (2) and (ii) that

$$\begin{aligned}|\langle x, y\rangle| &\leq \frac{1}{\lambda}\varphi(\lambda x, y)\\ &\to 0 \begin{cases}\text{as } \lambda \to \infty, & (\text{for } 0 < c < 1),\\ \text{as } \lambda \to 0, & (\text{for } c > 1),\end{cases}\end{aligned}$$

which finishes our proof. □

In the following lemma, we will prove the converse of Lemma 3, i.e., $f(0) = 0$ under an essential condition that the range of f is a finite-dimensional space.

Lemma 6. *It holds that* $f(0) = 0$.

Proof. We need to consider the case $c > 1$ only because D_n does not contain 0 for the other case $0 < c < 1$. For any $x \in D_n \setminus \{0\}$, by putting $y = 0$, Lemma 5 gives

$$\langle f(x), f(0)\rangle = 0. \tag{11}$$

Let $\{x_1, \ldots, x_n\}$ be an orthogonal basis for $\mathbb{R}^n$ with $x_i \in D_n \setminus \{0\}$. Then, Lemma 3 implies $f(x_i) \neq 0$ for $i = 1, \ldots, n$. Furthermore, Lemma 5 implies that $\langle f(x_i), f(x_j)\rangle = 0$ for any $i, j \in \{1, \ldots, n\}$ with $i \neq j$, i.e., $\{f(x_1), \ldots, f(x_n)\}$ is another orthogonal basis for $\mathbb{R}^n$.

Therefore, it follows from (11) that

$$\langle f(x_i), f(0)\rangle = 0$$

for $i = 1, \ldots, n$, and this relation implies $f(0) = 0$. □

By using ideas from Propositions 1 and 2 of [5], we can prove the following theorem.

Theorem 7. *Assume that a function* $f : D_n \to \mathbb{R}^n$ *satisfies the functional inequality*

$$\big||\langle f(x), f(y)\rangle| - |\langle x, y\rangle|\big| \leq \varphi(x, y)$$

for all $x, y \in D_n$. *We have for* $2 \leq k \leq n$:

(a) $x_1, \dots, x_k \in D_n$ *are linearly independent if and only if* $f(x_1), \dots, f(x_k)$ *are linearly independent*;
(b) *Let* $\mathcal{P}$ *be a* k*-dimensional subspace of* $\mathbb{R}^n$. *Then* f *transforms* $\mathcal{P} \cap D_n$ *into* k*-dimensional subspace* $\mathcal{P}'$ *of* $\mathbb{R}^n$ *spanned by the images of the elements of an arbitrary basis* $\mathcal{B}$ *of* $\mathcal{P}$ *with* $\mathcal{B} \subset \mathcal{P} \cap D_n$.

Proof.

(a) Let $x_1, \dots, x_k \in D_n$ be linearly independent and suppose that $f(x_1), \dots, f(x_k)$ are linearly dependent. Then, we can choose $\lambda_2, \dots, \lambda_k \in \mathbb{R}$ such that

$$f(x_1) = \lambda_2 f(x_2) + \cdots + \lambda_k f(x_k). \tag{12}$$

Let $x \in \text{lin}\{x_1, \dots, x_k\} \cap D_n$ be chosen with $x \neq 0$ and $\langle x, x_i \rangle = 0$ for $i = 2, \dots, k$. According to Lemma 5, it holds that $\langle f(x), f(x_i) \rangle = 0$ for $i = 2, \dots, k$, and hence (12) implies $\langle f(x), f(x_1) \rangle = 0$. By Lemma 5 again, $\langle x, x_i \rangle = 0$ for $i = 1, \dots, k$, and hence $x \notin \text{lin}\{x_1, \dots, x_k\}$, a contradiction.

In this paper, the converse of the above statement will not be used. But here we will introduce its proof for completion. Let $f(x_1), \dots, f(x_k)$ be linearly independent and $x_1, \dots, x_k$ be linearly dependent. Then, there are real numbers $\lambda_2, \dots, \lambda_k$ such that

$$x_1 = \lambda_2 x_2 + \cdots + \lambda_k x_k. \tag{13}$$

Choose $y \in \text{lin}\{f(x_1), \dots, f(x_k)\} \cap f(D_n)$ with $y \neq 0$ and $\langle y, f(x_i) \rangle = 0$ for $i = 2, \dots, k$. There exists an $x \in D_n \setminus \{0\}$ with $y = f(x)$. Due to Lemma 5, we have $\langle x, x_i \rangle = 0$ for $i = 2, \dots, k$, and (13) means $\langle x, x_1 \rangle = 0$. Using Lemma 5 again, we obtain $\langle f(x), f(x_i) \rangle = \langle y, f(x_i) \rangle = 0$ for $i = 1, \dots, k$. This implies that $y \notin \text{lin}\{f(x_1), \dots, f(x_k)\}$ which leads to a contradiction.

(b) Let $\{x_1, \dots, x_k\} \subset D_n$ be an orthogonal basis for a k-dimensional subspace $\mathcal{P}$ of $\mathbb{R}^n$ and let $\{x_1, \dots, x_k, x_{k+1}, \dots, x_n\} \subset D_n$ be an orthogonal basis for $\mathbb{R}^n$. On account of Lemmas 3 and 5, $\{f(x_1), \dots, f(x_n)\}$ is also an orthogonal basis for $\mathbb{R}^n$. Thus, for any $x \in \mathcal{P} \cap D_n$ there exist $\lambda_1, \dots, \lambda_k, \xi_1, \dots, \xi_n \in \mathbb{R}$ such that

$$x = \lambda_1 x_1 + \cdots + \lambda_k x_k \quad \text{and} \quad f(x) = \xi_1 f(x_1) + \cdots + \xi_n f(x_n). \tag{14}$$

Since $\langle x, x_i \rangle = 0$ for $i = k+1, \dots, n$, Lemma 5 implies that $\langle f(x), f(x_i) \rangle = 0$ for $i = k+1, \dots, n$. Hence, it follows from (14) that

$$\langle f(x), f(x_i) \rangle = \xi_i \, \|f(x_i)\|^2 = 0$$

for $i = k+1, \dots, n$, and we have $\xi_{k+1} = \cdots = \xi_n = 0$. Therefore, we conclude that

$$f(x) = \xi_1 f(x_1) + \cdots + \xi_k f(x_k)$$

or

$$f(\mathcal{P} \cap D_n) \subset \text{lin}\{f(x_1), \dots, f(x_k)\}.$$

If $\{y_1, \dots, y_k\} \subset D_n$ is a basis for $\mathcal{P}$, it then follows from (a) that $\{f(y_1), \dots, f(y_k)\}$ is a basis for $\text{lin}\{f(x_1), \dots, f(x_k)\}$, and this completes the proof. □

In the following lemma, we will modify Lemma 3 of [4] in order to be applicable to our case.

Lemma 8. *It holds that*

$$\lim_{k\to\infty} c^{2k}\,\|f(c^{-k}x)\|\,\|f(c^{-k}y)\| = \|x\|\,\|y\|$$

for all $x, y \in D_n$.

Proof. By Lemma 2, we get

$$\begin{aligned}
&\sqrt{\|x\|^2 - c^{2k}\varphi(c^{-k}x, c^{-k}x)}\,\sqrt{\|y\|^2 - c^{2k}\varphi(c^{-k}y, c^{-k}y)}\\
&\le\; c^{2k}\,\|f(c^{-k}x)\|\,\|f(c^{-k}y)\|\\
&\le \sqrt{\|x\|^2 + c^{2k}\varphi(c^{-k}x, c^{-k}x)}\,\sqrt{\|y\|^2 + c^{2k}\varphi(c^{-k}y, c^{-k}y)}\,,
\end{aligned}$$

and (iii) gives the validity of our assertion. □

3. Main results

We know that D_n is a subset of $\mathbb{R}^n$ defined by

$$D_n := \begin{cases} \{x \in \mathbb{R}^n\colon \|x\| \ge d\}, & \text{for } 0 < c < 1,\\ \{x \in \mathbb{R}^n\colon \|x\| < d\}, & \text{for } c > 1, \end{cases}$$

for given positive numbers $c \neq 1$ and $d > 0$. The function $\varphi\colon \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ was defined as a symmetric function which satisfies the following conditions:

(i) There exists a function $\phi\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ such that $\varphi(x, y) = \phi(\|x\|, \|y\|)$ for all $x, y \in \mathbb{R}^n$.

(ii) For all $x, y \in \mathbb{R}^n$,

$$\frac{1}{|\lambda|}\,\varphi(\lambda x, y) = O\left(-\frac{\ln c}{\ln|\lambda|}\right)$$

either as $|\lambda| \to \infty$ (for $0 < c < 1$) or as $|\lambda| \to 0$ (for $c > 1$).

(iii) If both of $|\lambda|$ and $|\mu|$ are different from 1, then for all $x, y \in \mathbb{R}^n$,

$$\frac{1}{|\lambda\mu|}\,\varphi(\lambda x, \mu y) = O\left(\left|\frac{\ln c}{\ln|\lambda|}\,\frac{\ln c}{\ln|\mu|}\right|\right)$$

either as $|\lambda\mu| \to \infty$ (for $0 < c < 1$) or as $|\lambda\mu| \to 0$ (for $c > 1$).

As assumed in the previous section, throughout this section also, let the function $f\colon D_n \to \mathbb{R}^n$ satisfy the functional inequality (2) for all $x, y \in D_n$ if there is no specification for f.

Lemma 9. *It holds that*

$$|\cos A(f(x), f(y))| = |\cos A(x, y)|$$

for any x *and* y *in* $D_n\backslash\{0\}$.

Proof. By making use of Lemmas 3 and 4, it is easy to see

$$|\cos A(f(x), f(y))| = |\cos A(f(c^{-k}x), f(c^{-k}y))| \tag{15}$$

for all $x, y \in D_n \backslash \{0\}$ and any $k \in \mathbb{N}$.

If we replace x, y in (2) by $c^{-k}x$ and $c^{-k}y$, respectively, and if we divide the resulting inequalities by c^{-2k}, then

$$\begin{aligned}
&\|x\|\,\|y\|\,|\cos A(x, y)| - c^{2k}\varphi(c^{-k}x, c^{-k}y)\\
&\quad \le c^{2k}\,\|f(c^{-k}x)\|\,\|f(c^{-k}y)\|\,|\cos A(f(c^{-k}x), f(c^{-k}y))|\\
&\quad \le \|x\|\,\|y\|\,|\cos A(x, y)| + c^{2k}\varphi(c^{-k}x, c^{-k}y).
\end{aligned}$$

Taking the limit as $k \to \infty$ in the above inequalities and using (iii), (15) and Lemma 8, we obtain

$$\|x\|\,\|y\|\,|\cos A(x, y)| = \|x\|\,\|y\|\,|\cos A(f(x), f(y))|,$$

which ends the proof. □

We now define an integer $k_0 \in \mathbb{N}_0$ by

$$k_0 := \min\{k \in \mathbb{N}_0 \colon c^{-k}e_i \in D_n \quad \text{for all} \quad i = 1, \ldots, n\},$$

and let

$$e_i' := c^{-k_0}e_i$$

for $i = 1, \ldots, n$, where $\{e_1, \ldots, e_n\}$ is the canonical basis for $\mathbb{R}^n$.

Lemma 10. *There exists an orthogonal automorphism $\psi \colon \mathbb{R}^n \to \mathbb{R}^n$ such that*

(a) *the composition $f' := \psi \circ f$ satisfies the inequality* (2) *for all $x, y \in D_n$;*
(b) *every element of $\{e_1', \ldots, e_n'\}$ is an eigenvector of f', i.e.,*

$$f'(e_i') = \lambda_i e_i',$$

where λ_i is a constant with $0 < \lambda_i \le \sqrt{1 + c^{2k_0}\varphi(e_i', e_i')}$ for $i = 1, \ldots, n$.

Proof.

(a) By Lemmas 3 and 5, $\{f(e_1'), \ldots, f(e_n')\}$ is an orthogonal basis for $\mathbb{R}^n$. We may define an orthogonal automorphism $\psi \colon \mathbb{R}^n \to \mathbb{R}^n$ by

$$\psi(x) := \lambda_1\,\|f(e_1')\|\,e_1 + \cdots + \lambda_n\,\|f(e_n')\|\,e_n \tag{16}$$

for any $x \in \mathbb{R}^n$ expressed by $x = \lambda_1 f(e_1') + \cdots + \lambda_n f(e_n')$. Since ψ is orthogonal, we have

$$\langle \psi(f(x)), \psi(f(y))\rangle = \langle f(x), f(y)\rangle$$

for all $x, y \in D_n$. Hence, it is obvious that $f' = \psi \circ f$ satisfies inequality (2) for all $x, y \in D_n$.

(b) By (16), we obtain

$$f'(e_i') = \psi(f(e_i')) = \|f(e_i')\|\, e_i = c^{k_0}\, \|f(e_i')\|\, e_i'.$$

Further, it follows from Lemmas 2 and 3 that

$$0 < c^{k_0}\, \|f(e_i')\| \leq \sqrt{1 + c^{2k_0}\varphi(e_i', e_i')}$$

for $i = 1, \ldots, n$. □

On the basis of Theorem 7(b), we are now ready to deal with a special case of $n = 2$ in the following lemma.

Lemma 11. *Let a function $f : D_2 \to \mathbb{R}^2$ satisfy the inequality* (2) *for all $x, y \in D_2$. If $f(e_i') = \lambda_i e_i'$ for $i = 1, 2$ with*

$$0 < \lambda_i \leq \sqrt{1 + c^{2k_0}\varphi(e_i', e_i')}\,,$$

then either $f(x) = x$, $f(x) = -x$, $f(x) = \overline{x}$, or $f(x) = -\overline{x}$ for each $x \in D_2$, where $\overline{x} = (x_1, -x_2)$ for $x = (x_1, x_2)$ and see Lemma 10 *for the e_i''s and k_0.*

Proof. According to Lemma 6, it holds $f(0) = 0$. This means the validity of our assertion for $x = 0$ (if 0 belongs to D_2).

Now, let $x \in D_2\backslash\{0\}$. Due to Lemma 9, we have

$$|\cos A(x, e_1')| = |\cos A(f(x), f(e_1'))| = |\cos A(f(x), e_1')|.$$

This implies that there exists a non-zero real number λ such that either $f(x) = \lambda x$ or $f(x) = \lambda\overline{x}$.

Let us define

$$\mathcal{A} := \{x \in D_2\text{: there exists a } \lambda \in \mathbb{R} \text{ with } f(x) = \lambda x\}$$

and

$$\mathcal{B} := \{x \in D_2\text{: there exists a } \lambda \in \mathbb{R} \text{ with } f(x) = \lambda\overline{x}\}.$$

On account of Lemma 4, it is not difficult to see $D_2 \cap (\mathrm{lin}\{e_1\} \cup \mathrm{lin}\{e_2\}) \subset \mathcal{A} \cap \mathcal{B}$. We set

$$D_2^* := D_2\backslash(\mathrm{lin}\{e_1\} \cup \mathrm{lin}\{e_2\}).$$

We assert that either $D_2^* \subset \mathcal{A}$ or $D_2^* \subset \mathcal{B}$. Suppose that there were $x, y \in D_2^*$ and $\lambda, \mu \in \mathbb{R}\backslash\{0\}$ with

$$f(x) = \lambda x \quad \text{and} \quad f(y) = \mu\overline{y}.$$

By Lemma 9, we would have

$$|\cos A(x, \overline{y})| = |\cos A(f(x), f(y))| = |\cos A(x, y)|.$$

This implies that x or y should belong to $D_2 \cap (\mathrm{lin}\{e_1\} \cup \mathrm{lin}\{e_2\})$, which leads to a contradiction.

From the above fact we can deduce that there exists a function λ: $D_2 \to \mathbb{R}$ such that either

$$f(x) = \lambda(x)x \quad \text{for all } x \in D_2 \tag{17}$$

or

$$f(x) = \lambda(x)\overline{x} \quad \text{for all } x \in D_2. \tag{18}$$

In (17), it follows from (2) that

$$||\langle\lambda(x)x, \lambda(x)x\rangle| - |\langle x, x\rangle|| \le \varphi(x, x)$$

for any $x \in D_2$, and hence

$$|\lambda(x)^2 - 1| \, \|x\|^2 \le \varphi(x, x)$$

for $x \in D_2$. If we replace x by $c^{-k}x$ in the last inequality, then we get

$$|\lambda(c^{-k}x)^2 - 1| \, \|x\|^2 \le c^{2k}\varphi(c^{-k}x, c^{-k}x),$$

and if we take the limit as $k \to \infty$, then (iii) means

$$\lim_{k\to\infty} \lambda(c^{-k}x)^2 = 1 \tag{19}$$

for any $x \in D_2\backslash\{0\}$. Choose $x, y \in D_2$ with $\langle x, y\rangle \neq 0$ and let $k \in \mathbb{N}$. It follows from (2) that

$$||\langle\lambda(x)x, \lambda(c^{-k}y)c^{-k}y\rangle| - |\langle x, c^{-k}y\rangle|| \le \varphi(x, c^{-k}y).$$

By making use of (ii) and (19) and by taking the limit as $k \to \infty$, we conclude that $|\lambda(x)| = 1$ for every $x \in D_2\backslash\{0\}$, i.e.,

$$f(x) = x \quad \text{or} \quad f(x) = -x,$$

for all $x \in D_2$, in view of (17) and Lemma 6.

In (18), we can analogously obtain the equality (19) for each $x \in D_2\backslash\{0\}$ because of the fact $\|\overline{x}\| = \|x\|$. The fact $\langle\overline{x}, \overline{y}\rangle = \langle x, y\rangle$ yields $|\lambda(x)| = 1$ for each $x \in D_2\backslash\{0\}$ and hence (18) and Lemma 6 give

$$f(x) = \overline{x} \quad \text{or} \quad f(x) = -\overline{x},$$

for all $x \in D_2$, which completes the proof. □

By making use of Lemmas 10 and 11 we can easily prove the following corollary. Hence, we omit the proof.

COROLLARY 12

If a function $f : D_2 \to \mathbb{R}^2$ satisfies the inequality (2) *for all $x, y \in D_2$, then*

$$\|f(x)\| = \|x\|$$

for every x in D_2.

In the following lemma, we will extend the last corollary to the spaces of higher dimensions.

Lemma 13. *If a function* $f : D_n \to \mathbb{R}^n$ *satisfies the inequality* (2) *for all* $x, y \in D_n$, *then*

$$\|f(x)\| = \|x\|$$

for any $x \in D_n$.

Proof. Lemma 6 says that $f(0) = 0$, and this means that our assertion holds true for $x = 0$ whenever $0 \in D_n$ (i.e., in the case $c > 1$).

We now choose $x, y \in D_n \setminus \{0\}$ with $\langle x, y\rangle = 0$. In view of Lemmas 3 and 5, we know that $f(x) \neq 0$, $f(y) \neq 0$ and $\langle f(x), f(y)\rangle = 0$. Due to Theorem 7(b) and Lemma 4, we obtain

$$f\left(D_n \cap \operatorname{lin}\left\{\frac{x}{\|x\|}, \frac{y}{\|y\|}\right\}\right) \subset \operatorname{lin}\left\{\frac{f(x)}{\|f(x)\|}, \frac{f(y)}{\|f(y)\|}\right\}.$$

This means that for each pair (λ_1, λ_2) of real numbers satisfying

$$\lambda_1 \frac{x}{\|x\|} + \lambda_2 \frac{y}{\|y\|} \in D_n, \tag{20}$$

there exists a unique pair (μ_1, μ_2) of real numbers such that

$$f\left(\lambda_1 \frac{x}{\|x\|} + \lambda_2 \frac{y}{\|y\|}\right) = \mu_1 \frac{f(x)}{\|f(x)\|} + \mu_2 \frac{f(y)}{\|f(y)\|}. \tag{21}$$

We observe

$$\left\|\lambda_1 \frac{x}{\|x\|} + \lambda_2 \frac{y}{\|y\|}\right\|^2 = \lambda_1^2 + \lambda_2^2 = \|(\lambda_1, \lambda_2)\|^2. \tag{22}$$

This implies that $(\lambda_1, \lambda_2) \in D_2$ if and only if (20) holds true. On the basis of this fact, let us define a function $f^*: D_2 \to \mathbb{R}^2$ by

$$f^*(\lambda) = \mu, \tag{23}$$

where $\lambda = (\lambda_1, \lambda_2) \in D_2$ and $\mu = (\mu_1, \mu_2)$ obey the relation (21). Let $\lambda = (\lambda_1, \lambda_2)$ and $\lambda' = (\lambda_1', \lambda_2')$ belong to D_2 and let

$$u = \lambda_1 \frac{x}{\|x\|} + \lambda_2 \frac{y}{\|y\|} \quad \text{and} \quad u' = \lambda_1' \frac{x}{\|x\|} + \lambda_2' \frac{y}{\|y\|}. \tag{24}$$

Then we have

$$|\langle u, u'\rangle| = |\lambda_1\lambda_1' + \lambda_2\lambda_2'| = |\langle \lambda, \lambda'\rangle|$$

and

$$\begin{aligned} |\langle f(u), f(u')\rangle| &= \left|\left\langle \mu_1 \frac{f(x)}{\|f(x)\|} + \mu_2 \frac{f(y)}{\|f(y)\|},\ \mu_1' \frac{f(x)}{\|f(x)\|} + \mu_2' \frac{f(y)}{\|f(y)\|}\right\rangle\right| \\ &= |\mu_1\mu_1' + \mu_2\mu_2'| \\ &= |\langle f^*(\lambda), f^*(\lambda')\rangle|. \end{aligned}$$

Since f satisfies the inequality (2) for all $x, y \in D_n$, we obtain by (22) and (24) that

$$\begin{aligned} ||\langle f^*(\lambda), f^*(\lambda')\rangle| - |\langle \lambda, \lambda'\rangle|| &= ||\langle f(u), f(u')\rangle| - |\langle u, u'\rangle|| \\ &\le \varphi(u, u') = \phi(\|u\|, \|u'\|) \\ &= \phi(\|\lambda\|, \|\lambda'\|) =: \tilde{\phi}(\lambda, \lambda') \end{aligned}$$

for all $\lambda, \lambda' \in D_2$, where we understand $\tilde{\phi}: \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ as a restriction $\varphi|_{\mathbb{R}^2 \times \mathbb{R}^2}$.

According to Corollary 12, we get

$$\|f^*(\lambda)\| = \|\lambda\| \tag{25}$$

for every $\lambda \in D_2$. If we put $\lambda = (\|x\|, 0)$, then $\lambda \in D_2$ in view of the assertion that was verified by (22). For this case, it follows from (21) and (23) that

$$f^*(\lambda) = (\|f(x)\|, 0).$$

And (25), together with Lemma 6, yields

$$\|x\| = \|\lambda\| = \|f^*(\lambda)\| = \|f(x)\|$$

for each x in D_n. □

At last, by making use of Lemmas 9 and 13, and considering Lemma 6, we can prove the main theorem of this paper.

Theorem 14. *If a function* $f : D_n \to \mathbb{R}^n$ *satisfies the inequality*

$$||\langle f(x), f(y)\rangle| - |\langle x, y\rangle|| \le \varphi(x, y)$$

for all $x, y \in D_n$, *then* f *satisfies the generalized orthogonality equation*

$$|\langle f(x), f(y)\rangle| = |\langle x, y\rangle|,$$

for all $x, y \in D_n$.

Let B be an open ball in $\mathbb{R}^n$ with radius $d > 0$ and centered at the origin, i.e.,

$$B := \{x \in \mathbb{R}^n : \|x\| < d\}.$$

In view of Theorem 14, the following corollaries are obvious.

COROLLARY 15

If a function $f : B \to \mathbb{R}^n$ *satisfies the inequality*

$$||\langle f(x), f(y)\rangle| - |\langle x, y\rangle|| \le \varepsilon\|x\|^p \, \|y\|^p$$

for some $\varepsilon \ge 0$, $p > 1$ *and for all* $x, y \in B$, *then* f *satisfies the generalized orthogonality equation* (1) *for all* $x, y \in B$.

COROLLARY 16

If a function $f : \mathbb{R}^n \setminus B \to \mathbb{R}^n$ *satisfies the inequality*

$$||\langle f(x), f(y)\rangle| - |\langle x, y\rangle|| \le \varepsilon\|x\|^p \, \|y\|^p$$

for some $\varepsilon \ge 0$, $p < 1$ *and for all* $x, y \in \mathbb{R}^n \setminus B$, *then* f *satisfies the generalized orthogonality equation* (1) *for all* $x, y \in \mathbb{R}^n \setminus B$.

If we assume $p = 0$ in Corollary 16, then we can extend the result of Chmieliński [4] which was introduced in §1 to the case of restricted (unbounded) domains.

4. Applications

In this section, we will still use the notations D_n and φ to denote the ones defined in §1. With these notations, we will prove the superstability of the orthogonality equation

$$\langle f(x), f(y)\rangle = \langle x, y\rangle \tag{26}$$

on restricted domains. Every solution of the orthogonality equation (26) is an isometry.

We will first improve Lemma 9 adequately for our purpose.

Lemma 17. *If a function $f\colon D_n \to \mathbb{R}^n$ satisfies the inequality*

$$|\langle f(x), f(y)\rangle - \langle x, y\rangle| \leq \varphi(x, y) \tag{27}$$

for all $x, y \in D_n$, then it holds $\cos A(f(x), f(y)) = \cos A(x, y)$ *for any $x, y \in D_n\backslash\{0\}$.*

Proof. Since the inequality (27) implies the validity of the inequality (2), all lemmas, theorems and corollaries in the previous sections hold true for this case.

Let $x \in D_n\backslash\{0\}$ be given. According to Lemma 4, there exists a function $\mu_x\colon \mathbb{R} \to \mathbb{R}$ with

$$f(c^{-k}x) = \mu_x(c^{-k})f(x) \tag{28}$$

for any $k \in \mathbb{N}$. If we replace x and y in (27) by $c^{-k}x$ and x, respectively, then it follows from (ii) that

$$\begin{aligned}|c^k\mu_x(c^{-k})\|f(x)\|^2 - \|x\|^2| &\leq c^k\varphi(c^{-k}x, x)\\ &\to 0 \quad \text{as } k \to \infty.\end{aligned}$$

Since $\|f(x)\|^2 > 0$ and $\|x\|^2 > 0$, we have

$$\mu_x(c^{-k}) > 0 \tag{29}$$

for any sufficiently large $k \in \mathbb{N}$.

By using Lemma 3, Lemma 4, (28) and (29), we get

$$\cos A(f(x), f(y)) = \cos A(f(c^{-k}x), f(c^{-k}y)) \tag{30}$$

for all $x, y \in D_n\backslash\{0\}$ and for all sufficiently large $k \in \mathbb{N}$.

If we replace x and y in (27) by $c^{-k}x$ and $c^{-k}y$, respectively, and if we multiply the resulting inequalities by c^{2k}, then we obtain

$$\begin{aligned}&\|x\|\,\|y\|\cos A(x, y) - c^{2k}\varphi(c^{-k}x, c^{-k}y)\\ &\quad\leq c^{2k}\,\|f(c^{-k}x)\|\,\|f(c^{-k}y)\|\,\cos A(f(c^{-k}x), f(c^{-k}y))\\ &\quad\leq \|x\|\,\|y\|\,\cos A(x, y) + c^{2k}\varphi(c^{-k}x, c^{-k}y).\end{aligned}$$

Taking the limit as $k \to \infty$ and using (iii), (30) and Lemma 8, we can conclude that our assertion is valid. □

By using Lemmas 13 and 17 and considering Lemma 6 also, we will prove the superstability of the orthogonality equation on restricted domains.

Theorem 18. *If a function $f : D_n \to \mathbb{R}^n$ satisfies the inequality*

$$|\langle f(x), f(y)\rangle - \langle x, y\rangle| \le \varphi(x, y)$$

for all $x, y \in D_n$, then f satisfies the orthogonality equation, $\langle f(x), f(y)\rangle = \langle x, y\rangle$, for all $x, y \in D_n$.

Let B be an open ball in $\mathbb{R}^n$ defined by $B = \{x \in \mathbb{R}^n : \|x\| < d\}$ for a given $d > 0$. The following corollaries are analogous versions of Corollaries 15 and 16 for the orthogonality equation.

COROLLARY 19

If a function $f : B \to \mathbb{R}^n$ satisfies the inequality

$$|\langle f(x), f(y)\rangle - \langle x, y\rangle| \le \varepsilon \, \|x\|^p \, \|y\|^p$$

for some $\varepsilon \ge 0$, $p > 1$ and for all $x, y \in B$, then f satisfies the orthogonality equation (26) *for all $x, y \in B$.*

COROLLARY 20

If a function $f : \mathbb{R}^n \setminus B \to \mathbb{R}^n$ satisfies the inequality

$$|\langle f(x), f(y)\rangle - \langle x, y\rangle| \le \varepsilon \, \|x\|^p \, \|y\|^p$$

for some $\varepsilon \ge 0$, $p < 1$ and for all $x, y \in \mathbb{R}^n \setminus B$, then f satisfies the orthogonality equation (26) *for all $x, y \in \mathbb{R}^n \setminus B$.*

It will be an interesting problem to investigate what happens if $p = 1$ in the above Corollary 15, 16, 19 or 20.

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