

## Cobordism independence of Grassmann manifolds

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**Abstract.** This note proves that, for  $F = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , the bordism classes of all non-bounding Grassmannian manifolds  $G_k(F^{n+k})$ , with  $k < n$  and having real dimension  $d$ , constitute a linearly independent set in the unoriented bordism group  $\mathfrak{N}_d$  regarded as a  $\mathbb{Z}_2$ -vector space.

**Keywords.** Grassmannians; bordism; Stiefel–Whitney class.

### 1. Introduction

This paper is a continuation of the ongoing study of cobordism of Grassmann manifolds. Let  $F$  denote one of the division rings  $\mathbb{R}$  of reals,  $\mathbb{C}$  of complex numbers, or  $\mathbb{H}$  of quaternions. Let  $t = \dim_{\mathbb{R}} F$ . Then the Grassmannian manifold  $G_k(F^{n+k})$  is defined to be the set of all  $k$ -dimensional (left) subspaces of  $F^{n+k}$ .  $G_k(F^{n+k})$  is a closed manifold of real dimension  $nk$ . Using the orthogonal complement of a subspace one identifies  $G_k(F^{n+k})$  with  $G_n(F^{n+k})$ .

In [8], Sankaran has proved that, for  $F = \mathbb{R}, \mathbb{C}$  or  $\mathbb{H}$ , the Grassmannian manifold  $G_k(F^{n+k})$  bounds if and only if  $\nu(n+k) > \nu(k)$ , where, given a positive integer  $m$ ,  $\nu(m)$  denotes the largest integer such that  $2^{\nu(m)}$  divides  $m$ .

Given a positive integer  $d$ , let  $\mathcal{G}(d)$  denote the set of bordism classes of all non-bounding Grassmannian manifolds  $G_k(F^{n+k})$  having real dimension  $d$  such that  $k < n$ . The restriction  $k < n$  is imposed because  $G_k(F^{n+k}) \approx G_n(F^{n+k})$  and, for  $k = n$ ,  $G_k(F^{n+k})$  bounds. Thus,  $\mathcal{G}(d) = \{[G_k(F^{n+k})] \in \mathfrak{N}_* \mid nkt = d, k < n, \text{ and } \nu(n+k) \leq \nu(k)\} \subset \mathfrak{N}_d$ .

The purpose of this paper is to prove the following:

**Theorem 1.1.**  $\mathcal{G}(d)$  is a linearly independent set in the  $\mathbb{Z}_2$ -vector space  $\mathfrak{N}_d$ .

Similar results for Dold and Milnor manifolds can be found in [6] and [1] respectively.

### 2. The real Grassmannians — a Brief review

The real Grassmannian manifold  $G_k(\mathbb{R}^{n+k})$  is an  $nk$ -dimensional closed manifold of  $k$ -planes in  $\mathbb{R}^{n+k}$ . It is well-known (see [3]) that the mod-2 cohomology of  $G_k(\mathbb{R}^{n+k})$  is given by

$$H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, w_2, \dots, w_k, \bar{w}_1, \bar{w}_2, \dots, \bar{w}_n] / \{w \cdot \bar{w} = 1\},$$

where  $w = 1 + w_1 + w_2 + \cdots + w_k$  and  $\bar{w} = 1 + \bar{w}_1 + \bar{w}_2 + \cdots + \bar{w}_n$  are the total Stiefel–Whitney classes of the universal  $k$ -plane bundle  $\gamma_k$  and the corresponding complementary bundle  $\gamma_k^\perp$ , both over  $G_k(\mathbb{R}^{n+k})$ , respectively.

For computational convenience in this cohomology one uses the flag manifold  $\text{Flag}(\mathbb{R}^{n+k})$  consisting of all ordered  $(n+k)$ -tuples  $(V_1, V_2, \dots, V_{n+k})$  of mutually orthogonal one-dimensional subspaces of  $\mathbb{R}^{n+k}$  with respect to the ‘standard’ inner product on  $\mathbb{R}^{n+k}$ . It is standard (see [4]) that the mod-2 cohomology of  $\text{Flag}(\mathbb{R}^{n+k})$  is given by

$$H^*(\text{Flag}(\mathbb{R}^{n+k}); \mathbb{Z}_2) \cong \mathbb{Z}_2[e_1, e_2, \dots, e_{n+k}] / \left\{ \prod_{i=1}^{n+k} (1 + e_i) = 1 \right\},$$

where  $e_1, e_2, \dots, e_{n+k}$  are one-dimensional classes. In fact each  $e_i$  is the first Stiefel–Whitney class of the line bundle  $\lambda_i$  over  $\text{Flag}(\mathbb{R}^{n+k})$  whose total space consists of pairs, a flag  $(V_1, V_2, \dots, V_{n+k})$  and a vector in  $V_i$ .

There is a map  $\pi_{n+k} : \text{Flag}(\mathbb{R}^{n+k}) \rightarrow G_k(\mathbb{R}^{n+k})$  which assigns to  $(V_1, V_2, \dots, V_{n+k})$ , the  $k$ -dimensional subspace  $V_1 \oplus V_2 \oplus \cdots \oplus V_k$ . In the cohomology,  $\pi_{n+k}^* : H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2) \rightarrow H^*(\text{Flag}(\mathbb{R}^{n+k}); \mathbb{Z}_2)$  is injective and is described by

$$\pi_{n+k}^*(w) = \prod_{i=1}^k (1 + e_i), \quad \pi_{n+k}^*(\bar{w}) = \prod_{i=k+1}^{n+k} (1 + e_i).$$

In [9], Stong has observed, among others, the following facts:

*Fact 2.1.* The value of the class  $u \in H^*(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2)$  on the fundamental class of  $G_k(\mathbb{R}^{n+k})$  is the same as the value of

$$\pi_{n+k}^*(u) e_1^{k-1} e_2^{k-2} \cdots e_{k-1} e_{k+1}^{n-1} e_{k+2}^{n-2} \cdots e_{n+k-1}$$

on the fundamental class of  $\text{Flag}(\mathbb{R}^{n+k})$ .

*Fact 2.2.* In  $H^*(\text{Flag}(\mathbb{R}^{n+k}); \mathbb{Z}_2)$  one has

$$e_{i_1}^{n+k-(r-1)} e_{i_2}^{n+k-(r-1)} \cdots e_{i_{r-1}}^{n+k-2} e_{i_r}^{n+k-1} = 0$$

if  $1 \leq r \leq n+k$  and the set  $\{i_1, i_2, \dots, i_r\} \subset \{1, 2, \dots, n+k\}$ . In particular  $e_i^{n+k} = 0$  for each  $i$ ,  $1 \leq i \leq n+k$ .

*Fact 2.3.* In the top dimensional cohomology of  $\text{Flag}(\mathbb{R}^{n+k})$ , a monomial  $e_{i_1}^{i_1} e_{i_2}^{i_2} \cdots e_{i_{n+k}}^{i_{n+k}}$  represents the non-zero class if and only if the set  $\{i_1, i_2, \dots, i_{n+k}\} = \{0, 1, \dots, n+k-1\}$ .

The tangent bundle  $\tau$  over  $G_k(\mathbb{R}^{n+k})$  is given (see [5]) by

$$\tau \oplus \gamma_k \otimes \gamma_k \cong (n+k)\gamma_k.$$

In particular, the total Stiefel–Whitney class  $W(G_k(\mathbb{R}^{n+k}))$  of the tangent bundle over  $G_k(\mathbb{R}^{n+k})$  maps under  $\pi_{n+k}^*$  to

$$\prod_{1 \leq i \leq k} (1 + e_i)^{n+k} \cdot \prod_{1 \leq i < j \leq k} (1 + e_i + e_j)^{-2}.$$

Choosing a positive integer  $\alpha$  such that  $2^\alpha \geq n + k$ , we have, using Fact 2.2,

$$\pi_{n+k}^*(W(G_k(\mathbb{R}^{n+k}))) = \prod_{1 \leq i \leq k} (1 + e_i)^{n+k} \cdot \prod_{1 \leq i < j \leq k} (1 + e_i + e_j)^{2^\alpha - 2}.$$

Thus, the  $m$ th Stiefel–Whitney class  $W_m = W_m(G_k(\mathbb{R}^{n+k}))$  maps under  $\pi_{n+k}^*$  to the  $m$ th elementary symmetric polynomial in  $e_i$ ,  $1 \leq i \leq k$ , each with multiplicity  $n+k$ , and  $e_i + e_j$ ,  $1 \leq i < j \leq k$ , each with multiplicity  $2^\alpha - 2$ . Therefore, if  $S_p(\sigma_1, \sigma_2, \dots, \sigma_p)$  denotes the expression of the power sum  $\sum_{m=1}^q y_m^p$  as a polynomial in elementary symmetric polynomials  $\sigma_m$ 's in  $q$  ‘unknowns’  $y_1, y_2, \dots, y_q$ ,  $q \geq p$ , we have (see [8])

$$S_p(\pi_{n+k}^*(W_1), \pi_{n+k}^*(W_2), \dots, \pi_{n+k}^*(W_p)) = \sum_{1 \leq i \leq k} (n+k)e_i^p.$$

Thus we have a polynomial

$$S_p(G_k(\mathbb{R}^{n+k})) = S_p(W_1, W_2, \dots, W_p) \in H^p(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2)$$

of Stiefel–Whitney classes of  $G_k(\mathbb{R}^{n+k})$  such that

$$\pi_{n+k}^*(S_p(G_k(\mathbb{R}^{n+k}))) = \begin{cases} \sum_{1 \leq i \leq k} e_i^p, & \text{if } n+k \text{ is odd and } p < n+k \\ 0, & \text{otherwise.} \end{cases} \quad (2.4)$$

### 3. Proof of Theorem 1.1

It is shown in [2] that

$$[G_{2k}(\mathbb{R}^{2n+2k})] = [G_k(\mathbb{R}^{n+k})]^4 \quad \text{in } \mathfrak{N}_{4nk}.$$

From this, we have, in particular,

$$[G_k(F^{n+k})] = [G_k(\mathbb{R}^{n+k})]^t \quad \text{in } \mathfrak{N}_{nkt}.$$

For this one has to simply observe that the mod-2 cohomology of the  $\mathbb{F}$ -Grassmannian is isomorphic as ring to that of the corresponding real Grassmannian by an obvious isomorphism that multiplies the degree by  $t$ . On the other hand, since  $\mathfrak{N}_*$  is a polynomial ring over the field  $\mathbb{Z}_2$ , we have the following:

*Remark 3.1.* A set  $\{[M_1], [M_2], \dots, [M_m]\}$  is linearly independent in  $\mathfrak{N}_d$  if and only if the set  $\{[M_1]^{2^\beta}, [M_2]^{2^\beta}, \dots, [M_m]^{2^\beta}\}$  is linearly independent in  $\mathfrak{N}_{d \cdot 2^\beta}$ ,  $\beta \geq 0$ .

Therefore, noting that  $t = 1, 2, \text{ or } 4$ , it is enough to prove Theorem 1.1 for real Grassmannians only. Thus, from now onwards, we shall take

$$\mathcal{G}(d) = \{[G_k(\mathbb{R}^{n+k})] \mid nk = d, k < n, \text{ and } v(n+k) \leq v(k)\}.$$

If  $G_k(\mathbb{R}^{n+k})$  is an odd-dimensional real Grassmannian manifold then both  $n$  and  $k$  must be odd, and so  $v(n+k) > v(k)$ . This means that  $G_k(\mathbb{R}^{n+k})$  bounds and so it follows that  $\mathcal{G}(d) = \emptyset$  if  $d$  is odd. Therefore we assume that  $d$  is even.

*Lemma 3.2.* In  $H^*(\text{Flag}(\mathbb{R}^{n+k}); \mathbb{Z}_2)$  one has, for  $1 \leq j \leq k$ ,

$$\begin{aligned} & \left( \sum_{1 \leq i \leq k} e_i^{n+k-(2j-1)} \right) \cdot e_1^{k-1} e_2^{k-2} \cdots e_{k-j}^j \cdot e_{k-(j-1)}^{j-1} \cdot e_{k-(j-2)}^{n+k-(j-1)} \cdots e_k^{n+k-1} \\ &= e_1^{k-1} e_2^{k-2} \cdots e_{k-j}^j \cdot e_{k-(j-1)}^{n+k-j} \cdot e_{k-(j-2)}^{n+k-(j-1)} \cdots e_k^{n+k-1}. \end{aligned}$$

*Proof.* Note that

(a) if  $i \neq k - (j - 1)$  then the exponent of  $e_i$  in the product

$$e_1^{k-1} e_2^{k-2} \cdots e_{k-j}^j \cdot e_{k-(j-1)}^{j-1} \cdot e_{k-(j-2)}^{n+k-(j-1)} \cdots e_k^{n+k-1}$$

is greater than or equal to  $j$ , and

(b)  $\{n + k - (2j - 1)\} + j = n + k - (j - 1)$ .

Therefore, invoking Fact 2.2, the lemma follows.  $\square$

### PROPOSITION 3.3

Let  $\mathcal{O}(d) = \{[G_k(\mathbb{R}^{n+k})] \in \mathcal{G}(d) \mid n + k \text{ is odd}\}$ . Then  $\mathcal{O}(d)$  is linearly independent in  $\mathfrak{N}_d$ .

*Proof.* Arrange the members of  $\mathcal{O}(d)$  in descending order of the values of  $n + k$ , so that

$$\mathcal{O}(d) = \{[G_{k_1}(\mathbb{R}^{n_1+k_1})], [G_{k_2}(\mathbb{R}^{n_2+k_2})], \dots, [G_{k_s}(\mathbb{R}^{n_s+k_s})]\},$$

where  $n_1 + k_1 > n_2 + k_2 > \cdots > n_s + k_s$ . Note that  $n_1 = d$  and  $k_1 = 1$ .

For a  $d$ -dimensional Grassmannian manifold  $G_k(\mathbb{R}^{n+k})$ , consider the polynomials

$$f_\ell(G_k(\mathbb{R}^{n+k})) = \prod_{1 \leq j \leq k_\ell} S_{n_\ell+k_\ell-(2j-1)}(G_k(\mathbb{R}^{n+k})) \in H^d(G_k(\mathbb{R}^{n+k}); \mathbb{Z}_2)$$

of Stiefel–Whitney classes of  $G_k(\mathbb{R}^{n+k})$ , where  $1 \leq \ell \leq s$ . Then, for each  $\ell$ ,  $1 \leq \ell \leq s$ , we have, using (2.4),

$$\begin{aligned} & \pi_{n_\ell+k_\ell}^* (f_\ell(G_{k_\ell}(\mathbb{R}^{n_\ell+k_\ell}))) e_1^{k_\ell-1} e_2^{k_\ell-2} \cdots e_{k_\ell-1} e_{k_\ell+1}^{n_\ell-1} e_{k_\ell+2}^{n_\ell-2} \cdots e_{n_\ell+k_\ell-1} \\ &= \left( \prod_{1 \leq j \leq k_\ell} \left( \sum_{1 \leq i \leq k_\ell} e_i^{n_\ell+k_\ell-(2j-1)} \right) \right) e_1^{k_\ell-1} e_2^{k_\ell-2} \cdots e_{k_\ell-1} e_{k_\ell+1}^{n_\ell-1} e_{k_\ell+2}^{n_\ell-2} \\ & \quad \cdots e_{n_\ell+k_\ell-1} \\ &= e_1^{n_\ell} e_2^{n_\ell+1} \cdots e_{k_\ell}^{n_\ell+k_\ell-1} e_{k_\ell+1}^{n_\ell-1} e_{k_\ell+2}^{n_\ell-2} \cdots e_{n_\ell+k_\ell-1}, \end{aligned}$$

applying Lemma 3.2 repeatedly for successive values of  $j$ .

Thus, in view of Facts 2.1 and 2.3, the Stiefel–Whitney number

$$\langle f_\ell(G_{k_\ell}(\mathbb{R}^{n_\ell+k_\ell})), [G_{k_\ell}(\mathbb{R}^{n_\ell+k_\ell})] \rangle \neq 0$$

for each  $\ell$ ,  $1 \leq \ell \leq s$ . On the other hand, using (2.4), it is clear that

$$\langle f_\ell(G_{k_h}(\mathbb{R}^{n_h+k_h})), [G_{k_h}(\mathbb{R}^{n_h+k_h})] \rangle = 0$$

for each  $h > \ell$ , since  $n_\ell + k_\ell - 1 \geq n_h + k_h$ . Therefore, it follows that the  $s \times s$  matrix

$$[\langle f_\ell(G_{k_h}(\mathbb{R}^{n_h+k_h})), [G_{k_h}(\mathbb{R}^{n_h+k_h})] \rangle]_{1 \leq \ell \leq s, 1 \leq h \leq s}$$

is non-singular; being lower triangular with 1's in the diagonal. This completes the proof.  $\square$

Now we shall complete the proof of Theorem 1.1 using induction on  $d$ . First note that

$$\mathcal{G}(2) = \{[G_1(\mathbb{R}^{2+1})]\} = \{[\mathbb{R}P^2]\},$$

$$\mathcal{G}(4) = \{[G_1(\mathbb{R}^{4+1})]\} = \{[\mathbb{R}P^4]\},$$

and so both are linearly independent in  $\mathfrak{N}_2, \mathfrak{N}_4$  respectively. Assume that the theorem holds for all dimensions less than  $d$ .

We have  $\mathcal{G}(d) = \mathcal{E}(d) \cup \mathcal{O}(d)$ , where

$$\mathcal{E}(d) = \{[G_k(\mathbb{R}^{n+k})] \in \mathcal{G}(d) | n+k \text{ is even}\}$$

and

$$\mathcal{O}(d) = \{[G_k(\mathbb{R}^{n+k})] \in \mathcal{G}(d) | n+k \text{ is odd}\}.$$

Observe that if  $[G_k(\mathbb{R}^{n+k})] \in \mathcal{E}(d)$  then both  $n$  and  $k$  are even with  $v(k) \neq v(n)$ . On the other hand,  $[G_2(\mathbb{R}^{\frac{d}{2}+2})] \in \mathcal{E}(d)$  if  $d \equiv 0 \pmod{8}$ . Thus,  $\mathcal{E}(d) \neq \emptyset$  if and only if  $d \equiv 0 \pmod{8}$ .

In view of Proposition 3.3, we may assume without any loss that  $\mathcal{E}(d) \neq \emptyset$ . Then, by the above observation and by Theorem 2.2 of [8] every member of  $\mathcal{E}(d)$  is of the form  $[G_{\frac{k}{2}}(\mathbb{R}^{\frac{n}{2}+\frac{k}{2}})]^4$ , where  $[G_{\frac{k}{2}}(\mathbb{R}^{\frac{n}{2}+\frac{k}{2}})] \in \mathcal{G}(\frac{d}{4})$ . By induction hypothesis,  $\mathcal{G}(\frac{d}{4})$  is linearly independent in  $\mathfrak{N}_{\frac{d}{4}}$ .

So, by Remark 3.1,

$$\mathcal{E}(d) \text{ is linearly independent in } \mathfrak{N}_d. \quad (3.4)$$

Again note that if  $[G_k(\mathbb{R}^{n+k})] \in \mathcal{E}(d)$ , then, by (2.4), the polynomial  $S_p(G_k(\mathbb{R}^{n+k})) = 0$ ,  $\forall p \geq 1$ . So, for each of the polynomials  $f_\ell$ ,  $1 \leq \ell \leq s$ , considered in Proposition 3.3, we have

$$\langle f_\ell(G_k(\mathbb{R}^{n+k})), [G_k(\mathbb{R}^{n+k})] \rangle = 0.$$

Therefore, writing

$$\mathcal{E}(d) = \{[G_{k_{s+1}}(\mathbb{R}^{n_{s+1}+k_{s+1}})], [G_{k_{s+2}}(\mathbb{R}^{n_{s+2}+k_{s+2}})], \dots, [G_{k_{s+q}}(\mathbb{R}^{n_{s+q}+k_{s+q}})]\},$$

where  $n_{s+1} + k_{s+1} > n_{s+2} + k_{s+2} > \dots > n_{s+q} + k_{s+q}$ , we see that the  $s \times (s+q)$  matrix

$$[\langle f_\ell(G_{k_h}(\mathbb{R}^{n_h+k_h})), [G_{k_h}(\mathbb{R}^{n_h+k_h})] \rangle]_{1 \leq \ell \leq s, 1 \leq h \leq s+q}$$

is of the form

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \star & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \star & \star & 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ - & - & - & \cdots & - & - & - & - & \cdots & - \\ - & - & - & \cdots & - & - & - & - & \cdots & - \\ \star & \star & \star & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}. \quad (3.5)$$

$\mathcal{O}(d)$ 
 $\mathcal{E}(d)$

Thus, no non-trivial linear combination of members of  $\mathcal{O}(d)$  can be expressed as a linear combination of the members of  $\mathcal{E}(d)$ . This, together with (3.4) and Proposition 3.3, proves that the set  $\mathcal{G}(d) = \mathcal{E}(d) \cup \mathcal{O}(d)$  is linearly independent in  $\mathfrak{N}_d$ . Hence, by induction, Theorem 1.1 is completely proved.

*Remark 3.6.* Using the decomposition of the members of  $\mathcal{E}(d)$ , and the polynomials  $f_\ell$ , in the lower dimensions together with the *doubling homomorphism* defined by Milnor [7], one can obtain a set of polynomials of Stiefel–Whitney classes which yield, as in Proposition 3.3, a lower triangular matrix for  $\mathcal{E}(d)$  with 1's in the diagonal. Thus using (3.5) we have a lower triangular matrix, with 1's in the diagonal, for the whole set  $\mathcal{G}(d)$ .

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