Proc. Indian Acad. Sci. (Math. Sci.) Vol. 113, No. 4, November 2003, pp. 451–456.


# On Howard's conjecture in heterogeneous shear flow problem

R G SHANDIL and JAGJIT SINGH*

Department of Mathematics, H.P. University, Shimla 171 005, India
*Sidharth Govt. Degree College, Nadaun, Dist. Hamirpur 177 033, India

**Abstract.** Howard's conjecture, which states that in the linear instability problem of inviscid heterogeneous parallel shear flow growth rate of an arbitrary unstable wave must approach zero as the wave length decreases to zero, is established in a mathematically rigorous fashion for plane parallel heterogeneous shear flows with negligible buoyancy force $g\beta \ll 1$ (Miles J W, *J. Fluid Mech.* **10** (1961) 496–508), where $\beta$ is the basic heterogeneity distribution function).

**Keywords.** Heterogeneous shear flows; linear stability.

## 1. Introduction

In the stability problem of inviscid heterogeneous parallel shear flows, Howard [4], making use of a novel transformation of the dependent variable was not only able to prove the validity of Taylor's conjecture without the restrictive conditions under which Miles [5] got the result but he was also able to obtain a semicircular region in the $c_r c_i$-plane in which the complex velocity of an unstable wave must lie. Howard also made a conjectural assertion to the effect that the growth rate of an arbitrary unstable mode must approach zero as the wave number approaches infinity. This conjecture of Howard has also drawn the attention of researchers. Banerjee *et al* [1] were able to validate the correctness of this conjecture for the special case of inviscid homogeneous parallel shear flows. Their approach consisted of combining the governing equations and boundary conditions in an innovative way and thereby deriving an upper bound of the growth rate under consideration. Banerjee *et al* [2] attempted Howard's conjecture in heterogeneous parallel shear flows but succeeded only in proving it in the case of the Garcia-type [3] flows wherein the basic velocity distribution has a point of inflexion in the domain of the flow while the vertical velocity gradient of the basic density distribution vanishes at this point.

In this paper we prove, in a mathematically rigorous fashion, that the growth rate of an arbitrary unstable wave must approach zero as the wave number approaches infinity in the linear instability of non-viscous heterogeneous parallel shear flows with negligible buoyancy force ($g\beta \ll 1$, [5]) so that second- and higher-order terms in $g\beta$, where $\beta$ is the basic heterogeneity distribution function, are negligible as compared to the first-order terms in $g\beta$.

## 2. Mathematical formulation of the problem

The basic equations governing the linear instability in a Boussinesq inviscid parallel shear flow which is confined between two rigid horizontal boundaries is given by

$$(D^2-\alpha^2)w-\left(\frac{U''}{U-c}\right)w+\left(\frac{g\beta}{(U-c)^2}\right)w=0, \tag{2.1}$$

where $z \in [z_1, z_2]$ is the real independent variable and stands for the vertical coordinate, $D \equiv \mathrm{d}/\mathrm{d}z$, $U(z)$ is a twice continuously differentiable function of $z$ and stands for the basic velocity distribution, $\beta(z)$ is a non-negative continuous function of $z$ and stands for the basic heterogeneity distribution, $w(z)$ is the dependent variable and stands for the $z$-component of the perturbation velocity, $c = c_r + ic_i$ is a complex constant in general and stands for the complex wave velocity of the perturbation wave with $c_r$ as the phase velocity and $c_i$ as the amplification factor, and $\alpha^2$ is a positive constant which satisfies $0 < \alpha^2 < \infty$ and stands for the square of the wave number of the perturbation wave. The boundary conditions associated with the problem are that $w(z)$ vanishes on the rigid horizontal boundaries at $z = z_1$ and $z = z_2$ i.e.,

$$w(z_1) = w(z_2) = 0. \tag{2.2}$$

(The boundaries in the limiting case may recede to $\pm\infty$.)

For the existence of a non-trivial solution of eqs (2.1) and (2.2) we have a double eigenvalue problem for the determination of $c_r$ and $c_i$ for the prescribed values of $\alpha^2$ and the flow is unstable if such solutions exist for which the imaginary part $c_i$ of $c$ is greater than zero.

## 3. Mathematical analysis

Firstly, we prove the following two lemmas:

*Lemma* 1. *A necessary condition for the existence of a non-trivial solution* $(w, c, \alpha^2)$ *with* $c_i > 0$ *of eqs* (2.1) *and* (2.2) *is that the integral relations*

$$\int_{z_1}^{z_2}(|Dw|^2+\alpha^2|w|^2)\mathrm{d}z+\int_{z_1}^{z_2}\frac{U''(U-c_r)}{(U-c_r)^2+c_i^2}|w|^2\mathrm{d}z \\ -\int_{z_1}^{z_2}\frac{g\beta\{(U-c_r)^2-c_i^2\}}{\{(U-c_r)^2+c_i^2\}^2}|w|^2=0 \tag{3.1}$$

*and*

$$\int_{z_1}^{z_2}\frac{U''}{(U-c_r)^2+c_i^2}|w|^2\mathrm{d}z-2\int_{z_1}^{z_2}\frac{g\beta(U-c_r)}{\{(U-c_r)^2+c_i^2\}^2}|w|^2=0, \tag{3.2}$$

*are true.*

*Proof.* We multiply eq. (2.1) by $w^*$ (the complex conjugate of $w$) throughout and integrate the resulting equation over the domain of $z$, to get

$$\int_{z_1}^{z_2}w^*(D^2-\alpha^2)w\mathrm{d}z-\int_{z_1}^{z_2}w^*\left(\frac{U''}{U-c}\right)w\mathrm{d}z \\ +\int_{z_1}^{z_2}w^*\left(\frac{g\beta}{(U-c)^2}\right)w\mathrm{d}z=0. \tag{3.3}$$

In order to calculate the first term of the first integral on the left hand side of eq. (3.3), we integrate it by parts once and use the boundary condition (2.2), to derive

$$\int_{z_1}^{z_2} (|Dw|^2 + \alpha^2|w|^2)\mathrm{d}z + \int_{z_1}^{z_2} \left(\frac{U''}{U-c} - \frac{g\beta}{(U-c)^2}\right)|w|^2\mathrm{d}z = 0. \quad (3.4)$$

Now equating real and imaginary parts of the two sides of eq. (3.4) and cancelling $c_i\,(> 0)$ throughout from the imaginary part, we get

$$\int_{z_1}^{z_2} (|Dw|^2 + \alpha^2|w|^2)\mathrm{d}z + \int_{z_1}^{z_2} \frac{U''(U-c_r)}{(U-c_r)^2 + c_i^2}|w|^2\mathrm{d}z$$
$$- \int_{z_1}^{z_2} \frac{g\beta\{(U-c_r)^2 - c_i^2\}}{\{(U-c_r)^2 + c_i^2\}^2}|w|^2 = 0 \quad (3.5)$$

and

$$\int_{z_1}^{z_2} \frac{U''}{(U-c_r)^2 + c_i^2}|w|^2\mathrm{d}z - 2\int_{z_1}^{z_2} \frac{g\beta(U-c_r)}{\{(U-c_r)^2 + c_i^2\}^2}|w|^2 = 0, \quad (3.6)$$

and this proves the lemma.

*Lemma* 2. *A necessary condition for the existence of a non-trivial solution* $(w, c, \alpha^2)$ *with* $c_i > 0$ *of eqs* (2.1) *and* (2.2) *is that the integral relation*

$$\int_{z_1}^{z_2} (|D^2w|^2 + \alpha^2|Dw|^2)\mathrm{d}z - \alpha^2\int_{z_1}^{z_2} \frac{U''(U-c_r)}{(U-c_r)^2 + c_i^2}|w|^2\mathrm{d}z$$
$$+ \alpha^2\int_{z_1}^{z_2} \frac{g\beta\{(U-c_r)^2 - c_i^2\}}{\{(U-c_r)^2 + c_i^2\}^2}|w|^2\mathrm{d}z - \int_{z_1}^{z_2} \frac{(U'')^2}{(U-c_r)^2 + c_i^2}|w|^2\mathrm{d}z$$
$$+ 2\int_{z_1}^{z_2} \frac{g\beta(U-c_r)}{\{(U-c_r)^2 + c_i^2\}^2}|w|^2\mathrm{d}z - \int_{z_1}^{z_2} \frac{g^2\beta^2}{\{(U-c_r)^2 + c_i^2\}^2}|w|^2\mathrm{d}z = 0$$
$$(3.7)$$

*and*

$$\int_{z_1}^{z_2} (|D^2w|^2 + 2\alpha^2|Dw|^2 + \alpha^4|w|^2)\mathrm{d}z - \int_{z_1}^{z_2} \frac{(U'')^2}{\{(U-c_r)^2 + c_i^2\}}|w|^2\mathrm{d}z$$
$$+ 2\int_{z_1}^{z_2} \frac{g\beta U''((U-c_r))}{\{(U-c_r)^2 + c_i^2\}^2}|w|^2\mathrm{d}z - \int_{z_1}^{z_2} \frac{g^2\beta^2}{\{(U-c_r)^2 + c_i^2\}^2}|w|^2\mathrm{d}z = 0$$
$$(3.8)$$

*are true.*

*Proof.* We multiply eq. (2.1) by $D^2w^*$ (the complex conjugate of $D^2w$) throughout and integrate the resulting equation over the domain of $z$, to get

$$\int_{z_1}^{z_2} D^2w^*(D^2 - \alpha^2)w\mathrm{d}z - \int_{z_1}^{z_2} D^2w^*\left(\frac{U''}{U-c}\right)w\mathrm{d}z$$
$$+ \int_{z_1}^{z_2} D^2w^*\left(\frac{g\beta}{(U-c)^2}\right)w\mathrm{d}z = 0. \quad (3.9)$$

In order to calculate the first term of the first integral on the left hand side of eq. (3.9), we integrate it by parts once and use the boundary condition (2.2), to derive

$$\int_{z_1}^{z_2}(|D^2w|^2+\alpha^2|Dw|^2)\mathrm{d}z-\int_{z_1}^{z_2}D^2w^*\left(\frac{U''}{U-c}-\frac{g\beta}{(U-c)^2}\right)w\mathrm{d}z=0. \tag{3.10}$$

Now from eq. (2.1), we have

$$D^2w=\alpha^2w+\left(\frac{U''}{U-c}\right)w-\left(\frac{g\beta}{(U-c)^2}\right)w. \tag{3.11}$$

Taking the complex conjugate of both sides of eq. (3.11), we derive

$$D^2w^*=\alpha^2w^*+\left(\frac{U''}{U-c^*}\right)w^*-\left(\frac{g\beta}{(U-c^*)^2}\right)w^*. \tag{3.12}$$

Substituting the value of $D^2w^*$ from eq. (3.12) into eq. (3.10), we derive

$$\int_{z_1}^{z_2}(|D^2w|^2+\alpha^2|Dw|^2)\mathrm{d}z-\int_{z_1}^{z_2}\left(\alpha^2+\frac{U''}{(U-c^*)}-\frac{g\beta}{(U-c^*)^2}\right) \times w^*\left(\frac{U''}{U-c}-\frac{g\beta}{(U-c)^2}\right)w\mathrm{d}z=0. \tag{3.13}$$

On simplification, eq. (3.13) can be written in the form

$$\begin{aligned}
&\int_{z_1}^{z_2}(|D^2w|^2+\alpha^2|Dw|^2)\mathrm{d}z-\alpha^2\int_{z_1}^{z_2}\frac{U''(U-c_r+ic_i)}{(U-c_r)^2+c_i^2}|w|^2\mathrm{d}z\\
&\quad-\int_{z_1}^{z_2}\frac{(U'')^2}{\{(U-c_r)^2+c_i^2\}}|w|^2\mathrm{d}z\\
&\quad+\alpha^2\int_{z_1}^{z^2}\frac{g\beta\{(U-c_r)^2-c_i^2+2ic_i(U-c_r)\}}{\{(U-c_r)^2+c_i^2\}^2}|w|^2\mathrm{d}z\\
&\quad+\int_{z_1}^{z_2}\frac{g\beta U''((U-c_r)-ic_i)}{\{(U-c_r)^2+c_i^2\}^2}|w|^2\mathrm{d}z\\
&\quad+\int_{z_1}^{z_2}\frac{g\beta U''\{(U-c_r)+ic_i\}}{\{(U-c_r)^2+c_i^2\}^2}|w|^2\mathrm{d}z\\
&\quad+\int_{z_1}^{z_2}\frac{g^2\beta^2}{\{(U-c_r)^2+c_i^2\}^2}|w|^2\mathrm{d}z=0.
\end{aligned} \tag{3.14}$$

Equating the real parts on both sides of eq. (3.14) we obtain

$$\int_{z_1}^{z_2}(|D^2w|^2+\alpha^2|Dw|^2)\mathrm{d}z-\alpha^2\int_{z_1}^{z_2}\frac{U''(U-c_r)}{(U-c_r)^2+c_i^2}|w|^2\mathrm{d}z$$
$$-\int_{z_1}^{z_2}\frac{(U'')^2}{\{(U-c_r)^2+c_i^2\}}|w|^2\mathrm{d}z+\alpha^2\int_{z_1}^{z_2}\frac{g\beta\{(U-c_r)^2-c_i^2\}}{\{(U-c_r)^2+c_i^2\}^2}|w|^2\mathrm{d}z$$
$$+2\int_{z_1}^{z_2}\frac{g\beta U''((U-c_r))}{\{(U-c_r)^2+c_i^2\}^2}|w|^2\mathrm{d}z-\int_{z_1}^{z_2}\frac{g^2\beta^2}{\{(U-c_r)^2+c_i^2\}^2}|w|^2\mathrm{d}z=0. \tag{3.15}$$

Multiplying eq. (3.1) by $\alpha^2$ and adding to eq. (3.15) we get

$$\int_{z_1}^{z_2}(|D^2w|^2+2\alpha^2|Dw|^2+\alpha^4|w|^2)\mathrm{d}z-\int_{z_1}^{z_2}\frac{(U'')^2}{\{(U-c_r)^2+c_i^2\}}|w|^2\mathrm{d}z$$
$$+2\int_{z_1}^{z_2}\frac{g\beta U''((U-c_r))}{\{(U-c_r)^2+c_i^2\}^2}|w|^2\mathrm{d}z-\int_{z_1}^{z_2}\frac{g^2\beta^2}{\{(U-c_r)^2+c_i^2\}^2}|w|^2\mathrm{d}z=0, \tag{3.16}$$

which completes the proof of the lemma.

**Theorem 5.1.** *If* $(w, c, \alpha^2)$ *is a non-trivial solution of eqs* (2.1) *and* (2.2) *with* $c_i > 0$ *and heterogeneity factor is small so that second- and higher-order terms in* $g\beta$, *are negligible as compared to the first-order terms in* $g\beta$ *then* $\alpha c_i \to 0$ *as* $\alpha \to \infty$.

*Proof.* Since heterogeneity factor is small so that second- and higher-order terms in $g\beta$, where $\beta$ is the basic heterogeneity distribution function, are negligible as compared to the first-order terms in $g\beta$, therefore eq. (3.8) reduces to

$$\int_{z_1}^{z_2}(|D^2w|^2+2\alpha^2|Dw|^2+\alpha^4|w|^2)\mathrm{d}z-\int_{z_1}^{z_2}\frac{(U'')^2}{\{(U-c_r)^2+c_i^2\}}|w|^2\mathrm{d}z$$
$$+2\int_{z_1}^{z_2}\frac{g\beta U''((U-c_r))}{\{(U-c_r)^2+c_i^2\}^2}|w|^2\mathrm{d}z=0. \tag{3.17}$$

Since $c_i > 0$ and $1/\{(U-c_r)^2+c_i^2\} \le 1/c_i^2$, therefore, from eq. (3.17) we derive that

$$\int_{z_1}^{z_2}(|D^2w|^2+2\alpha^2|Dw|^2+\alpha^4|w|^2)\mathrm{d}z-\int_{z_1}^{z_2}\frac{[(U'')^2]_{\max}}{c_i^2}|w|^2\mathrm{d}z$$
$$+2\int_{z_1}^{z_2}\frac{g\beta U''(U-c_r)c_i}{\{(U-c_r)^2+c_i^2\}^2c_i}|w|^2\mathrm{d}z\le 0. \tag{3.18}$$

where $[(U'')^2]_{\max}$ stands for the maximum value of the bracketed expression over the interval $[z_1, z_2]$. Further, since $(U-c_r)^2+c_i^2 \ge 2(U-c_r)c_i$, therefore from inequality (3.18) we derive that

$$\int_{z_1}^{z_2}(|D^2w|^2+2\alpha^2|Dw|^2+\alpha^4|w|^2)\mathrm{d}z-\int_{z_1}^{z_2}\frac{[(U'')^2]_{\max}}{c_i^2}|w|^2\mathrm{d}z$$
$$-\int_{z_1}^{z_2}\frac{g\beta|U''|((U-c_r)^2+c_i^2)}{\{(U-c_r)^2+c_i^2\}^2c_i}|w|^2\mathrm{d}z\leq 0, \tag{3.19}$$

or

$$\int_{z_1}^{z_2}(|D^2w|^2+2\alpha^2|Dw|^2)\mathrm{d}z$$
$$+\int_{z_1}^{z_2}\left(\alpha^4-\frac{[(U'')^2]_{\max}c_i}{c_i^3}-\frac{[g\beta|U''|]_{\max}}{c_i^3}\right)|w|^2\mathrm{d}z\leq 0. \tag{3.20}$$

Now, on account of the non-negativity of first integral in (3.20) we get

$$\left(\alpha^4-\frac{[(U'')^2]_{\max}c_i}{c_i^3}-\frac{[g\beta|U''|]_{\max}}{c_i^3}\right)\leq 0 \tag{3.21}$$

or

$$\alpha^3c_i^3\leq\frac{[(U'')^2]_{\max}c_i}{\alpha}+\frac{[g\beta|U''|]_{\max}}{\alpha}. \tag{3.22}$$

Inequality (3.22) implies that $\alpha c_i \to 0$ as $\alpha \to \infty$. This completes the proof of the theorem.

Here, it is to be noted that for the case $\beta = 0$, from inequality (3.22), we get the result of Banerjee *et al* [1] for the special case of inviscid homogeneous parallel shear flows.

## References

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