

A criterion for regular sequences

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Abstract. Let R be a commutative noetherian ring and $f_1, \dots, f_r \in R$. In this article we give (cf. the Theorem in §2) a criterion for f_1, \dots, f_r to be regular sequence for a finitely generated module over R which strengthens and generalises a result in [2]. As an immediate consequence we deduce that if $V(g_1, \dots, g_r) \subseteq V(f_1, \dots, f_r)$ in $\text{Spec } R$ and if f_1, \dots, f_r is a regular sequence in R , then g_1, \dots, g_r is also a regular sequence in R .

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1. Regular sequences

As there is no uniformity about the concept of regular sequence, we first recall the following definitions that we shall use in this note.

DEFINITION 1

Let R be a commutative noetherian ring and $f_1, \dots, f_r \in R$. We say that f_1, \dots, f_r is a *strongly regular sequence* on a R -module M , if for every $i = 1, \dots, r$ the element f_i is a non-zero divisor for $M/(f_1, \dots, f_{i-1})M$. The sequence f_1, \dots, f_r is called a *regular sequence* on a R -module M , if for every $\mathfrak{p} \in \text{Supp}(M/(f_1, \dots, f_r)M)$, the sequence f_1, \dots, f_r in the local ring $R_{\mathfrak{p}}$ is a strongly regular sequence on the $R_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$.

Note that, in contrast to most of the standard text books, we do not assume the $M \neq (f_1, \dots, f_r)M$ for a strongly regular sequence f_1, \dots, f_r . For general notations in commutative algebra we also refer to [1].

If the sequence f_1, \dots, f_r is strongly regular respectively regular on the R -module M , then the same is true for the sequence $f_1 \cdot 1_S, \dots, f_r \cdot 1_S$ on the S -module $S \otimes_R M$, where S is an arbitrary flat noetherian R -algebra.

Note that every sequence is a strongly regular as well as regular sequence on the zero module. Further, it is clear that a strongly regular sequence is a regular sequence but not conversely. For example:

Example. Let $P := k[X, Y, Z]$ be the polynomial ring in three indeterminates over a field k , $\mathfrak{p} := P(X - 1) + PZ$, $\mathfrak{q} := PY$ and let $R := P/\mathfrak{p} \cap \mathfrak{q} = P/PY(X - 1) + PYZ$. Then Z, X is a regular sequence on the P -module R but not a strongly regular sequence.

The difference between regular and strongly regular sequences is well-illustrated in the following statement given in Chapter II, 6.1 of [4].

PROPOSITION

Let M be a finitely generated module over a noetherian ring R and let $f_1, \dots, f_r \in R$. Then the following conditions are equivalent:

- (i) f_1, \dots, f_r is a strongly regular sequence on M .
- (ii) For every $s = 1, \dots, r$ the sequence f_1, \dots, f_s is a regular sequence on M .

It can be easily seen that (see the proof of Proposition 3, Chapter IV, A, §1 of [5]) a sequence f_1, \dots, f_r in a commutative noetherian ring R is a regular sequence for a finitely generated R -module M if and only if the Koszul complex $K_\bullet(f_1, \dots, f_r; M)$ gives a resolution of $M/(f_1, \dots, f_r)M$. In particular, if f_1, \dots, f_r is a regular sequence on M , then for every permutation $\sigma \in \mathfrak{S}_r$ the sequence $f_{\sigma_1}, \dots, f_{\sigma_r}$ is also regular for M . Further, the above proposition implies that the sequence $f_{\sigma_1}, \dots, f_{\sigma_r}$ is strongly regular on M for every $\sigma \in \mathfrak{S}_r$ if and only if all subsequences of f_1, \dots, f_r are regular on M . For the sake of completeness let us recall Definition 2.

DEFINITION 2

Let (R, \mathfrak{m}_R) be a noetherian local ring and let M be a non-zero R -module. Then the length of a maximal regular sequence on M in the maximal ideal \mathfrak{m}_R is called the depth of M over R and is denoted by $\text{depth}_R(M)$.

If M is finitely generated then depth can be (cf. [5], Proposition and Definition 3, Chapter IV, A, §2) characterized by

$$(\ddagger) \quad \text{depth}_R(M) = \min\{i \in \mathbb{N} \mid \text{Ext}_R^i(R/\mathfrak{m}_R, M) \neq 0\}.$$

A finitely generated R -module is called a *Cohen–Macaulay module* if $\dim_R(M) = \text{depth}_R(M)$.

2. Theorem

The following theorem is the main result of this note.

Theorem. Let R be a commutative noetherian ring, $f_1, \dots, f_r \in R$ and let M be a finitely generated R -module. Then the following statements are equivalent:

- (i) f_1, \dots, f_r is a regular sequence on M .
- (ii) $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq r$ for every $\mathfrak{p} \in \text{Supp}(M/(f_1, \dots, f_r)M)$.
- (iii) $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq r$ for every $\mathfrak{p} \in \text{Ass}(M/(f_1, \dots, f_r)M)$.

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are trivial.

(ii) \Rightarrow (i): We may assume that R is local and $f_1, \dots, f_r \in \mathfrak{m}_R$. Let $\mathfrak{p} \in \text{Ass}(M)$ and let \mathfrak{q} be a minimal prime ideal in $\mathbb{V}(\mathfrak{p} + Rf_1 + \dots + Rf_r)$. Then $\mathfrak{q} \in \text{Supp}(M/(f_1, \dots, f_r)M) = \text{Supp}(M) \cap \mathbb{V}(f_1, \dots, f_r)$ and so $\text{depth}_{R_{\mathfrak{q}}}M_{\mathfrak{q}} \geq r$ by (ii). Since $\mathfrak{p} \in \text{Ass}(M)$, we have

$\text{Hom}_{R_{\mathfrak{p}}}(k(\mathfrak{p}), M_{\mathfrak{p}}) \neq 0$ and so $\text{Ext}_{R_{\mathfrak{q}}}^h(k(\mathfrak{q}), M_{\mathfrak{q}}) \neq 0$ by Chapter 6, §18, Lemma 4 of [3], where $h := ht_{R_{\mathfrak{p}}}(\mathfrak{q}/\mathfrak{p})$. Therefore $r \leq \text{depth}_{R_{\mathfrak{q}}} M_{\mathfrak{q}} \leq h$ (see (\ddagger) in §1). But then $f_1 \notin \mathfrak{p}$, since otherwise $h \leq r - 1$ by the (generalised) Krull's theorem (see [5], Corollary 4, Chapter III, B, §2). This proves that f_1 is a non-zero divisor for M . Now, induction on r completes the proof.

The implication (iii) \Rightarrow (i) is proved in the lemma which is given below. (In the proof of the lemma we use the implication (ii) \Rightarrow (i).) ■

COROLLARY 1 ([2], Corollary 1)

Let R be a commutative noetherian ring, $f_1, \dots, f_r \in R$ and let M be a finitely generated R -module. Then f_1, \dots, f_r is a regular sequence on M if and only if f_1, \dots, f_r is a regular sequence on $M_{\mathfrak{p}}$ for every $\mathfrak{p} \in \text{Ass}(M/f_1, \dots, f_r)M$.

COROLLARY 2

Let R be a commutative noetherian ring and let $f_1, \dots, f_r, g_1, \dots, g_r \in R$. Let M be a finitely generated R -module such that $\text{Supp}(M/(g_1, \dots, g_r)M) \subseteq \text{Supp}(M/(f_1, \dots, f_r)M)$. Suppose that f_1, \dots, f_r is a regular sequence on M . Then g_1, \dots, g_r is also a regular sequence on M . In particular, if $V(g_1, \dots, g_r) \subseteq V(f_1, \dots, f_r)$ and if f_1, \dots, f_r is a regular sequence in R , then g_1, \dots, g_r is also a regular sequence in R .

From the above equivalence we can also deduce the following well-known fact:

COROLLARY 3 (cf. [5], Theorem 2, Chapter IV, B, §2)

If M is a finitely generated Cohen–Macaulay module over a noetherian local ring R , then every system of parameters of M is a regular sequence on M . In particular, in a Cohen–Macaulay local ring every system of parameters is a regular sequence.

Finally, we give a proof of the lemma which we have already used for the proof of the implication (iii) \Rightarrow (i) of the theorem.

Lemma. *Let R be a commutative noetherian ring, $f_1, \dots, f_r \in R$ and let M be a finitely generated R -module. Suppose that $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq r$ for every $\mathfrak{p} \in \text{Ass}(M/(f_1, \dots, f_r)M)$. Then f_1, \dots, f_r is a regular sequence on M .*

Proof. We shall prove by induction on r the following implication:

$(*)_r$: If $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq r$ for every $\mathfrak{p} \in \text{Ass}(M/(f_1, \dots, f_r)M)$, then f_1, \dots, f_r is a regular sequence on M .

Proof of $()_1$.* Put $f := f_1$ and suppose that $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq 1$ for every $\mathfrak{p} \in \text{Ass}(M/fM)$. Then $\text{Ass}(M) \cap \text{Ass}(M/fM) = \emptyset$. We shall show that f is a non-zero divisor for M . Suppose on the contrary that f is a zero divisor on M . By localising at a minimal prime ideal in $\text{Ass}(M) \cap V(Rf)$, we may assume that R is a local ring, $\text{depth}_R(M) = 0$ and that $\text{Ass}(M) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m, \mathfrak{m}_R\}$ with $\mathfrak{p}_i \notin V(Rf)$ for all $i = 1, \dots, m$. Then $m \geq 1$. Let Q_1, \dots, Q_m and Q be the primary components corresponding to $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ and \mathfrak{m}_R respectively and let $0 = Q_1 \cap \dots \cap Q_m \cap Q$ be an irredundant primary decomposition of the zero module in M . Let $N := Q_1 \cap \dots \cap Q_m$. Then $N \neq 0$, $\text{Ass}(M/N) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_m\}$ and f is a non-zero divisor for M/N , since $f \notin \mathfrak{p}_i$ for all $i = 1, \dots, m$. This implies

that the canonical homomorphism $N/fN \longrightarrow M/fM$ is injective. Further, since Q is \mathfrak{m}_R -primary in M , we have $\mathfrak{m}_R^n N \subseteq N \cap \mathfrak{m}_R^n M \subseteq N \cap Q = 0$ for some $n \in \mathbb{N}^+$, and hence N has finite length. Therefore N/fN has finite length. But $\text{depth}_R(M/fM) \geq 1$, since $\mathfrak{m}_R \notin \text{Ass}(M/fM)$ and therefore cannot contain any submodules of finite length. This proves that $N/fN = 0$ and then $N = 0$ by Nakayama's lemma, which contradicts $N \neq 0$.

Proof of $()_r \Rightarrow (*)_{r+1}$.* We may assume that R is local, $f_1, \dots, f_{r+1} \in \mathfrak{m}_R$ and $M \neq 0$. Now, we shall prove this implication by induction on $\dim(R)$. Clearly the induction starts at $\dim(R) = 0$. Put $\overline{M}_r := M/(f_1, \dots, f_r)M$ and $\overline{M}_{r+1} := M/(f_1, \dots, f_{r+1})M$. Then by induction hypothesis.

(\dagger) f_1, \dots, f_{r+1} is a regular sequence on $M_{\mathfrak{p}}$ for every $\mathfrak{p} \in \text{Supp}(\overline{M}_{r+1}) \setminus \{\mathfrak{m}_R\}$.

In particular, we have:

($\dagger\dagger$) $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq r + 1$ for every $\mathfrak{p} \in \text{Supp}(\overline{M}_{r+1}) \setminus \{\mathfrak{m}_R\}$.

We consider two cases:

Case 1. $\mathfrak{m}_R \in \text{Ass}(\overline{M}_{r+1})$. In this case, by assumption in $(*)_{r+1}$, $\text{depth}_R(M) \geq r + 1$. Now, use (ii) \Rightarrow (i) of the theorem to conclude that f_1, \dots, f_{r+1} is a regular sequence on M .

Case 2. $\mathfrak{m}_R \notin \text{Ass}(\overline{M}_{r+1})$. In this case $\text{Ass}(\overline{M}_r) \cap \text{Ass}(\overline{M}_{r+1}) = \emptyset$, since $\text{depth}_{R_{\mathfrak{p}}}(\overline{M}_r)_{\mathfrak{p}} \geq 1$ for every $\mathfrak{p} \in \text{Ass}(\overline{M}_{r+1}) \setminus \{\mathfrak{m}_R\}$ by ($\dagger\dagger$). Therefore by $(*)_1$, f_{r+1} is a non-zero divisor on \overline{M}_r . Now, it remains to show that the sequence f_1, \dots, f_r is a regular sequence on M . For this, let $\mathfrak{p} \in \text{Ass}(\overline{M}_r)$. Since f_{r+1} is a non-zero divisor for \overline{M}_r , there exists $\mathfrak{q} \in \text{Ass}(\overline{M}_{r+1})$ such that $\mathfrak{p} \subseteq \mathfrak{q}$. Note that $\mathfrak{q} \neq \mathfrak{m}_R$ and that f_1, \dots, f_r is a regular sequence on $M_{\mathfrak{q}}$ by (\dagger) and hence in particular for $M_{\mathfrak{p}}$. This proves that $\text{depth}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \geq r$ for every $\mathfrak{p} \in \text{Ass}(\overline{M}_r)$ and hence f_1, \dots, f_r is a regular sequence on M by $(*)_r$. ■

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