

## Multilinear integral operators and mean oscillation

LANZHE LIU

College of Mathematics and Computer, Changsha University of Science and  
Technology, Changsha 410077, People's Republic of China  
E-mail: lanzheliu@263.net

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**Abstract.** In this paper, the boundedness properties for some multilinear operators related to certain integral operators from Lebesgue spaces to Orlicz spaces are obtained. The operators include Calderón–Zygmund singular integral operator, fractional integral operator, Littlewood–Paley operator and Marcinkiewicz operator.

**Keywords.** Multilinear operator; Calderón–Zygmund operators; fractional integral operator; Littlewood–Paley operator; Marcinkiewicz operator; BMO space; Orlicz space.

### 1. Introduction and theorems

Let  $b \in \text{BMO}(R^n)$  and  $T$  be the Calderón–Zygmund singular integral operator. The commutator  $[b, T]$  generated by  $b$  and  $T$  is defined by  $[b, T]f(x) = b(x)Tf(x) - T(bf)(x)$ . By a classical result of Coifman *et al* [6], we know that the commutator is bounded on  $L^p(R^n)$  for  $1 < p < \infty$ . Chanillo [1] proves a similar result when  $T$  is replaced by the fractional integral operators. In [9], the boundedness properties for the commutators from Lebesgue spaces to Orlicz spaces are obtained. As the development of Calderón–Zygmund singular integral operators, fractional integral operators and their commutators (see [7,10,11,15]), multilinear singular integral operators have been well-studied. In this paper, we are going to consider some integral operators and their multilinear operator as follows.

Let  $m$  be a positive integer and  $A$  be a function on  $R^n$ . We denote that

$$R_{m+1}(A; x, y) = A(x) - \sum_{|\alpha| \leq m} \frac{1}{\alpha!} D^\alpha A(y)(x - y)^\alpha.$$

#### DEFINITION 1

Let  $T: S \rightarrow S'$  be a linear operator and there exists a locally integrable function  $K(x, y)$  on  $R^n \times R^n$  such that

$$Tf(x) = \int_{R^n} K(x, y)f(y)dy$$

for every bounded and compactly supported function  $f$ , where  $K$  satisfies, for fixed  $\varepsilon > 0$  and  $\delta \geq 0$ ,

$$|K(x, y)| \leq C|x - y|^{-n+\delta}$$

and

$$|K(y, x) - K(z, x)| \leq C|y - z|^\epsilon |x - z|^{-n-\epsilon+\delta},$$

if  $2|y - z| \leq |x - z|$ . The multilinear operator related to the integral operator  $T$  is defined by

$$T^A(f)(x) = \int \frac{R_{m+1}(A; x, y)}{|x - y|^m} K(x, y) f(y) dy.$$

DEFINITION 2

Let  $F_t(x, y)$  define on  $R^n \times R^n \times [0, +\infty)$ . Hence we denote that

$$F_t(f)(x) = \int_{R^n} F_t(x, y) f(y) dy$$

for every bounded and compactly supported function  $f$  and

$$F_t^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A; x, y)}{|x - y|^m} F_t(x, y) f(y) dy.$$

Let  $H$  be the Banach space and  $H = \{h : \|h\| < \infty\}$ . For each fixed  $x \in R^n$ , we view  $F_t(f)(x)$  and  $F_t^A(f)(x)$  as a mapping from  $[0, +\infty)$  to  $H$ . Then, the multilinear operators related to  $F_t$  is defined by

$$S^A(f)(x) = \|F_t^A(f)(x)\|,$$

where  $F_t$  satisfies, for fixed  $\epsilon > 0$  and  $\delta \geq 0$ ,

$$\|F_t(x, y)\| \leq C|x - y|^{-n+\delta}$$

and

$$\|F_t(y, x) - F_t(z, x)\| \leq C|y - z|^\epsilon |x - z|^{-n-\epsilon+\delta},$$

if  $2|y - z| \leq |x - z|$ . We also define that  $S(f)(x) = \|F_t(f)(x)\|$ .

Note that when  $m = 0$ ,  $T^A$  and  $S^A$  are just the commutators of  $T$  and  $S$  with  $A$  (see [9,12,14,18]). When  $m > 0$ , it is a non-trivial generalization of the commutators. It is well-known that multilinear operators are of great interest in harmonic analysis and have been widely studied by many authors [2-5,7,13]. The main purpose of this paper is to prove the boundedness properties for the multilinear operators  $T^A$  and  $S^A$  from Lebesgue spaces to Orlicz spaces.

Let us introduce some notations. Throughout this paper,  $Q$  will denote a cube of  $R^n$  with sides parallel to the axes. For any locally integrable function  $f$ , the sharp function of  $f$  is defined by

$$f^\#(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, and in what follows,  $f_Q = |Q|^{-1} \int_Q f(x) dx$ . It is well-known that (see [8])

$$f^\#(x) \approx \sup_{x \in Q} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

For  $1 \leq r < \infty$  and  $0 \leq \delta < n$ , let

$$M_{\delta,r}(f)(x) = \sup_{x \in Q} \left( \frac{1}{|Q|^{1-r\delta/n}} \int_Q |f(y)|^r dy \right)^{1/r}.$$

We say that  $f$  belongs to  $BMO(R^n)$  if  $f^\#$  belongs to  $L^\infty(R^n)$  and  $\|f\|_{BMO} = \|f\|_{L^\infty}$ . More generally, let  $\varphi$  be a non-decreasing positive function and define  $BMO_\varphi(R^n)$  as the space of all functions  $f$  such that

$$\frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y) - f_Q| dy \leq C\varphi(r).$$

For  $\beta > 0$ , the Lipschitz space  $Lip_\beta(R^n)$  is the space of functions  $f$  such that

$$\|f\|_{Lip_\beta} = \sup_{x \neq y} |f(x) - f(y)|/|x - y|^\beta < \infty.$$

For  $f$ ,  $m_f$  denotes the distribution function of  $f$ , that is  $m_f(t) = |\{x \in R^n : |f(x)| > t\}|$ .

Let  $\psi$  be a non-decreasing convex function on  $R^+$  with  $\psi(0) = 0$ .  $\psi^{-1}$  denotes the inverse function of  $\psi$ . The Orlicz space  $L_\psi(R^n)$  is defined by the set of functions  $f$  such that  $\int \psi(\lambda|f(x)|)dx < \infty$  for some  $\lambda > 0$ . The norm is given by  $\|f\|_{L_\psi} = \inf_{\lambda > 0} \lambda^{-1}(1 + \int \psi(\lambda|f(x)|)dx)$ .

We shall prove the following theorems in §2.

**Theorem 1.** *Let  $0 \leq \delta < n$ ,  $1 < p < n/\delta$  and  $\varphi, \psi$  be two non-decreasing positive functions on  $R^+$  with  $\varphi(t) = t^{n/p}\psi^{-1}(t^{-n})$  (or equivalently  $\psi^{-1}(t) = t^{1/p}\varphi(t^{-1/n})$ ). Suppose that  $\psi$  is convex,  $\psi(0) = 0$ ,  $\psi(2t) \leq C\psi(t)$ . Let  $T$  be the same as in Definition 1 such that  $T$  is bounded from  $L^r(R^n)$  to  $L^s(R^n)$  for any  $1 < r < n/\delta$  and  $1/s = 1/r - \delta/n$ . Then  $T^A$  is bounded from  $L^p(R^n)$  to  $L_\psi(R^n)$  if  $D^\alpha A \in BMO_\varphi(R^n)$  for all  $\alpha$  with  $|\alpha| = m$ .*

**Theorem 2.** *Let  $0 \leq \delta < n$ ,  $1 < p < n/\delta$  and  $\varphi, \psi$  be two non-decreasing positive functions on  $R^+$  with  $\varphi(t) = t^{n/p}\psi^{-1}(t^{-n})$  (or equivalently  $\psi^{-1}(t) = t^{1/p}\varphi(t^{-1/n})$ ). Suppose that  $\psi$  is convex,  $\psi(0) = 0$ ,  $\psi(2t) \leq C\psi(t)$ . Let  $S$  be the same as in Definition 2 such that  $S$  is bounded from  $L^r(R^n)$  to  $L^s(R^n)$  for any  $1 < r < n/\delta$  and  $1/s = 1/r - \delta/n$ . Then  $S^A$  is bounded from  $L^p(R^n)$  to  $L_\psi(R^n)$  if  $D^\alpha A \in BMO_\varphi(R^n)$  for all  $\alpha$  with  $|\alpha| = m$ .*

*Remark.*

- (i) If  $\varphi(t) \equiv 1$  and  $\psi(t) = t^p$  for  $1 < p < \infty$ , then  $T^A$  and  $S^A$  are all bounded on  $L^p(R^n)$  if  $D^\alpha A \in BMO(R^n)$  for all  $\alpha$  with  $|\alpha| = m$ .
- (ii) If  $\psi(t) = t^q$  and  $\varphi(t) = t^{n(1/p-1/q)}$  for  $1 < p < q < \infty$ , then, by  $BMO_{t^\beta} = Lip_\beta$  (see Lemma 4 of [9]),  $T^A$  and  $S^A$  are all bounded from  $L^p(R^n)$  to  $L^q(R^n)$  if  $D^\alpha A \in Lip_{n(1/p-1/q)}$  for all  $\alpha$  with  $|\alpha| = m$ .

## 2. Proofs of theorems

We begin with the following preliminary lemmas.

**Lemma 1** [9]. Let  $\varphi$  be a non-decreasing positive function on  $R^+$  and  $\eta$  be an infinitely differentiable function on  $R^n$  with compact support such that  $\int \eta(x)dx = 1$ . Denote that  $b_t(x) = \int_{R^n} b(x - ty)\eta(y)dy$ . Then  $\|b - b_t\|_{\text{BMO}} \leq C\varphi(t)\|b\|_{\text{BMO}_\varphi}$ .

**Lemma 2** [9]. Let  $0 < \beta < 1$  and  $\varphi$  be a non-decreasing positive function on  $R^+$  or  $\beta = 1$ . Then  $\|b_t\|_{\text{Lip}_\beta} \leq Ct^{-\beta}\varphi(t)\|b\|_{\text{BMO}_\varphi}$ .

**Lemma 3** [9]. Suppose  $1 \leq p_2 < p < p_1 < \infty$ ,  $\rho$  is a non-increasing function on  $R^+$ ,  $B$  is a linear operator such that  $m_{B(f)}(t^{1/p_1}\rho(t)) \leq Ct^{-1}$  if  $\|f\|_{L^{p_1}} \leq 1$  and  $m_{B(f)}(t^{1/p_2}\rho(t)) \leq Ct^{-1}$  if  $\|f\|_{L^{p_2}} \leq 1$ . Then  $\int_0^\infty m_{B(f)}(t^{1/p}\rho(t))dt \leq C$  if  $\|f\|_{L^p} \leq (p/p_1)^{1/p}$ .

**Lemma 4** [1]. Suppose that  $1 \leq r < p < n/\beta$  and  $1/q = 1/p - \beta/n$ . Then  $\|M_{\beta,r}(f)\|_{L^q} \leq C\|f\|_{L^p}$ .

**Lemma 5** [15]. Suppose that  $1 \leq r < \infty$  and  $b \in \text{Lip}_\beta$ . Then

$$\|(b - b_Q)f\chi_{2Q}\|_{L^r} \leq C|Q|^{1/r}\|b\|_{\text{Lip}_\beta}M_{\beta,r}(f)(x).$$

**Lemma 6** [4]. Let  $A$  be a function on  $R^n$  and  $D^\alpha A \in L^q(R^n)$  for all  $\alpha$  with  $|\alpha| = m$  and some  $q > n$ . Then

$$|R_m(A; x, y)| \leq C|x - y|^m \sum_{|\alpha|=m} \left( \frac{1}{|\tilde{Q}(x, y)|} \int_{\tilde{Q}(x, y)} |D^\alpha A(z)|^q dz \right)^{1/q},$$

where  $\tilde{Q}$  is the cube centered at  $x$  and having side length  $5\sqrt{n}|x - y|$ .

To prove the theorems of the paper, we need the following:

**Key Lemma.** Let  $T$  and  $S$  be the same as in Definitions 1 and 2. Suppose that  $Q = Q(x_0, d)$  is a cube with  $\text{supp } f \subset (2Q)^c$  and  $x, \tilde{x} \in Q$ .

(a) If  $0 < \delta < n$  and  $D^\alpha A \in \text{BMO}(R^n)$  for all  $\alpha$  with  $|\alpha| = m$ , then

$$\begin{aligned} & |T^A(f)(x) - T^A(f)(x_0)| \\ & \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}}(M_{\delta,1}(f)(\tilde{x}) + M_{\delta,r}(f)(\tilde{x})) \quad \text{for any } r > 1. \end{aligned}$$

(b) If  $0 < \beta + \delta < n$  and  $D^\alpha A \in \text{Lip}_\beta(R^n)$  for all  $\alpha$  with  $|\alpha| = m$ , then

$$|T^A(f)(x) - T^A(f)(x_0)| \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{Lip}_\beta} M_{\beta+\delta,1}(f)(\tilde{x}).$$

(c) If  $0 < \delta < n$  and  $D^\alpha A \in \text{BMO}(R^n)$  for all  $\alpha$  with  $|\alpha| = m$ , then

$$\begin{aligned} & \|F_t^A(f)(x) - F_t^A(f)(x_0)\| \\ & \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}}(M_{\delta,1}(f)(\tilde{x}) + M_{\delta,r}(f)(\tilde{x})) \quad \text{for any } r > 1. \end{aligned}$$

(d) If  $0 < \beta + \delta < n$  and  $D^\alpha A \in \text{Lip}_\beta(R^n)$  for all  $\alpha$  with  $|\alpha| = m$ , then

$$\|F_t^A(f)(x) - F_t^A(f)(x_0)\| \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{Lip}_\beta} M_{\beta+\delta,1}(f)(\tilde{x}).$$

*Proof.* Let  $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A)_Q x^\alpha$ , then  $R_{m+1}(A; x, y) = R_{m+1}(\tilde{A}; x, y)$  and  $D^\alpha \tilde{A} = D^\alpha A - (D^\alpha A)_Q$  for  $|\alpha| = m$ . Suppose  $\text{supp } f \subset (2Q)^c$  and  $x, \tilde{x} \in Q = Q(x_0, d)$ . Note that  $|x_0 - y| \approx |x - y|$  for  $y \in (2Q)^c$ . We write

$$\begin{aligned} T^A(f)(x) - T^A(f)(x_0) &= \int_{R^n} \left[ \frac{K(x, y)}{|x - y|^m} - \frac{K(x_0, y)}{|x_0 - y|^m} \right] R_m(\tilde{A}; x, y) f(y) dy \\ &\quad + \int_{R^n} \frac{K(x_0, y) f(y)}{|x_0 - y|^m} [R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y)] dy \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left( \frac{K(x, y)(x - y)^\alpha}{|x - y|^m} - \frac{K(x_0, y)(x_0 - y)^\alpha}{|x_0 - y|^m} \right) D^\alpha \tilde{A}(y) f(y) dy \\ &:= I + II + III. \end{aligned}$$

(a) By Lemma 6 and the following inequality (see [15]), for  $b \in \text{BMO}(R^n)$ ,

$$|b_{Q_1} - b_{Q_2}| \leq C \log(|Q_2|/|Q_1|) \|b\|_{\text{BMO}} \quad \text{for } Q_1 \subset Q_2,$$

we know that, for  $x \in Q$  and  $y \in 2^{k+1}Q \setminus 2^kQ$  with  $k \geq 1$ ,

$$\begin{aligned} |R_m(\tilde{A}; x, y)| &\leq C|x - y|^m \sum_{|\alpha|=m} (\|D^\alpha A\|_{\text{BMO}} + |(D^\alpha A)_Q - (D^\alpha A)_{Q(x,y)}|) \\ &\leq Ck|x - y|^m \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}}; \end{aligned}$$

thus

$$\begin{aligned} |I| &\leq C \int_{R^n \setminus 2Q} \left( \frac{|x - x_0|}{|x_0 - y|^{m+n+1-\delta}} + \frac{|x - x_0|^\varepsilon}{|x_0 - y|^{m+n+\varepsilon-\delta}} \right) \\ &\quad \times |R_m(\tilde{A}; x, y)| |f(y)| dy \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} \sum_{k=1}^\infty \int_{2^{k+1}Q \setminus 2^kQ} k \left( \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta}} \right. \\ &\quad \left. + \frac{|x - x_0|^\varepsilon}{|x_0 - y|^{n+\varepsilon-\delta}} \right) |f(y)| dy \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} \sum_{k=1}^\infty k(2^{-k} + 2^{-k\varepsilon}) \left( \frac{1}{|2^{k+1}Q|^{1-\delta/n}} \int_{2^{k+1}Q} |f(y)| dy \right) \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} M_{\delta,1}(f)(\tilde{x}). \end{aligned}$$

For *II*, by the formula (see [4])

$$R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y) = \sum_{|\eta| < m} \frac{1}{\eta!} R_{m-|\eta|}(D^\eta \tilde{A}; x, x_0)(x - y)^\eta$$

and Lemma 6, we get

$$\begin{aligned} |II| &\leq C \int_{R^n \setminus 2Q} \frac{|R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y)|}{|x_0 - y|^{m+n-\delta}} |f(y)| dy \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} \sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta}} |f(y)| dy \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} M_{\delta,1}(f)(\tilde{x}). \end{aligned}$$

For *III*, similar to the estimates of *I*, we obtain, for any  $r > 1$  with  $1/r + 1/r' = 1$ ,

$$\begin{aligned} |III| &\leq C \int_{R^n \setminus 2Q} \left( \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta}} + \frac{|x - x_0|^\varepsilon}{|x_0 - y|^{n+\varepsilon-\delta}} \right) \\ &\quad \times |D^\alpha A(y) - (D^\alpha A)_Q| |f(y)| dy \\ &\leq C \sum_{|\alpha|=m} \sum_{k=1}^{\infty} (2^{-k} + 2^{-k\varepsilon}) \left( \frac{1}{|2^{k+1}Q|^{1-r\delta/n}} \int_{2^{k+1}Q} |f(y)|^r dy \right)^{1/r} \\ &\quad \times \left( \frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |D^\alpha A(x) - (D^\alpha A)_Q|^{r'} dx \right)^{1/r'} \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} M_{\delta,r}(f)(\tilde{x}). \end{aligned}$$

Thus

$$\begin{aligned} &|T^A(f)(x) - T^A(f)(x_0)| \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} (M_{\delta,1}(f)(\tilde{x}) + M_{\delta,r}(f)(\tilde{x})). \end{aligned}$$

(b) By Lemma 6 and the following inequality, for  $b \in \text{Lip}_\beta$ ,

$$|b(x) - b_Q| \leq \frac{1}{|Q|} \int_Q \|b\|_{\text{Lip}_\beta} |x - y|^\beta dy \leq \|b\|_{\text{Lip}_\beta} (|x - x_0| + d)^\beta,$$

we get

$$|R_m(\tilde{A}; x, y)| \leq \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{Lip}_\beta} (|x - y| + d)^{m+\beta},$$

thus

$$\begin{aligned}
 |I| &\leq C \int_{R^n \setminus 2Q} \left( \frac{|x - x_0|}{|x_0 - y|^{m+n+1-\delta}} + \frac{|x - x_0|^\varepsilon}{|x_0 - y|^{m+n+\varepsilon-\delta}} \right) |R_m(\tilde{A}; x, y)| |f(y)| dy \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{Lip}_\beta} \sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} \left( \frac{|x - x_0|}{|x_0 - y|^{n+1-\beta-\delta}} + \frac{|x - x_0|^\varepsilon}{|x_0 - y|^{n+\varepsilon-\beta-\delta}} \right) |f(y)| dy \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{Lip}_\beta} \sum_{k=1}^{\infty} (2^{-k} + 2^{-k\varepsilon}) \frac{1}{|2^{k+1}Q|^{1-(\beta+\delta)/n}} \int_{2^{k+1}Q} |f(y)| dy \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{Lip}_\beta} M_{\beta+\delta,1}(f)(\tilde{x}), \\
 |II| &\leq C \int_{R^n \setminus 2Q} \frac{|R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y)|}{|x_0 - y|^{m+n-\delta}} |f(y)| dy \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{Lip}_\beta} \sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^kQ} \frac{|x - x_0|}{|x_0 - y|^{n+1-\beta-\delta}} |f(y)| dy \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{Lip}_\beta} M_{\beta+\delta,1}(f)(\tilde{x}), \\
 |III| &\leq C \int_{R^n \setminus 2Q} \left( \frac{|x - x_0|}{|x_0 - y|^{n+1-\beta-\delta}} + \frac{|x - x_0|^\varepsilon}{|x_0 - y|^{n+\varepsilon-\beta-\delta}} \right) \\
 &\quad \times |D^\alpha A(y) - (D^\alpha A)_Q| |f(y)| dy \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{Lip}_\beta} \sum_{k=1}^{\infty} (2^{-k} + 2^{-k\varepsilon}) \frac{1}{|2^{k+1}Q|^{1-(\beta+\delta)/n}} \int_{2^{k+1}Q} |f(y)| dy \\
 &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{Lip}_\beta} M_{\beta+\delta,1}(f)(\tilde{x}).
 \end{aligned}$$

Thus

$$|T^A(f)(x) - T^A(f)(x_0)| \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{Lip}_\beta} M_{\beta+\delta,1}(f)(\tilde{x}).$$

Similar argument as in the proof of (a) and (b) will give the proof of (c) and (d), and so we omit the details.

Now we are in position to prove our theorems.

*Proof of Theorem 1.* We prove the theorem in several steps. First, we prove

$$(T^A(f))^\# \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} (M_{\delta,1}(f) + M_{\delta,r}(f)) \tag{1}$$

for any  $1 < r < n/\delta$ . Fix a cube  $Q = Q(x_0, d)$  and  $\tilde{x} \in Q$ . Let  $\tilde{A}(x) = A(x) - \sum_{|\alpha|=m} \frac{1}{\alpha!} (D^\alpha A)_Q x^\alpha$ . We write, for  $f_1 = f \chi_{2Q}$  and  $f_2 = f \chi_{R^n \setminus 2Q}$ ,

$$\begin{aligned} T^A(f)(x) &= \int_{R^n} \frac{R_{m+1}(A; x, y)}{|x - y|^m} K(x, y) f(y) dy \\ &= \int_{R^n} \frac{R_{m+1}(A; x, y)}{|x - y|^m} K(x, y) f_2(y) dy \\ &\quad + \int_{R^n} \frac{R_m(\tilde{A}; x, y)}{|x - y|^m} K(x, y) f_1(y) dy \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{K(x, y)(x - y)^\alpha}{|x - y|^m} D^\alpha \tilde{A}(y) f_1(y) dy, \end{aligned}$$

then

$$\begin{aligned} &|T^A(f)(x) - T^A(f_2)(x_0)| \\ &\leq \left| T \left( \frac{R_m(\tilde{A}; x, \cdot)}{|x - \cdot|^m} f_1 \right) (x) \right| + \sum_{|\alpha|=m} \frac{1}{\alpha!} \left| T \left( \frac{(x - \cdot)^\alpha}{|x - \cdot|^m} D^\alpha \tilde{A} f_1 \right) (x) \right| \\ &\quad + |T^A(f_2)(x) - T^A(f_2)(x_0)| \\ &:= I_1(x) + I_2(x) + I_3(x), \end{aligned}$$

thus,

$$\begin{aligned} &\frac{1}{|Q|} \int_Q |T^A(f)(x) - T^A(f_2)(x_0)| dx \\ &\leq \frac{1}{|Q|} \int_Q I_1(x) dx + \frac{1}{|Q|} \int_Q I_2(x) dx + \frac{1}{|Q|} \int_Q I_3(x) dx \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

Now, for  $I_1$ , if  $x \in Q$  and  $y \in 2Q$ , using Lemma 6, we get

$$R_m(\tilde{A}; x, y) \leq C|x - y|^m \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}}.$$

Thus, by the  $(L^r, L^s)$ -boundedness of  $T$  for  $1/s = 1/r - \delta/n$  and Holder's inequality, we obtain

$$\begin{aligned} I_1 &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} \frac{1}{|Q|} \int_Q |T(f_1)(x)| dx \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} \|T(f_1)\|_{L^s} |Q|^{-1/s} \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} \|f_1\|_{L^r} |Q|^{-1/s} \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} M_{\delta,r}(f)(\tilde{x}). \end{aligned}$$



For  $I_2$ , taking  $q > 1, l > 1$  such that  $1/s = 1/q - \delta/n$  and denoting  $r = ql$ , by the  $(L^q, L^s)$ -boundedness of  $T$ , we gain

$$\begin{aligned} I_2 &\leq \frac{C}{|Q|} \int_Q |T(\sum_{|\alpha|=m} (D^\alpha A - (D^\alpha A)_Q) f_1)(x)| dx \\ &\leq C \sum_{|\alpha|=m} \left( \frac{1}{|Q|} \int_Q |T((D^\alpha A - (D^\alpha A)_Q) f_1)(x)|^s dx \right)^{1/s} \\ &\leq C |Q|^{-1/s} \sum_{|\alpha|=m} \|(D^\alpha A - (D^\alpha A)_Q) f_1\|_{L^q} \\ &\leq C \sum_{|\alpha|=m} \left( \frac{1}{|Q|} \int_Q |D^\alpha A(y) - (D^\alpha A)_Q|^{ql'} dy \right)^{1/ql'} \\ &\quad \times \left( \frac{1}{|Q|^{1-r\delta/n}} \int_Q |f(y)|^{ql} dy \right)^{1/ql} \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} M_{\delta,r}(f)(\tilde{x}). \end{aligned}$$

For  $I_3$ , by using Key Lemma, we have

$$I_3 \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} (M_{\delta,1}(f)(\tilde{x}) + M_{\delta,r}(f)(\tilde{x})).$$

We now put these estimates together, and taking the supremum over all  $Q$  such that  $\tilde{x} \in Q$ , we obtain

$$(T^A(f))^\#(\tilde{x}) \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} (M_{\delta,1}(f)(\tilde{x}) + M_{\delta,r}(f)(\tilde{x})).$$

Thus, taking  $1 \leq r < p < n/\delta, 1/q = 1/p - \delta/n$  and by Lemma 4, we obtain

$$\begin{aligned} \|T^A(f)\|_{L^q} &\leq C \|(T^A(f))^\#\|_{L^q} \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} (\|M_{\delta,1}(f)\|_{L^q} + \|M_{\delta,r}(f)\|_{L^q}) \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} \|f\|_{L^p}. \end{aligned} \tag{2}$$

Secondly, we prove that, for  $D^\alpha A \in \text{Lip}_\beta(R^n)$  with  $|\alpha| = m$ ,

$$(T^A(f))^\# \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{Lip}_\beta} (M_{\beta+\delta,r}(f) + M_{\beta+\delta,1}(f)) \tag{3}$$

for any  $1 \leq r < n/(\beta + \delta)$ . In fact, by Lemma 6, we have, for  $x \in Q$  and  $y \in 2Q$ ,

$$\begin{aligned} |R_m(\tilde{A}; x, y)| &\leq C|x - y|^m \\ &\times \sum_{|\alpha|=m} \sup_{z \in 2Q} |D^\alpha A(z) - (D^\alpha A)_Q| \leq C|x - y|^m |Q|^{\beta/n} \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{Lip}_\beta} \end{aligned}$$

and by Lemma 5, we have

$$\begin{aligned} & \| (D^\alpha A - (D^\alpha A)_{2Q}) f \chi_{2Q} \|_{L^r} \\ & \leq C |Q|^{1/r} \| D^\alpha A \|_{\text{Lip}_\beta} \left( \frac{1}{|Q|^{1-r\beta/n}} \int_Q |f(y)|^r dy \right)^{1/r}, \end{aligned}$$

by the  $(L^r, L^s)$ -boundedness of  $T$  for  $1/s = 1/r - \delta/n$ , we obtain

$$\begin{aligned} & \frac{1}{|Q|} \int_Q |T^A(f)(x) - T^A(f)(x_0)| dx \\ & \leq \frac{1}{|Q|} \int_Q \left| T \left( \frac{R_m(\tilde{A}; x, \cdot)}{|x - \cdot|^m} f_1 \right) (x) \right| dx \\ & \quad + \frac{1}{|Q|} \int_Q \sum_{|\alpha|=m} \frac{1}{\alpha!} \left| T \left( \frac{(x - \cdot)^\alpha}{|x - \cdot|^m} D^\alpha \tilde{A} f_1 \right) (x) \right| dx \\ & \quad + \frac{1}{|Q|} \int_Q |T^A(f_2)(x) - T^A(f_2)(x_0)| dx \\ & \leq \sum_{|\alpha|=m} \| D^\alpha A \|_{\text{Lip}_\beta} \frac{C}{|Q|^{1/s-\beta/n}} \left( \int_Q |T(f_1)(x)|^s dx \right)^{1/s} \\ & \quad + \sum_{|\alpha|=m} \left( \frac{C}{|Q|} \int_Q |T(D^\alpha \tilde{A} f \chi_{2Q})(x)|^s dx \right)^{1/s} \\ & \quad + \frac{1}{|Q|} \int_Q |T^A(f_2)(x) - T^A(f_2)(x_0)| dx \\ & \leq C \sum_{|\alpha|=m} \| D^\alpha A \|_{\text{Lip}_\beta} \frac{1}{|Q|^{1/r-(\beta+\delta)/n}} \| f_1 \|_{L^r} \\ & \quad + \sum_{|\alpha|=m} \frac{C}{|Q|^{1/s}} \left( \int_{R^n} |(D^\alpha A(x) - (D^\alpha A)_Q) f(x) \chi_{2Q}(x)|^r dx \right)^{1/r} \\ & \quad + \frac{1}{|Q|} \int_Q |T^A(f_2)(x) - T^A(f_2)(x_0)| dx \\ & \leq C \sum_{|\alpha|=m} \| D^\alpha A \|_{\text{Lip}_\beta} (M_{\beta+\delta,1}(f)(\tilde{x}) + M_{\beta+\delta,r}(f)(\tilde{x})). \end{aligned}$$

Thus, taking  $1 \leq r < p < n/(\beta + \delta)$ ,  $1/q = 1/p - (\beta + \delta)/n$  and by Lemma 4, we obtain

$$\begin{aligned} \| T^A(f) \|_{L^q} & \leq C \| (T^A(f))^\# \|_{L^q} \\ & \leq C \sum_{|\alpha|=m} \| D^\alpha A \|_{\text{Lip}_\beta} (\| M_{\beta+\delta,r}(f) \|_{L^q} + \| M_{\beta+\delta,1}(f) \|_{L^q}) \\ & \leq C \sum_{|\alpha|=m} \| D^\alpha A \|_{\text{Lip}_\beta} \| f \|_{L^p}. \end{aligned} \tag{4}$$

Now we verify that  $T^A$  satisfies the conditions of Lemma 3. In fact, for any  $1 < p_i < n/(\beta + \delta)$  with  $1/s_i = 1/p_i - \delta/n$ ,  $1/q_i = 1/p_i - (\beta + \delta)/n$  ( $i = 1, 2$ ) and  $\|f\|_{L^{p_i}} \leq 1$ , note that  $T^A(f)(x) = T^{A-A_r}(f)(x) + T^{A_r}(f)(x)$  and  $D^\alpha(A_r) = (D^\alpha A)_r$ . By (2) and Lemma 1, we obtain

$$\begin{aligned} \|T^{A-A_r}(f)\|_{L^{s_i}} &\leq C \sum_{|\alpha|=m} \|D^\alpha(A - A_r)\|_{\text{BMO}} \|f\|_{L^{p_i}} \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A - (D^\alpha A)_r\|_{\text{BMO}} \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}_\varphi} \varphi(r). \end{aligned}$$

and by (4) and Lemma 2, we obtain

$$\|T^{A_r}(f)\|_{L^{q_i}} \leq C \sum_{|\alpha|=m} \|(D^\alpha A)_r\|_{\text{Lip}_\beta} \|f\|_{L^{p_i}} \leq Cr^{-\beta} \varphi(r) \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}_\varphi}.$$

Thus, for  $r = t^{-1/n}$ ,

$$\begin{aligned} m_{T^A(f)}(t^{1/s_i} \varphi(t^{-1/n})) &\leq m_{T^{A-A_r}(f)}(t^{1/s_i} \varphi(t^{-1/n})/2) \\ &\quad + m_{T^{A_r}(f)}(t^{1/s_i} \varphi(t^{-1/n})/2) \\ &\leq C \left[ \left( \frac{\varphi(r)}{t^{1/s_i} \varphi(r)} \right)^{s_i} + \left( \frac{r^{-\beta} \varphi(r)}{t^{1/s_i} \varphi(r)} \right)^{q_i} \right] = Ct^{-1}. \end{aligned}$$

Taking  $1 < s_2 < p < s_1 < n/(\beta + \delta)$  and by Lemma 3, we obtain, for  $\|f\|_{L^p} \leq (p/s_1)^{1/p}$ ,

$$\int_{R^n} \psi(|T^A(f)(x)|) dx = \int_0^\infty m_{T^A(f)}(\psi^{-1}(t)) dt \leq C,$$

thus,  $\|T^A(f)\|_{L_\psi} \leq C$ . This completes the proof of Theorem 1.

*Proof of Theorem 2.* Let  $Q, \tilde{A}(x), f_1$  and  $f_2$  be the same as in the proof of Theorem 1, we write

$$\begin{aligned} F_t^A(f)(x) &= \int_{R^n} \frac{R_{m+1}(\tilde{A}; x, y)}{|x - y|^m} F_t(x, y) f(y) dy \\ &= \int_{R^n} \frac{R_{m+1}(\tilde{A}; x, y)}{|x - y|^m} F_t(x, y) f(y) dy \\ &\quad + \int_{R^n} \frac{R_m(\tilde{A}; x, y)}{|x - y|^m} F_t(x, y) f_1(y) dy \\ &\quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \frac{F_t(x, y)(x - y)^\alpha}{|x - y|^m} D^\alpha \tilde{A}(y) f_1(y) dy, \end{aligned}$$

then

$$\begin{aligned} & \frac{1}{|Q|} \int_Q |S^A(f)(x) - S^A(f_2)(x_0)| dx \\ &= \frac{1}{|Q|} \int_Q \left| \|F_t^A(f)(x)\| - \|F_t^A(f_2)(x_0)\| \right| dx \\ &\leq \frac{1}{|Q|} \int_Q \left\| F_t \left( \frac{R_m(\tilde{A}; x, \cdot)}{|x - \cdot|^m} f_1 \right) (x) \right\| dx \\ &\quad + \sum_{|\alpha|=m} \frac{1}{\alpha!} \frac{1}{|Q|} \int_Q \left\| F_t \left( \frac{(x - \cdot)^\alpha}{|x - \cdot|^m} D^\alpha \tilde{A} f_1 \right) (x) \right\| dx \\ &\quad + \frac{1}{|Q|} \int_Q \|F_t^A(f_2)(x) - F_t^A(f_2)(x_0)\| dx. \end{aligned}$$

Using the same argument as in the proof of Theorem 1 will give the proof of Theorem 2. Hence we omit the details.

### 3. Applications

In this section we shall apply Theorems 1 and 2 of the paper to some particular operators such as the Calderón–Zygmund singular integral operator, fractional integral operator, Littlewood–Paley operator and Marcinkiewicz operator.

*Application 1. Calderón–Zygmund singular integral operator*

Let  $T$  be the Calderón–Zygmund operator (see [5,8,16]). The multilinear operator related to  $T$  is defined by

$$T^A(f)(x) = \int \frac{R_{m+1}(A; x, y)}{|x - y|^m} K(x, y) f(y) dy.$$

Then it is easy to verify that Key Lemma holds for  $T^A$  with  $\delta = 0$ , and thus  $T$  satisfies the conditions in Theorem 1. The conclusion of Theorem 1 holds for  $T^A$  with  $\delta = 0$ .

*Application 2. Fractional integral operator with rough kernel*

For  $0 < \delta < n$ , let  $T_\delta$  be the fractional integral operator with rough kernel defined by (see [1,7])

$$T_\delta f(x) = \int_{R^n} \frac{\Omega(x - y)}{|x - y|^{n-\delta}} f(y) dy.$$

The multilinear operator related to  $T_\delta$  is defined by

$$T_\delta^A f(x) = \int_{R^n} \frac{R_{m+1}(A; x, y)}{|x - y|^{m+n-\delta}} \Omega(x - y) f(y) dy,$$

where  $\Omega$  is homogeneous of degree zero on  $R^n$ ,  $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$  and  $\Omega \in \text{Lip}_\varepsilon(S^{n-1})$  for some  $0 < \varepsilon \leq 1$ , that is there exists a constant  $M > 0$  such that for any  $x, y \in S^{n-1}$ ,  $|\Omega(x) - \Omega(y)| \leq M|x - y|^\varepsilon$ . Then  $T_\delta$  satisfies the conditions in Theorem 1. Thus, the conclusion of Theorem 1 holds for  $T_\delta^A$ .

Application 3. Littlewood–Paley operator

Let  $\varepsilon > 0$ ,  $n > \delta \geq 0$  and  $\psi$  be a fixed function which satisfies the following properties:

- (1)  $|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$ ,
- (2)  $|\psi(x + y) - \psi(x)| \leq C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon-\delta)}$ , when  $2|y| < |x|$ .

The multilinear Littlewood–Paley operator is defined by

$$g_\psi^A(f)(x) = \left( \int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where

$$F_t^A(f)(x) = \int_{R^n} \frac{R_{m+1}(A; x, y)}{|x - y|^m} \psi_t(x - y) f(y) dy$$

and  $\psi_t(x) = t^{-n+\delta} \psi(x/t)$  for  $t > 0$ . We write that  $F_t(f) = \psi_t * f$ . We also define that

$$g_\psi(f)(x) = \left( \int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

which is the Littlewood–Paley  $g$  function (see [17]).

Let  $H$  be the space  $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t)^{1/2} < \infty\}$ , then, for each fixed  $x \in R^n$ ,  $F_t^A(f)(x)$  may be viewed as a mapping from  $[0, +\infty)$  to  $H$ , and it is clear that

$$g_\psi(f)(x) = \|F_t(f)(x)\| \quad \text{and} \quad g_\psi^A(f)(x) = \|F_t^A(f)(x)\|.$$

It is only to verify that Key Lemma holds for  $g_\psi^A$ . In fact, for  $D^\alpha A \in \text{BMO}(R^n)$  with  $|\alpha| = m$ , we write, for a cube  $Q = Q(x_0, d)$  with  $\text{supp } f \subset (2Q)^c$ ,  $x, \tilde{x} \in Q = Q(x_0, d)$ ,

$$\begin{aligned} & F_t^A(f)(x) - F_t^A(f)(x_0) \\ &= \int_{R^n} \left( \frac{\psi_t(x - y)}{|x - y|^m} - \frac{\psi_t(x_0 - y)}{|x_0 - y|^m} \right) R_m(\tilde{A}; x, y) f(y) dy \\ & \quad + \int_{R^n} \frac{\psi_t(x_0 - y)}{|x_0 - y|^m} (R_m(\tilde{A}; x, y) - R_m(\tilde{A}; x_0, y)) f(y) dy \\ & \quad - \sum_{|\alpha|=m} \frac{1}{\alpha!} \int_{R^n} \left( \frac{(x - y)^\alpha \psi_t(x - y)}{|x - y|^m} - \frac{(x_0 - y)^\alpha \psi_t(x_0 - y)}{|x_0 - y|^m} \right) \\ & \quad \times D^\alpha \tilde{A}(y) f(y) dy \\ & := J_1 + J_2 + J_3. \end{aligned}$$

By the condition of  $\psi$  and Minkowski's inequality, we obtain, for any  $r > 1$ ,

$$\begin{aligned} \|J_1\| &\leq C \int_{R^n} \frac{|R_m(\tilde{A}; x, y)||f(y)|}{|x_0 - y|^m} \left[ \int_0^\infty \left( \frac{t|x - x_0|}{|x_0 - y|(t + |x_0 - y|)^{n+1-\delta}} \right. \right. \\ &\quad \left. \left. + \frac{t|x - x_0|^\varepsilon}{(t + |x_0 - y|)^{n+1+\varepsilon-\delta}} \right)^2 \frac{dt}{t} \right]^{1/2} dy \\ &\leq C \int_{(2Q)^c} \left( \frac{|x - x_0|}{|x_0 - y|^{m+n+1-\delta}} + \frac{|x - x_0|^\varepsilon}{|x_0 - y|^{m+n+\varepsilon-\delta}} \right) \\ &\quad \times |R_m(\tilde{A}; x, y)||f(y)| dy \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}M_{\delta,1}}(f)(\tilde{x}), \\ \|J_2\| &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} \sum_{k=1}^\infty \int_{2^{k+1}\setminus 2^k Q} \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta}} |f(y)| dy \\ &\leq C \|D^\alpha A\|_{\text{BMO}M_{\delta,1}}(f)(\tilde{x}), \\ \|J_3\| &\leq C \sum_{|\alpha|=m} \sum_{k=1}^\infty \int_{2^{k+1}\setminus 2^k Q} \left( \frac{|x - x_0|}{|x_0 - y|^{n+1-\delta}} + \frac{|x - x_0|^\varepsilon}{|x_0 - y|^{n+\varepsilon-\delta}} \right) \\ &\quad \times |D^\alpha \tilde{A}(y)||f(y)| dy \\ &\leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}M_{\delta,r}}(f)(\tilde{x}). \end{aligned}$$

Similarly, for  $D^\alpha A \in \text{Lip}_\beta(R^n)$  with  $|\alpha| = m$ , we get

$$\|F_t^A(f)(x) - F_t^A(f)(x_0)\| \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{Lip}_\beta} M_{\beta+\delta,1}(f)(\tilde{x}).$$

From the above estimates, we know that Theorem 2 holds for  $g_\psi^A$ .

*Application 4. Marcinkiewicz operator*

Let  $0 \leq \delta < n$ ,  $0 < \varepsilon \leq 1$  and  $\Omega$  be homogeneous of degree zero on  $R^n$  and  $\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0$ . Assume that  $\Omega \in \text{Lip}_\varepsilon(S^{n-1})$ , that is there exists a constant  $M > 0$  such that for any  $x, y \in S^{n-1}$ ,  $|\Omega(x) - \Omega(y)| \leq M|x - y|^\varepsilon$ . The multilinear Marcinkiewicz operator is defined by

$$\mu_\Omega^A(f)(x) = \left( \int_0^\infty |F_t^A(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_t^A(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} \frac{R_{m+1}(A; x, y)}{|x-y|^m} f(y) dy.$$

We write that

$$F_t(f)(x) = \int_{|x-y|\leq t} \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} f(y) dy.$$

We also define that

$$\mu_{\Omega}(f)(x) = \left( \int_0^{\infty} |F_t(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

which is the Marcinkiewicz operator (see [18]).

Let  $H$  be the space  $H = \{h : \|h\| = (\int_0^{\infty} |h(t)|^2 dt/t^3)^{1/2} < \infty\}$ . Then, it is clear that

$$\mu_{\Omega}(f)(x) = \|F_t(f)(x)\| \quad \text{and} \quad \mu_{\Omega}^A(f)(x) = \|F_t^A(f)(x)\|.$$

Now, it is only to verify that Key Lemma holds for  $\mu_{\Omega}^A$ . In fact, for  $D^{\alpha} A \in \text{BMO}(R^n)$  with  $|\alpha| = m$ , a cube  $Q = Q(x_0, d)$  with  $\text{supp } f \subset (2Q)^c$ ,  $x, \tilde{x} \in Q = Q(x_0, d)$  and  $r > 1$ , we have

$$\begin{aligned} & \|F_t^A(f)(x) - F_t^A(f)(x_0)\| \\ & \leq \left( \int_0^{\infty} \left| \int_{|x-y|\leq t} \frac{\Omega(x-y)R_m(\tilde{A}; x, y)}{|x-y|^{m+n-1-\delta}} f(y)dy \right. \right. \\ & \quad \left. \left. - \int_{|x_0-y|\leq t} \frac{\Omega(x_0-y)R_m(\tilde{A}; x_0, y)}{|x_0-y|^{m+n-1-\delta}} f(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ & \quad + \sum_{|\alpha|=m} \left( \int_0^{\infty} \left| \int_{|x-y|\leq t} \left( \frac{\Omega(x-y)(x-y)^{\alpha}}{|x-y|^{m+n-1-\delta}} \right. \right. \right. \\ & \quad \left. \left. \left. - \int_{|x_0-y|\leq t} \frac{\Omega(x_0-y)(x_0-y)^{\alpha}}{|x_0-y|^{m+n-1-\delta}} \right) D^{\alpha} \tilde{A}(y) f(y)dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\ & \leq \left( \int_0^{\infty} \left[ \int_{|x-y|\leq t, |x_0-y|>t} \frac{|\Omega(x-y)||R_m(\tilde{A}; x, y)|}{|x-y|^{m+n-1-\delta}} |f(y)|dy \right]^2 \frac{dt}{t^3} \right)^{1/2} \\ & \quad + \left( \int_0^{\infty} \left[ \int_{|x-y|>t, |x_0-y|\leq t} \frac{|\Omega(x_0-y)||R_m(\tilde{A}; x_0, y)|}{|x_0-y|^{m+n-1-\delta}} |f(y)|dy \right]^2 \frac{dt}{t^3} \right)^{1/2} \\ & \quad + \left( \int_0^{\infty} \left[ \int_{|x-y|\leq t, |x_0-y|\leq t} \left| \frac{\Omega(x-y)R_m(\tilde{A}; x, y)}{|x-y|^{m+n-1-\delta}} \right. \right. \right. \\ & \quad \left. \left. \left. - \frac{\Omega(x_0-y)R_m(\tilde{A}; x_0, y)}{|x_0-y|^{m+n-1-\delta}} \right| |f(y)|dy \right]^2 \frac{dt}{t^3} \right)^{1/2} \end{aligned}$$

$$\begin{aligned}
& + \sum_{|\alpha|=m} \left( \int_0^\infty \left| \int_{|x-y|\leq t} \left( \frac{\Omega(x-y)(x-y)^\alpha}{|x-y|^{m+n-1-\delta}} \right. \right. \right. \\
& \quad \left. \left. \left. - \int_{|x_0-y|\leq t} \frac{\Omega(x_0-y)(x_0-y)^\alpha}{|x_0-y|^{m+n-1-\delta}} \right) D^\alpha \tilde{A}(y) f(y) dy \right|^2 \frac{dt}{t^3} \right)^{1/2} \\
& := L_1 + L_2 + L_3 + L_4
\end{aligned}$$

and

$$\begin{aligned}
L_1 & \leq C \int_{R^n} \frac{|f(y)| |R_m(\tilde{A}; x, y)|}{|x-y|^{m+n-1-\delta}} \left( \int_{|x-y|\leq t < |x_0-y|} \frac{dt}{t^3} \right)^{1/2} dy \\
& \leq C \int_{R^n} \frac{|f(y)| |R_m(\tilde{A}; x, y)|}{|x-y|^{m+n-1-\delta}} \left( \frac{1}{|x-y|^2} - \frac{1}{|x_0-y|^2} \right)^{1/2} dy \\
& \leq C \int_{(2Q)^c} \frac{|f(y)| |R_m(\tilde{A}; x, y)|}{|x-y|^{m+n-1-\delta}} \frac{|x_0-x|^{1/2}}{|x-y|^{3/2}} dy \\
& \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}M_{\delta,1}}(f)(\tilde{x}).
\end{aligned}$$

Similarly, we have  $L_2 \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}M_{\delta,1}}(f)(\tilde{x})$ .

For  $L_3$ , by the following inequality (see [18]):

$$\left| \frac{\Omega(x-y)}{|x-y|^{n-1-\delta}} - \frac{\Omega(x_0-y)}{|x_0-y|^{n-1-\delta}} \right| \leq C \left( \frac{|x-x_0|}{|x_0-y|^{n-\delta}} + \frac{|x-x_0|^\gamma}{|x_0-y|^{n-1-\delta+\gamma}} \right),$$

we gain

$$\begin{aligned}
L_3 & \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}} \int_{(2Q)^c} \left( \frac{|x-x_0|}{|x_0-y|^{n-\delta}} + \frac{|x-x_0|^\gamma}{|x_0-y|^{n-1-\delta+\gamma}} \right) \\
& \quad \times \left( \int_{|x_0-y|\leq t, |x-y|\leq t} \frac{dt}{t^3} \right)^{1/2} |f(y)| dy \\
& \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}M_{\delta,1}}(f)(\tilde{x}).
\end{aligned}$$

For  $L_4$ , similar to the proof of  $L_1$ ,  $L_2$  and  $L_3$ , we obtain

$$\begin{aligned}
L_4 & \leq C \sum_{|\alpha|=m} \sum_{k=1}^\infty \int_{2^{k+1}Q \setminus 2^kQ} \left( \frac{|x-x_0|}{|x_0-y|^{n+1-\delta}} + \frac{|x-x_0|^{1/2}}{|x_0-y|^{n+1/2-\delta}} \right. \\
& \quad \left. + \frac{|x-x_0|^\gamma}{|x_0-y|^{n+\gamma-\delta}} \right) |D^\alpha \tilde{A}(y)| |f(y)| dy \\
& \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{BMO}M_{\delta,r}}(f)(\tilde{x}).
\end{aligned}$$



Similarly, for  $D^\alpha A \in \text{Lip}_\beta(\mathbb{R}^n)$  with  $|\alpha| = m$ , we get

$$\|F_t^A(f)(x) - F_t^A(f)(x_0)\| \leq C \sum_{|\alpha|=m} \|D^\alpha A\|_{\text{Lip}_\beta} M_{\beta+\delta,1}(f)(\tilde{x}).$$

Thus, Theorem 2 holds for  $\mu_\Omega^A$ .

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