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# Containment of $c_0$ and $\ell_1$ in $\Pi_1(E, F)$

MOHSEN ALIMOHAMMADY

Department of Mathematics, University of Mazandran, Faculty of Basic Sciences, Babolsar, Iran
E-mail: amohsen@umz.ac.ir

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**Abstract.** Suppose $\Pi_1(E, F)$ is the space of all absolutely 1-summing operators between two Banach spaces $E$ and $F$. We show that if $F$ has a copy of $c_0$, then $\Pi_1(E, F)$ will have a copy of $c_0$, and under some conditions if $E$ has a copy of $\ell_1$ then $\Pi_1(E, F)$ would have a complemented copy of $\ell_1$.

**Keywords.** Absolutely 1-suuming operators; copy of $c_0$; copy of $\ell_1$.

## 1. Introduction

Many studies on copy of $c_0$ or $\ell_1$ in spaces of bounded operators, weakly compact operators, and compact operators have been made [2–5, 7]. Here we intend to prove similar results in the space of absolutely 1-summing operators.

Absolutely 1-summing operators were introduced and studied by Grothendiéck in his famous résume [6]. For two Banach spaces $E$ and $F$, the operator $T\colon E \to F$ is called absolutely 1-summing operator, if given any sequence $(x_n)_n$ of $E$ for which $\sum_{n=1}^{\infty} |x^* x_n| < \infty$, for each $x^* \in E^*$, we have $\sum_{n=1}^{\infty} \|T x_n\| < \infty$. $\Pi_1(E, F)$ is the Banach space of all absolutely 1-summing operators from $E$ to $F$, endowed with $\pi_1$-norm as follows:

$$\pi_1(T) = \inf \left\{ \rho > 0 : \sum_{i=1}^{n} \|T x_i\| \right.$$
$$\left. \leq \rho \sup_{x^* \in B_E^*} \left( \sum_{i=1}^{n} |x^*(x_i)| \right);\ x_i \in E, 1 \leq i \leq n \right\}.$$

By Grothendiéck–Pietsch Dominated Theorem [1], for $T \in \Pi_1(E, F)$ there exists a regular probability measure $\mu$ defined on $B_{E^*}$ (with its weak$^*$ topology) for which

$$\|T(x)\| \leq \pi_1(T) \int_{B_{E^*}} |x^* x| \, \mathrm{d}\mu(x^*),$$

holds for each $x \in E$. Therefore, $\|T(x)\| \leq \pi_1(T)\, \|x\|$.

The following lemma discusses the main results of our study.

*Lemma* 1. [1]. *Let $x^*$ and $y$ be, in $B_{E^*}$ and $B_F$, the unit balls of two Banach spaces $E^*$ and $F$ respectively. Then $x^* \otimes y \in \Pi_1(E, F)$ and $\pi_1(x^* \otimes y) \leq 1$.* □

## 2. Main results

Here, we present a result showing complemented copy of $\ell_1$ in $\Pi_1(E, F)$.

**Theorem 2.** *Suppose $(x_n^*)_n$ is a sequence in $B_{E^*}$ such that $x_n^* x_n = 1$ and $x^{**}x_n^* > \epsilon$ for each $n \in N$, where $(x_n)_n$ is a weakly unconditionally Cauchy sequence in E and $x^{**} \in E^{**}$ (for example $E = c_0, e_n = x_n, x_n^* = e_n \in \ell_1, x^{**} = \chi N \in \ell_\infty$). Then $\Pi_1(E, F)$ has a complemented copy of $\ell_1$ if F has a copy of $\ell_1$.*

*Proof.* From the assumption there is an $\ell_1$-basic sequence $(y_n)_n$ in $F$ and so $M_1, M_2 > 0$ such that for each $(a_n)_n \in \ell_1$ we have

$$M_1 \sum_{n=1}^{\infty} |a_n| \leq \left\| \sum_{n=1}^{\infty} a_n y_n \right\| \leq M_2 \sum_{n=1}^{\infty} |a_n|.$$

Therefore,

$$\begin{aligned} \pi_1\left(\sum_{n=1}^{m} a_n x_n^* \otimes y_n\right) &\leq \sum_{n=1}^{m} |a_n| \pi_1(x_n^* \otimes y_n) \\ &\leq \sum_{n=1}^{m} |a_n| \|x_n^*\| \cdot \|y_n\| \leq M_2 \sum_{n=1}^{m} |a_n|. \end{aligned}$$

On the other hand,

$$\begin{aligned} \pi_1\left(\sum_{n=1}^{m} a_n x_n^* \otimes y_n\right) &\geq \left\| \sum_{n=1}^{m} x_n^* \otimes y_n \right\| \\ &= \sup_{x^{**} \in B_{E^{**}}} \left\| \sum_{n=1}^{m} a_n x^{**}(x_n^*) y_n \right\| \\ &\geq M_1 \sum_{n=1}^{m} |a_n| \cdot |x_0^{**} x_n^*| \geq C M_1 \sum_{n=1}^{m} |a_n|. \end{aligned}$$

This shows that $\Pi_1(E, F)$ contains $(x_n^* \otimes y_n)_n$ as a copy of $\ell_1$. We show this copy is complemented in $\Pi_1(E, F)$. Define $P\colon \Pi_1(E, F) \to [x_n^* \otimes y_n]$ by $P(T) = \sum_{n=1}^{\infty} (x_n^* \otimes y_n) y_n^* T(x_n)$, where $[x_n^* \otimes y_n]$ is the closed linear span of $(x_n^* \otimes y_n)$ in $\Pi_1(E, F)$. Since $(x_n)_n$ is $wuC$ and $T$ is the absolutely 1-summing operator, $P$ is a well-defined linear map. From the closed graph theorem one can easily show that $P$ is a projection from $\Pi_1(E, F)$ onto $[x_n^* \otimes y_n]$ which completes the proof. □

Now we explain this result for asymptotically isomorphic copy of $\ell_1$.

PROPOSITION 3

*$\Pi(E, F)$ has an asymptotically isomorphic copy of $\ell_1$ if F has an asymptotically isomorphic copy of $\ell_1$ too.*

*Proof.* Suppose for $(y_n)$ in $F$ there is a positive null sequence $(\epsilon_n)$ such that

$$\sum (1 - \epsilon_n)|a_n| \leq \left\| \sum a_n y_n \right\| \leq \sum |a_n|.$$

Consider $x_0^*$ and $x_0$ in the unit sphere of $E^*$ and $E$ respectively such that $x_0^* x_0 = 1$. Then

$$\sum (1-\epsilon_n)|a_n| \leq \left\| \sum a_n x_0^* x_0 y_n \right\| \leq \pi_1 \left( \sum a_n x_0^* \otimes y_n \right) \leq \sum |a_n|$$

which completes the proof. □

**Theorem 4.** *Suppose $L(E, F)$ contains a sequence $(T_n)$ equivalent to the standard unit vector basis of $c_0$ such that for $x_0 \in E$ (respectively $y_0^* \in F^*$)$T_n x_0$ (respectively $T_n^* y_0^*$) are basic sequences. Then $(F)$ or $E^{**}$ has a copy of $c_0$.*

*Proof.* Suppose $(T_n x_0)$ is a basic sequence and $(y_n^*)$ is its coefficient functional. We can assume $\|y_n^*\| \leq M$. On the other hand,

$$C_1 \mathrm{sup}_n |a_n| \leq \left\| \sum_{n+1}^{\infty} a_n T_n \right\| \leq C_2 \mathrm{sup}_n |a_n|.$$

Therefore,

$$(C_1/M)|a_n| = (1/M) \left| y_n^* \left( \sum_{n=1}^{\infty} a_n T_n(x_0) \right) \right| \leq \left\| \sum_{n=1}^{\infty} a_n T_n \right\| \leq C - 2 \mathrm{sup}_n |a_n|.$$

This shows that $(T_n x_0)$ is also a copy of $c_0$. □

COROLLARY 5

*Let $(T_n)$ be a copy of $c_0$ in $\Pi(E, F)$ and $x_0 \in B_E$ such that $T_n(x_0)$ is a semi-normalized sequence in $F$. Then $F$ would have a copy of $c_0$.*

*Proof.* $M_1, M_2 > 0$ such that for any $(a_n) \in c_0$,

$$M_1 \ \mathrm{sup}_n |a_n| \leq \pi_1 \left( \sum_{n=1}^{\infty} a_n T_n \leq M_2 \ \mathrm{sup}_n |a_n| \right).$$

But it is easy to see that $T_n(x_0)$ is weakly null. From the assumption $\liminf \|T_n x_0\| = C > 0$, it follows from Bessaga–Pełczynski' Theorem [1] that it would be a basic sequence with coefficient functionals $(y_n^*)$ such that $\|y_n^*\| < 1/C$. We would have,

$$\begin{aligned} |a_n| = \left| y_n^* \left( \sum_{m=1}^{\infty} a_m T_m(x_0) \right) \right| &\leq (1/C) \left\| \sum_{n=1}^{\infty} a_m T_m(x_0) \right\| \\ &\leq (1/C) \left\| \sum_{n=1}^{\infty} a_m T_m \right\| \\ &\leq (1/C) \pi_i \left( \sum_{n=1}^{\infty} a_m T_m \right) \\ &\leq M - (2/C) \ \mathrm{sup}_n |a_n|, \end{aligned}$$

which shows that $(T_n(x_0))$ is a $c_0$-basic sequence in $F$. □

**Theorem 6.** *$\Pi_1(E, F)$ has a complemented copy of $c_0$ if $F$ has a complemented copy of $c_0$.*

*Proof.* Similar to the proof of Theorem 2 we may assume $(y_n)_n$ is a complemented copy of $c_0$ in $F$ and $(y_n^*)_n$ the coefficient functional of it in $[y_n]$ the closed linear span of $(y_n^*)_n$. Since $[y_n]$ is complemented in $F$, we can assume $(y_n^*)$ is a weak* null convergence sequence in $F^*$. Define $P\colon \Pi_1(E, F) \to [x^* \otimes y_n]$ by $P(T)(x) = \sum_{n=1}^{\infty}(x^* \otimes y_n)y_n^* T(x)$, where $x^*$ is an arbitrary element in the unit sphere of $E^*$ and $[x^* \otimes y_n]$ is the closed linear subspace generated by $(x^* \otimes y_n)$ in $\Pi_1(E, F)$. For any $(x_i)_{i=1}^n$ in $E$, we have

$$\begin{aligned}\sum_{i=1}^{n} \|PT(x_i)\| &= \sum_{i=1}^{n} \left\| \sum_{n=1}^{\infty}(y_n^* T(x_i))x^* \otimes y_n \right\| \\ &\le \sum_{i=1}^{n} C_2 \sup_n |(y_n^* T(x_i))| \\ &= C_2 \sum_{i=1}^{n} \|T(x_i)\| \le C_2 \pi_1(T) \sum_{i=1}^{n} |x^* x_i|.\end{aligned}$$

This shows that $\pi_1(PT) \le C_2\pi_1(T)$. Therefore, $P$ is a bounded projection, which completes the proof. □

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