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Vibrations of thin piezoelectric shallow shells: Two-dimensional approximation

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Abstract. In this paper we consider the eigenvalue problem for piezoelectric shallow shells and we show that, as the thickness of the shell goes to zero, the eigensolutions of the three-dimensional piezoelectric shells converge to the eigensolutions of a two-dimensional eigenvalue problem.

Keywords. Vibrations; piezoelectricity; shallow shells.

1. Introduction

Lower dimensional models of shells are preferred in numerical computations to three-dimensional models when the thickness of the shells is 'very small'. A lot of work has been done on the lower dimensional approximation of boundary value and eigenvalue problem for elastic plates and shells (cf. [2–6, 8, 9]). Recently some work has been done on the lower dimensional approximation of boundary value problem for piezoelectric shells (cf. [1]).

In this paper, we would like to study the limiting behaviour of the eigenvalue problems for thin piezoelectric shallow shells. We begin with a brief description of the problem and describe the results obtained.

Let $\hat{\Omega}^\epsilon = \Phi^\epsilon(\Omega^\epsilon)$, $\Omega^\epsilon = \omega \times (-\epsilon, \epsilon)$ with $\omega \subset \mathbb{R}^2$, and the mapping $\Phi^\epsilon : \overline{\Omega}^\epsilon \to \mathbb{R}^3$ is given by

$$\Phi^\epsilon(x^\epsilon) = (x_1, x_2, \epsilon\theta(x_1, x_2)) + x_3^\epsilon a_3^\epsilon(x_1, x_2)$$

for all $x^\epsilon = (x_1, x_2, x_3^\epsilon) \in \overline{\Omega}^\epsilon$, where θ is an injective mapping of class C^3 and a_3^ϵ is a unit normal vector to the middle surface $\Phi^\epsilon(\overline{\omega})$ of the shell. Let $\gamma_0, \gamma_e \subset \partial\omega$ with $\text{meas}(\gamma_0) > 0$ and $\text{meas}(\gamma_e) > 0$. Let $\hat{\Gamma}_0^\epsilon = \Phi^\epsilon(\gamma_0 \times (-\epsilon, \epsilon))$ and let $\hat{\Gamma}_e^\epsilon = \Phi^\epsilon(\gamma_e \times (-\epsilon, \epsilon))$. The shell is clamped along the portion $\hat{\Gamma}_0^\epsilon$ of the lateral surface.

Then the variational form of the eigenvalue problem consists of finding the displacement vector u^ϵ, the electric potential φ^ϵ and $\xi^\epsilon \in \mathbb{R}$ satisfying eq. (2.21). We then show that the component of the eigenvector involving the electric potential φ^ϵ can be uniquely determined in terms of the displacement vector u^ϵ and the problem thus reduces to finding $(u^\epsilon, \xi^\epsilon)$ satisfying equations (2.43) and (2.44).

After making appropriate scalings on the data and the unknowns, we transfer the problem to a domain $\Omega = \omega \times (-1, 1)$ which is independent of ϵ. Then we show that the scaled eigensolutions converge to the solutions of a two-dimensional eigenvalue problem (6.50).

2. The three-dimensional problem

Throughout this paper, Latin indices vary over the set $\{1, 2, 3\}$ and Greek indices over the set $\{1, 2\}$ for the components of vectors and tensors. The summation over repeated indices will be used.

Let $\omega \subset \mathbb{R}^2$ be a bounded domain with a Lipschitz continuous boundary γ and let ω lie locally on one side of γ. Let $\gamma_0, \gamma_e \subset \partial\omega$ with meas$(\gamma_0) > 0$ and meas$(\gamma_e) > 0$. Let $\gamma_1 = \partial\omega \backslash \gamma_0$ and $\gamma_s = \partial\omega \backslash \gamma_e$. For each $\epsilon > 0$, we define the sets

$$\Omega^\epsilon = \omega \times (-\epsilon, \epsilon), \quad \Gamma^{\pm,\epsilon} = \omega \times \{\pm\epsilon\}, \quad \Gamma_0^\epsilon = \gamma_0 \times (-\epsilon, \epsilon),$$
$$\Gamma_1^\epsilon = \gamma_1 \times (-\epsilon, \epsilon), \quad \Gamma_e^\epsilon = \gamma_e \times (-\epsilon, \epsilon), \quad \Gamma_s^\epsilon = \gamma_s \times (-\epsilon, \epsilon).$$

Let $x^\epsilon = (x_1, x_2, x_3^\epsilon)$ be a generic point on Ω^ϵ and let $\partial_\alpha = \partial_\alpha^\epsilon = \frac{\partial}{\partial x_\alpha}$ and $\partial_3^\epsilon = \frac{\partial}{\partial x_3^\epsilon}$.

We assume that for each ϵ, we are given a function $\theta^\epsilon : \omega \to \mathbb{R}$ of class C^3. We then define the map $\phi^\epsilon : \omega \to \mathbb{R}^3$ by

$$\phi^\epsilon(x_1, x_2) = (x_1, x_2, \theta^\epsilon(x_1, x_2)) \quad \text{for all } (x_1, x_2) \in \omega. \tag{2.1}$$

At each point of the surface $S^\epsilon = \phi^\epsilon(\omega)$, we define the normal vector

$$a^\epsilon = (|\partial_1\theta^\epsilon|^2 + |\partial_2\theta^\epsilon|^2 + 1)^{-1/2}(-\partial_1\theta^\epsilon, -\partial_2\theta^\epsilon, 1).$$

For each $\epsilon > 0$, we define the mapping $\Phi^\epsilon : \Omega^\epsilon \to \mathbb{R}^3$ by

$$\Phi^\epsilon(x^\epsilon) = \phi^\epsilon(x_1, x_2) + x_3^\epsilon a^\epsilon(x_1, x_2) \quad \text{for all } x^\epsilon \in \Omega^\epsilon. \tag{2.2}$$

It can be shown that there exists an $\epsilon_0 > 0$ such that the mappings $\Phi^\epsilon : \Omega^\epsilon \to \Phi^\epsilon(\Omega^\epsilon)$ are C^1 diffeomorphisms for all $0 < \epsilon \le \epsilon_0$. The set $\hat{\Omega}^\epsilon = \Phi^\epsilon(\Omega^\epsilon)$ is the reference configuration of the shell. For $0 < \epsilon \le \epsilon_0$, we define the sets

$$\hat{\Gamma}^{\pm,\epsilon} = \Phi^\epsilon(\Gamma^{\pm,\epsilon}), \quad \hat{\Gamma}_0^\epsilon = \Phi^\epsilon(\Gamma_0^\epsilon), \quad \hat{\Gamma}_1^\epsilon = \Phi(\Gamma_1^\epsilon), \quad \hat{\Gamma}_N^\epsilon = \hat{\Gamma}_i^\epsilon \cup \hat{\Gamma}^{\pm\epsilon},$$
$$\hat{\Gamma}_e^\epsilon = \Phi(\Gamma_e^\epsilon), \quad \hat{\Gamma}_s^\epsilon = \Phi(\Gamma_s^\epsilon), \quad \hat{\Gamma}_{eD}^\epsilon = \hat{\Gamma}_e^\epsilon \cup \hat{\Gamma}^{\pm\epsilon}$$

and we define vectors g_i^ϵ and $g^{i,\epsilon}$ by the relations

$$g_i^\epsilon = \partial_i^\epsilon \Phi^\epsilon \quad \text{and} \quad g^{j,\epsilon} \cdot g_i^\epsilon = \delta_i^j$$

which form the covariant and contravariant basis respectively of the tangent plane of $\Phi^\epsilon(\Omega^\epsilon)$ at $\Phi^\epsilon(x^\epsilon)$. The covariant and contravariant metric tensors are given respectively by

$$g_{ij}^\epsilon = g_i^\epsilon \cdot g_j^\epsilon \quad \text{and} \quad g^{ij,\epsilon} = g^{i,\epsilon} \cdot g^{j,\epsilon}.$$

The Christoffel symbols are defined by

$$\Gamma_{ij}^{p,\epsilon} = g^{p,\epsilon} \cdot \partial_j^\epsilon g_i^\epsilon.$$

Note however that when the set Ω^ϵ is of the special form $\Omega^\epsilon = \omega \times (-\epsilon, \epsilon)$ and the mapping Φ^ϵ is of the form (2.2), the following relations hold:

$$\Gamma_{\alpha 3}^{3,\epsilon} = \Gamma_{33}^{p,\epsilon} = 0.$$

The volume element is given by $\sqrt{g^\epsilon}\mathrm{d}x^\epsilon$ where

$$g^\epsilon = \det(g^\epsilon_{ij}).$$

It can be shown that there exist constants g_1 and g_2 such that

$$0 < g_1 \le g^\epsilon \le g_2 \tag{2.3}$$

for $0 \le \epsilon \le \epsilon_0$.

Let $\hat{A}^{ijkl,\epsilon}$, $\hat{P}^{ijk,\epsilon}$ and $\hat{\mathcal{E}}^{ij,\epsilon}$ be the elastic, piezoelectric and dielectric tensors respectively. We assume that the material of the shell is *homogeneous and isotropic*. Then the elasticity tensor is given by

$$\hat{A}^{ijkl,\epsilon} = \lambda g^{ij}g^{kl} + \mu(g^{ik}g^{jl} + g^{il}g^{jk}), \tag{2.4}$$

where λ and μ are the Lamè constants of the material.

These tensors satisfy the following coercive relations. There exists a constant $C > 0$ such that for all symmetric tensors (M_{ij}) and for any vector $(t_i) \in \mathbb{R}^3$,

$$\hat{A}^{ijkl,\epsilon}M_{kl}M_{ij} \ge C\sum_{i,j=1}^{3}(M_{ij})^2, \tag{2.5}$$

$$\hat{\mathcal{E}}^{kl,\epsilon}t_k t_l \ge C\sum_{j=1}^{3}t_j^2. \tag{2.6}$$

Moreover we have the symmetries

$$\hat{A}^{ijkl,\epsilon} = \hat{A}^{klij,\epsilon} = \hat{A}^{jikl,\epsilon}, \quad \hat{\mathcal{E}}^{kl,\epsilon} = \hat{\mathcal{E}}^{kl,\epsilon}, \quad \hat{P}^{ijk,\epsilon} = \hat{P}^{kij,\epsilon}.$$

Then the eigenvalue problem consists of finding $(\hat{u}^\epsilon, \hat{\varphi}^\epsilon, \xi^\epsilon)$ such that

$$\left.\begin{array}{l} -\mathrm{div}\hat{\sigma}^\epsilon(\hat{u}^\epsilon, \hat{\varphi}^\epsilon) = \xi^\epsilon\hat{u}^\epsilon \text{ in } \hat{\Omega}^\epsilon \\ \hat{\sigma}^\epsilon(\hat{u}^\epsilon, \hat{\varphi}^\epsilon)\nu = 0 \text{ on } \hat{\Gamma}^\epsilon_N \\ \hat{u}^\epsilon = 0 \text{ on } \hat{\Gamma}^\epsilon_0 \end{array}\right\}, \tag{2.7}$$

$$\left.\begin{array}{l} \mathrm{div}\hat{D}^\epsilon(\hat{u}^\epsilon, \hat{\varphi}^\epsilon) = 0 \text{ in } \hat{\Omega}^\epsilon \\ \hat{D}^\epsilon(\hat{u}^\epsilon, \hat{\varphi}^\epsilon)\nu = 0 \text{ on } \hat{\Gamma}^\epsilon_s \\ \hat{\varphi}^\epsilon = 0 \text{ on } \hat{\Gamma}^\epsilon_{eD}. \end{array}\right\}, \tag{2.8}$$

where

$$\hat{\sigma}^\epsilon_{ij} = \hat{A}^{ijkl,\epsilon}\hat{e}^\epsilon_{ij} - \hat{P}^{kij,\epsilon}\hat{E}_k, \tag{2.9}$$

$$\hat{D}_k = \hat{P}^{kij,\epsilon}\hat{e}^\epsilon_{ij} + \hat{\mathcal{E}}^{kl,\epsilon}\hat{E}_l, \tag{2.10}$$

where $\hat{e}^\epsilon_{ij}(\hat{u}^\epsilon) = \frac{1}{2}(\hat{\partial}^\epsilon_i\hat{u}^\epsilon_j + \hat{\partial}^\epsilon_j\hat{u}^\epsilon_i)$, $\hat{\partial}^\epsilon_i = \partial/\partial\hat{x}^\epsilon_i$ and $\hat{E}_k(\hat{\varphi}^\epsilon) = -\hat{\nabla}(\hat{\varphi}^\epsilon)$.

We define the spaces

$$\hat{V}^\epsilon = \{\hat{v} \in (H^1(\hat{\Omega}^\epsilon))^3, \hat{v}|_{\hat{\Gamma}^\epsilon_0} = 0\}, \tag{2.11}$$

$$\hat{\Psi}^\epsilon = \{\hat{\psi} \in H^1(\hat{\Omega}^\epsilon), \hat{\psi}|_{\hat{\Gamma}^\epsilon_{eD}} = 0\}. \tag{2.12}$$

Then the variational form of systems (2.7) and (2.8) is to find $(\hat{u}^\epsilon, \hat{\varphi}^\epsilon, \xi^\epsilon) \in \hat{V}^\epsilon \times \hat{\Psi}^\epsilon \times \mathbb{R}$ such that

$$\hat{a}^\epsilon((\hat{u}^\epsilon, \hat{\varphi}^\epsilon), (\hat{v}^\epsilon, \hat{\psi}^\epsilon)) = \xi^\epsilon \hat{l}^\epsilon(\hat{v}^\epsilon, \hat{\psi}^\epsilon) \quad \text{for all } (\hat{v}^\epsilon, \hat{\psi}^\epsilon) \in \hat{V}^\epsilon \times \hat{\Psi}^\epsilon, \tag{2.13}$$

where

$$\begin{aligned}\hat{a}^\epsilon((\hat{u}^\epsilon, \hat{\varphi}^\epsilon), (\hat{v}^\epsilon, \hat{\psi}^\epsilon)) = &\int_{\hat{\Omega}^\epsilon} \hat{A}^{ijkl,\epsilon} \hat{e}^\epsilon_{kl}(\hat{u}^\epsilon)\hat{e}^\epsilon_{ij}(\hat{v}^\epsilon)\mathrm{d}\hat{x}^\epsilon \\ &+ \int_{\hat{\Omega}^\epsilon} \hat{\mathcal{E}}^{ij,\epsilon}\hat{\partial}^\epsilon_i \hat{\varphi}^\epsilon \hat{\partial}^\epsilon_j \hat{\psi}^\epsilon \mathrm{d}\hat{x}^\epsilon \\ &+ \int_{\hat{\Omega}^\epsilon} \hat{P}^{mij,\epsilon}(\hat{\partial}^\epsilon_m \hat{\varphi}^\epsilon \hat{e}^\epsilon_{ij}(\hat{v}^\epsilon) - \hat{\partial}^\epsilon_m \hat{\psi}^\epsilon \hat{e}^\epsilon_{ij}(\hat{u}^\epsilon))\mathrm{d}\hat{x}^\epsilon,\end{aligned} \tag{2.14}$$

$$\hat{l}^\epsilon(\hat{v}^\epsilon, \hat{\psi}^\epsilon) = \int_{\hat{\Omega}^\epsilon} \hat{u}^\epsilon \cdot \hat{v}^\epsilon \mathrm{d}\hat{x}^\epsilon. \tag{2.15}$$

Since the mappings $\Phi^\epsilon : \overline{\Omega}^\epsilon \to \overline{\hat{\Omega}}^\epsilon$ are assumed to be C^1 diffeomorphisms, the correspondences that associate with every element $\hat{v}^\epsilon \in \hat{V}^\epsilon$, the vector

$$v^\epsilon = \hat{v}^\epsilon \cdot \Phi^\epsilon : \Omega^\epsilon \to \mathbb{R}^3$$

and with every element $\hat{\psi}^\epsilon \in \hat{\Psi}^\epsilon$, the function

$$\psi^\epsilon = \hat{\psi}^\epsilon \cdot \Phi^\epsilon : \Omega^\epsilon \to \mathbb{R}$$

induce bijections between the spaces $\hat{V}^\epsilon$ and V^ϵ, and the spaces $\hat{\Psi}^\epsilon$ and Ψ^ϵ respectively, where

$$V^\epsilon = \{v^\epsilon \in (H^1(\Omega^\epsilon))^3 | v^\epsilon = 0 \text{ on } \Gamma^\epsilon_0\}, \tag{2.16}$$

$$\Psi^\epsilon = \{\psi^\epsilon \in H^1(\Omega^\epsilon) | \psi^\epsilon = 0 \text{ on } \Gamma^\epsilon_{eD}\}. \tag{2.17}$$

Then we have

$$\hat{\partial}^\epsilon_j \hat{v}^\epsilon(\hat{x}^\epsilon) = (\partial^\epsilon_i v^\epsilon)(g^{i,\epsilon})_j, \tag{2.18}$$

$$\hat{e}_{ij}(\hat{v})(\hat{x}^\epsilon) = e^\epsilon_{k\|l}(v^\epsilon)(g^{k,\epsilon})_i(g^{l,\epsilon})_j, \tag{2.19}$$

where

$$e^\epsilon_{i\|j}(v^\epsilon) = \frac{1}{2}(\partial^\epsilon_i v^\epsilon_j + \partial^\epsilon_j v^\epsilon_i) - \Gamma^{p,\epsilon}_{ij} v^\epsilon_p. \tag{2.20}$$

Then the variational form (2.13) posed on the domain Ω^ϵ is to find $(u^\epsilon, \varphi^\epsilon, \xi^\epsilon) \in V^\epsilon \times \Psi^\epsilon \times \mathbb{R}$ such that

$$a^\epsilon((u^\epsilon, \varphi^\epsilon), (v^\epsilon, \psi^\epsilon)) = \xi^\epsilon l^\epsilon(v^\epsilon, \psi^\epsilon) \quad \text{for all } (v^\epsilon, \psi^\epsilon) \in V^\epsilon \times \Psi^\epsilon, \tag{2.21}$$

where

$$a^\epsilon((u^\epsilon,\varphi^\epsilon),(v^\epsilon,\psi^\epsilon)) = \int_{\Omega^\epsilon} A^{ijkl,\epsilon} e^\epsilon_{k\|l}(v^\epsilon) e^\epsilon_{i\|j}(v^\epsilon)\sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon + \int_{\Omega^\epsilon} \mathcal{E}^{ij,\epsilon}\partial^\epsilon_i\varphi^\epsilon\partial^\epsilon_j\psi^\epsilon\sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon + \int_{\Omega^\epsilon} P^{mij,\epsilon}(\partial^\epsilon_m\varphi^\epsilon e^\epsilon_{i\|j}(v^\epsilon) - \partial^\epsilon_m\psi^\epsilon e^\epsilon_{i\|j}(u^\epsilon))\sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon, \tag{2.22}$$

$$l^\epsilon(v^\epsilon,\psi^\epsilon) = \int_{\Omega^\epsilon} u^\epsilon\cdot v^\epsilon\sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon, \tag{2.23}$$

$$A^{pqrs,\epsilon} = \hat{A}^{ijkl,\epsilon}(g^{p,\epsilon})_i\cdot(g^{q,\epsilon})_j\cdot(g^{r,\epsilon})_k\cdot(g^{s,\epsilon})_l, \tag{2.24}$$

$$\mathcal{E}^{pq,\epsilon} = \hat{\mathcal{E}}^{ij,\epsilon}(g^{p,\epsilon})_i\cdot(g^{q,\epsilon})_j, \tag{2.25}$$

$$P^{pqr,\epsilon} = \hat{P}^{ijk,\epsilon}(g^{p,\epsilon})_i\cdot(g^{q,\epsilon})_j\cdot(g^{r,\epsilon})_k. \tag{2.26}$$

Using the relations (2.3), (2.5) and (2.6), it can be shown that there exists a constant $C > 0$ such that for all symmetric tensor (M_{ij}) and for any vector $(t_i) \in \mathbb{R}^3$,

$$A^{ijkl,\epsilon}M_{kl}M_{ij} \geq C\sum_{i,j=1}^{3}(M_{ij})^2, \tag{2.27}$$

$$\mathcal{E}^{ij,\epsilon}t_it_j \geq C\sum_{i=1}^{3}t_i^2. \tag{2.28}$$

Clearly the bilinear form associated with the left-hand side of (2.21) is elliptic. Hence by Lax–Milgram theorem, given $f^\epsilon \in V'^\epsilon$ and $h^\epsilon \in \Psi'^\epsilon$, there exists a unique $(u^\epsilon,\varphi^\epsilon) \in V^\epsilon \times \Psi^\epsilon$ such that

$$a^\epsilon((u^\epsilon,\varphi^\epsilon),(v^\epsilon,\psi^\epsilon)) = \langle (f^\epsilon,h^\epsilon),(v^\epsilon,\psi^\epsilon)\rangle \quad \forall V^\epsilon\times\Psi^\epsilon \in V^\epsilon\times\Psi^\epsilon. \tag{2.29}$$

In particular, for each $f^\epsilon \in (L^2(\Omega^\epsilon))^3$, there exists a unique solution $(u^\epsilon,\varphi^\epsilon) \in V^\epsilon \times \Psi^\epsilon$ such that

$$a^\epsilon((u^\epsilon,\varphi^\epsilon),(v^\epsilon,\psi^\epsilon)) = \int_{\Omega^\epsilon} f^\epsilon v^\epsilon\sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon \quad \forall v^\epsilon\times\psi^\epsilon \in V^\epsilon\times\Psi^\epsilon. \tag{2.30}$$

This is equivalent to the following equations.

$$\int_{\Omega^\epsilon} A^{ijkl,\epsilon}e^\epsilon_{k\|l}(u^\epsilon)e^\epsilon_{i\|j}(v^\epsilon)\sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon + \int_{\Omega^\epsilon} P^{mij,\epsilon}\partial^\epsilon_m(\varphi^\epsilon)e^\epsilon_{i\|j}(v^\epsilon)\sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon = \int_{\Omega^\epsilon} f^\epsilon v^\epsilon\sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon \quad \forall v^\epsilon \in V^\epsilon \tag{2.31}$$

and

$$\int_{\Omega^\epsilon} \mathcal{E}^{ij,\epsilon}\partial^\epsilon_i\varphi^\epsilon\partial^\epsilon_j\psi^\epsilon\sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon = \int_{\Omega^\epsilon} P^{mij,\epsilon}\partial^\epsilon_m\psi^\epsilon e^\epsilon_{i\|j}(u^\epsilon)\sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon \quad \forall\psi^\epsilon \in \Psi^\epsilon. \tag{2.32}$$

From relation (2.28), it follows that the bilinear form associated with the left-hand side of (2.32) is Ψ^ϵ-elliptic.

Also for each $h^\epsilon \in V^\epsilon$, the mapping

$$\psi^\epsilon \to \int_\Omega^\epsilon P^{mij,\epsilon} \partial_m \psi^\epsilon e^\epsilon_{i\|j}(h^\epsilon)\sqrt{g^\epsilon}\mathrm{d}x^\epsilon$$

defines a linear functional on Ψ^ϵ. Hence for each $h^\epsilon \in V^\epsilon$, there exists a unique $T^\epsilon(h^\epsilon) \in \Psi^\epsilon$ such that

$$\int_{\Omega^\epsilon} \mathcal{E}^{ij,\epsilon}\partial^\epsilon_i T^\epsilon(h^\epsilon)\partial^\epsilon_j \psi^\epsilon \sqrt{g^\epsilon}\mathrm{d}x^\epsilon = \int_{\Omega^\epsilon} P^{mij,\epsilon}\partial^\epsilon_m \psi^\epsilon e^\epsilon_{i\|j}(h^\epsilon)\sqrt{g^\epsilon}\mathrm{d}x^\epsilon \quad \forall \psi^\epsilon \in \Psi^\epsilon \tag{2.33}$$

and that $T^\epsilon : V^\epsilon \to \Psi^\epsilon$ is continuous.

In particular, it follows from (2.32) and the above equation that $\varphi^\epsilon = T^\epsilon(u^\epsilon)$ and eqs (2.31) and (2.32) become

$$\int_{\Omega^\epsilon} A^{ijkl,\epsilon} e^\epsilon_{k\|l}(u^\epsilon) e^\epsilon_{i\|j}(v^\epsilon)\sqrt{g^\epsilon}\mathrm{d}x^\epsilon + \int_{\Omega^\epsilon} P^{mij,\epsilon}\partial^\epsilon_m (T^\epsilon(u^\epsilon)) e^\epsilon_{i\|j}(v^\epsilon)\sqrt{g^\epsilon}\mathrm{d}x^\epsilon$$
$$= \int_{\Omega^\epsilon} f^\epsilon v^\epsilon \sqrt{g^\epsilon}\mathrm{d}x^\epsilon \quad \forall v^\epsilon \in V^\epsilon, \tag{2.34}$$

$$\int_{\Omega^\epsilon} \mathcal{E}^{ij,\epsilon}\partial^\epsilon_i (T^\epsilon(u^\epsilon))\partial^\epsilon_j \psi^\epsilon \sqrt{g^\epsilon}\mathrm{d}x^\epsilon = \int_{\Omega^\epsilon} P^{mij,\epsilon}\partial^\epsilon_m \psi^\epsilon e^\epsilon_{i\|j}(u^\epsilon)\sqrt{g^\epsilon}\mathrm{d}x^\epsilon$$
$$\forall \psi^\epsilon \in \Psi^\epsilon. \tag{2.35}$$

Lemma 2.1. *For each $h^\epsilon \in (L^2(\Omega^\epsilon))^3$, there exists a unique $G^\epsilon(h^\epsilon) \in V^\epsilon$ such that*

$$\int_{\Omega^\epsilon} A^{ijkl,\epsilon} e^\epsilon_{k\|l}(G^\epsilon(h^\epsilon)) e^\epsilon_{i\|j}(v^\epsilon)\sqrt{g^\epsilon}\mathrm{d}x^\epsilon + \int_{\Omega^\epsilon} P^{mij,\epsilon}\partial^\epsilon_m (T^\epsilon(G^\epsilon(h^\epsilon))) e^\epsilon_{i\|j}(v^\epsilon)\sqrt{g^\epsilon}\mathrm{d}x^\epsilon$$
$$= \int_{\Omega^\epsilon} h^\epsilon v^\epsilon \sqrt{g^\epsilon}\mathrm{d}x^\epsilon \quad \forall v^\epsilon \in V^\epsilon \tag{2.36}$$

and that $G^\epsilon : (L^2(\Omega^\epsilon))^3 \to V^\epsilon$ is continuous.

Proof. Let $B^\epsilon(u^\epsilon, v^\epsilon)$ denotes the bilinear form associated with the left-hand side of eq. (2.34). Using (2.35), we have

$$\begin{aligned} B^\epsilon(u^\epsilon, v^\epsilon) &= \int_{\Omega^\epsilon} A^{ijkl,\epsilon} e^\epsilon_{k\|l}(u^\epsilon) e^\epsilon_{i\|j}(v^\epsilon)\sqrt{g^\epsilon}\mathrm{d}x^\epsilon \\ &\quad + \int_{\Omega^\epsilon} P^{mij,\epsilon}\partial^\epsilon_m (T^\epsilon(u^\epsilon)) e^\epsilon_{i\|j}(v^\epsilon)\sqrt{g^\epsilon}\mathrm{d}x^\epsilon \\ &= \int_{\Omega^\epsilon} A^{ijkl,\epsilon} e^\epsilon_{k\|l}(u^\epsilon) e^\epsilon_{i\|j}(v^\epsilon)\sqrt{g^\epsilon}\mathrm{d}x^\epsilon \\ &\quad + \int_{\Omega^\epsilon} \mathcal{E}^{ij,\epsilon}\partial^\epsilon_i (T^\epsilon(u^\epsilon))\partial^\epsilon_j (T^\epsilon(v^\epsilon))\sqrt{g^\epsilon}\mathrm{d}x^\epsilon \end{aligned}$$

$$= \int_{\Omega^\epsilon} A^{ijkl,\epsilon} e^\epsilon_{k\|l}(v^\epsilon) e^\epsilon_{i\|j}(u^\epsilon)\sqrt{g^\epsilon}\mathrm{d}x^\epsilon$$
$$+ \int_{\Omega^\epsilon} \mathcal{E}^{ij,\epsilon}\partial^\epsilon_i(T^\epsilon(v^\epsilon))\partial^\epsilon_j(T^\epsilon(u^\epsilon))\sqrt{g^\epsilon}\mathrm{d}x^\epsilon$$
$$= B^\epsilon(v^\epsilon, u^\epsilon). \tag{2.37}$$

Also, using (2.35) and the relations (2.27) and (2.28), we have

$$B^\epsilon(u^\epsilon, u^\epsilon) = \int_{\Omega^\epsilon} A^{ijkl,\epsilon} e^\epsilon_{k\|l}(u^\epsilon) e^\epsilon_{i\|j}(u^\epsilon)\sqrt{g^\epsilon}\mathrm{d}x^\epsilon$$
$$+ \int_{\Omega^\epsilon} P^{mij,\epsilon}\partial^\epsilon_m(T^\epsilon(u^\epsilon)) e^\epsilon_{i\|j}(u^\epsilon)\sqrt{g^\epsilon}\mathrm{d}x^\epsilon$$
$$= \int_{\Omega^\epsilon} A^{ijkl,\epsilon} e^\epsilon_{k\|l}(u^\epsilon) e^\epsilon_{i\|j}(u^\epsilon)\sqrt{g^\epsilon}\mathrm{d}x^\epsilon$$
$$+ \int_{\Omega^\epsilon} \mathcal{E}^{ij,\epsilon}\partial^\epsilon_i(T^\epsilon(u^\epsilon))\partial^\epsilon_j(T^\epsilon(u^\epsilon))\sqrt{g^\epsilon}\mathrm{d}x^\epsilon$$
$$\geq C\|u^\epsilon\|^2_{V^\epsilon}. \tag{2.38}$$

Hence $B^\epsilon(\cdots)$ is symmetric and V^ϵ-elliptic. Hence by Lax–Milgram theorem, there exists a unique $G^\epsilon(h^\epsilon)$ satisfying (2.36). Letting $v^\epsilon = G^\epsilon(h^\epsilon)$ in (2.36), we get

$$\int_{\Omega^\epsilon} A^{ijkl,\epsilon} e^\epsilon_{k\|l}(G^\epsilon(h^\epsilon)) e^\epsilon_{i\|j}(G^\epsilon(h^\epsilon))\sqrt{g^\epsilon}\mathrm{d}x^\epsilon$$
$$+ \int_{\Omega^\epsilon} P^{mij,\epsilon}\partial^\epsilon_m(T^\epsilon(G^\epsilon(h^\epsilon))) e^\epsilon_{i\|j}(G^\epsilon(h^\epsilon))\sqrt{g^\epsilon}\mathrm{d}x^\epsilon$$
$$= \int_{\Omega^\epsilon} h^\epsilon G^\epsilon(h^\epsilon)\sqrt{g^\epsilon}\mathrm{d}x^\epsilon. \tag{2.39}$$

Using (2.35), it becomes

$$\int_{\Omega^\epsilon} A^{ijkl,\epsilon} e^\epsilon_{k\|l}(G^\epsilon(h^\epsilon)) e^\epsilon_{i\|j}(G^\epsilon(h^\epsilon))\sqrt{g^\epsilon}\mathrm{d}x^\epsilon$$
$$+ \int_{\Omega^\epsilon} \mathcal{E}^{ij,\epsilon}\partial^\epsilon_i(T^\epsilon(G^\epsilon(h^\epsilon)))\partial^\epsilon_j(T^\epsilon(G^\epsilon(h^\epsilon)))\sqrt{g^\epsilon}\mathrm{d}x^\epsilon$$
$$= \int_{\Omega^\epsilon} h^\epsilon G^\epsilon(h^\epsilon)\sqrt{g^\epsilon}\mathrm{d}x^\epsilon. \tag{2.40}$$

Using the relations (2.27) and (2.28), we have

$$\|G^\epsilon(h^\epsilon)\|^2_{V^\epsilon} \leq C^\epsilon\|G^\epsilon(h^\epsilon)\|_{V^\epsilon}\|h^\epsilon\|_{(L^2(\Omega^\epsilon))^3}. \tag{2.41}$$

Hence

$$\|G^\epsilon(h^\epsilon)\|_{V^\epsilon} \leq C^\epsilon\|h^\epsilon\|_{(L^2(\Omega^\epsilon))^3} \tag{2.42}$$

which implies that G^ϵ is continuous. ■

It follows from (2.34) and the above lemma that $u^\epsilon = G^\epsilon(f^\epsilon)$. Since the inclusion $(H^1(\Omega^\epsilon))^3 \hookrightarrow (L^2(\Omega^\epsilon))^3$ is compact, it follows that $G^\epsilon : (L^2(\Omega^\epsilon))^3 \to (L^2(\Omega^\epsilon))^3$ is compact. Also since the bilinear form $B^\epsilon(\cdots)$ is symmetric, it follows that G^ϵ is self-adjoint. Hence from the spectral theory of compact, self-adjoint operators, it follows that there exists a sequence of eigenpairs $(u^{m,\epsilon}, \xi^{m,\epsilon})_{m=1}^{\infty}$ such that

$$\begin{aligned}
&\int_{\Omega^\epsilon} A^{ijkl,\epsilon} e^\epsilon_{k\|l}(u^{m,\epsilon}) e^\epsilon_{i\|j}(v^\epsilon)\sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon \\
&\quad + \int_{\Omega^\epsilon} P^{mij,\epsilon}\partial^\epsilon_m(T^\epsilon(u^{m,\epsilon})) e^\epsilon_{i\|j}(v^\epsilon)\sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon \\
&= \xi^{m,\epsilon}\int_{\Omega^\epsilon} u^{m,\epsilon} v^\epsilon \sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon \quad \forall v^\epsilon \in V^\epsilon,
\end{aligned} \tag{2.43}$$

$$\begin{aligned}
&\int_{\Omega^\epsilon} \mathcal{E}^{ij,\epsilon}\partial^\epsilon_i(T^\epsilon(u^{m,\epsilon}))\partial^\epsilon_j\psi^\epsilon\sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon \\
&= \int_{\Omega^\epsilon} P^{mij,\epsilon}\partial^\epsilon_m\psi^\epsilon e^\epsilon_{i\|j}(u^{m,\epsilon})\sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon \quad \forall\psi^\epsilon \in \Psi^\epsilon,
\end{aligned} \tag{2.44}$$

$$0 < \xi^{1,\epsilon} \le \xi^{2,\epsilon} \le \cdots \le \xi^{m,\epsilon} \le \cdots \to \infty, \tag{2.45}$$

$$\int_{\Omega^\epsilon} u^{m,\epsilon}_i u^{n,\epsilon}_i \sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon = \epsilon^3\delta_{mn}. \tag{2.46}$$

The sequence $\{u^{m,\epsilon}\}$ forms a complete orthonormal basis for $(L^2(\Omega))^3$.

Define the Rayleigh quotient $R(\epsilon)(v^\epsilon)$ for $v^\epsilon \in V^\epsilon$ by

$$R^\epsilon(v^\epsilon) = \frac{\int_{\Omega^\epsilon} A^{ijkl,\epsilon} e_{k\|l}(v^\epsilon) e_{i\|j}(v^\epsilon)\sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon + \int_{\Omega^\epsilon} P^{mij,\epsilon}\partial^\epsilon_m(T^\epsilon(v^\epsilon)) e^\epsilon_{i\|j}(v^\epsilon)\sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon}{\int_{\Omega^\epsilon} v^\epsilon_i v^\epsilon_i \sqrt{g^\epsilon}\,\mathrm{d}x^\epsilon}. \tag{2.47}$$

Then

$$\xi^{m,\epsilon} = \min_{W^\epsilon \in W^\epsilon_m} \max_{v^\epsilon \in W^\epsilon\setminus\{0\}} R^\epsilon(v^\epsilon), \tag{2.48}$$

where W^ϵ_m denotes the collection of all m-dimensional subspaces of V^ϵ.

3. The scaled problem

We now perform a change of variable so that the domain no longer depends on ϵ. With $x = (x_1, x_2, x_3) \in \Omega$, we associate $x^\epsilon = (x_1, x_2, \epsilon x_3) \in \Omega^\epsilon$. Let

$$\begin{aligned}
&\Gamma_0 = \gamma_0 \times (-1,1), \quad \Gamma_1 = \gamma_1 \times (-1,1), \quad \Gamma^\pm = \omega \times \{\pm 1\}, \\
&\Gamma_e = \gamma_e \times (-1,1), \quad \Gamma_s = \gamma_s \times (-1,1), \\
&\Gamma_N = \Gamma_1 \cup \Gamma^+ \cup \Gamma^-, \quad \Gamma_{eD} = \Gamma^+ \cup \Gamma^- \cup \Gamma_e.
\end{aligned}$$

With the functions $\Gamma^{p,\epsilon}, g^\epsilon, A^{ijkl,\epsilon}, P^{ijk,\epsilon}, \mathcal{E}^{ij,\epsilon} : \Omega^\epsilon \to \mathbb{R}$, we associate the functions $\Gamma^p(\epsilon), g^\epsilon, A^{ijkl}(\epsilon), P^{ijk}(\epsilon), \mathcal{E}^{ij}(\epsilon) : \Omega \to \mathbb{R}$ defined by

$$\Gamma^p(\epsilon)(x) := \Gamma^{p,\epsilon}(x^\epsilon), \quad g(\epsilon)(x) = g^\epsilon(x^\epsilon), \quad A^{ijkl}(\epsilon)(x) = A^{ijkl,\epsilon}(x^\epsilon), \tag{3.1}$$

$$P^{ijk}(\epsilon)(x) = P^{ijk,\epsilon}(x^\epsilon), \quad \mathcal{E}^{ij}(\epsilon)(x) = \mathcal{E}^{ij,\epsilon}(x^\epsilon). \tag{3.2}$$

Assumption. We assume that the shell is a shallow shell, i.e. there exists a function $\theta \in C^3(\omega)$ such that

$$\phi^\epsilon(x_1, x_2) = (x_1, x_2, \epsilon\theta(x_1, x_2)) \quad \text{for all} \quad (x_1, x_2) \in \omega, \tag{3.3}$$

i.e., the curvature of the shell is of the order of the thickness of the shell.

We make the following scalings on the eigensolutions.

$$u_\alpha^{m,\epsilon}(x^\epsilon) = \epsilon^2 u_\alpha^m(\epsilon)(x), \quad v_\alpha(x^\epsilon) = \epsilon^2 v_\alpha(x), \tag{3.4}$$

$$u_3^{m,\epsilon}(x^\epsilon) = \epsilon u_3^m(\epsilon)(x), \quad v_3(x^\epsilon) = \epsilon v_3(x), \tag{3.5}$$

$$T^\epsilon(u^{m,\epsilon}(x^\epsilon)) = \epsilon^3 T(\epsilon)(u^m(\epsilon)(x)), \quad T^\epsilon(v(x^\epsilon)) = \epsilon^3 T(\epsilon)(v(x)), \tag{3.6}$$

$$\xi^{m,\epsilon} = \epsilon^2 \xi^m(\epsilon). \tag{3.7}$$

With the tensors $e_{i\|j}^\epsilon$, we associate the tensors $e_{i\|j}(\epsilon)$ through the relation

$$e_{i\|j}^\epsilon(v^\epsilon)(x^\epsilon) = \epsilon^2 e_{i\|j}(\epsilon; v)(x). \tag{3.8}$$

We define the spaces

$$V(\Omega) = \{v \in (H^1(\Omega))^3, v|_{\Gamma_0} = 0\}, \tag{3.9}$$

$$\Psi(\Omega) = \{\psi \in H^1(\Omega), \psi|_{\Gamma_{eD}} = 0\}. \tag{3.10}$$

We denote $\varphi^m(\epsilon) = T(\epsilon)(u^m(\epsilon))$. Then the variational equations (eqs (2.43)–(2.46)) become

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\epsilon) e_{k\|l}(\epsilon, u^m(\epsilon)) e_{i\|j}(\epsilon, v)\sqrt{g(\epsilon)}\mathrm{d}x \\
&\quad + \int_\Omega P^{3kl}\partial_3\varphi^m(\epsilon) e_{k\|l}(\epsilon, v)\sqrt{g(\epsilon)}\mathrm{d}x \\
&\quad + \epsilon\int_\Omega P^{\alpha kl}(\epsilon)\partial_\alpha\varphi^m(\epsilon) e_{k\|l}(\epsilon, v)\sqrt{g(\epsilon)}\mathrm{d}x \\
&= \xi^m(\epsilon)\int_\Omega [\epsilon^2 u_\alpha^m(\epsilon) v_\alpha + u_3^m(\epsilon) v_3]\sqrt{g(\epsilon)}\mathrm{d}x \quad \text{for all } v \in V(\Omega).
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
&\int_\Omega \mathcal{E}^{33}(\epsilon)\partial_3\varphi^m(\epsilon)\partial_3\psi\sqrt{g(\epsilon)}\mathrm{d}x \\
&\quad + \epsilon\int_\Omega [\mathcal{E}^{3\alpha}(\epsilon)(\partial_\alpha\varphi^m(\epsilon)\partial_3\psi + \partial_3\varphi^m(\epsilon)\partial_\alpha\psi)]\sqrt{g(\epsilon)}\mathrm{d}x \\
&\quad + \epsilon^2\int_\Omega \mathcal{E}^{\alpha\beta}(\epsilon)\partial_\alpha\varphi^m(\epsilon)\partial_\beta\psi\sqrt{g(\epsilon)}\mathrm{d}x \\
&= \int_\Omega P^{3kl}(\epsilon)\partial_3\psi e_{k\|l}(\epsilon, u^m(\epsilon))\sqrt{g(\epsilon)}\mathrm{d}x \\
&\quad + \epsilon\int_\Omega [P^{\alpha kl}(\epsilon)\partial_\alpha\psi e_{k\|l}(\epsilon, u^m(\epsilon))]\sqrt{g(\epsilon)}\mathrm{d}x \quad \text{for all } \psi \in \Psi(\Omega),
\end{aligned} \tag{3.12}$$

$$\int_\Omega [\epsilon^2 u_\alpha^m(\epsilon) u_\alpha^n(\epsilon) + u_3^m(\epsilon) u_3^n(\epsilon)]\sqrt{g(\epsilon)}\mathrm{d}x = \delta_{mn}. \tag{3.13}$$

4. Technical preliminaries

The following two lemmas are crucial; they play an important role in the proof of the convergence of the scaled unknowns as $\epsilon \to 0$. In the sequel, we denote by $C_1, C_2, \ldots, C_n$ various constants whose values do not depend on ϵ but may depend on θ.

Lemma 4.1. *The functions $e_{i\|j}(\epsilon, v)$ defined in* (3.8) *are of the form*

$$e_{\alpha\|\beta}(\epsilon; v) = \tilde{e}_{\alpha\beta}(v) + \epsilon^2 e^{\sharp}_{\alpha\|\beta}(\epsilon; v), \tag{4.1}$$

$$e_{\alpha\|3}(\epsilon; v) = \frac{1}{\epsilon}\{\tilde{e}_{\alpha 3}(v) + \epsilon^2 e^{\sharp}_{\alpha\|3}(\epsilon; v)\}, \tag{4.2}$$

$$e_{3\|3}(\epsilon; v) = \frac{1}{\epsilon^2}\tilde{e}_{33}(v), \tag{4.3}$$

where

$$\tilde{e}_{\alpha\beta}(v) = \frac{1}{2}(\partial_\alpha v_\beta + \partial_\beta v_\alpha) - v_3 \partial_{\alpha\beta}\theta, \tag{4.4}$$

$$\tilde{e}_{\alpha 3}(v) = \frac{1}{2}(\partial_\alpha v_3 + \partial_3 v_\alpha), \tag{4.5}$$

$$\tilde{e}_{33}(v) = \partial_3 v_3 \tag{4.6}$$

and there exists constant C_1 such that

$$\sup_{0<\epsilon\le\epsilon_0} \max_{\alpha, j} \|e^{\sharp}_{\alpha,j}(\epsilon; v)\|_{0,\Omega} \le C_1 \|v\|_{1,\Omega} \quad \text{for all } \; v \in V. \tag{4.7}$$

Also there exist constants C_2, C_3 and C_4 such that

$$\sup_{0<\epsilon\le\epsilon_0} \max_{x\in\Omega} |g(x) - 1| \le C_2 \epsilon^2, \tag{4.8}$$

$$\sup_{0<\epsilon\le\epsilon_0} \max_{x\in\Omega} |A^{ijkl}(\epsilon) - A^{ijkl}| \le C_3 \epsilon^2, \tag{4.9}$$

where

$$A^{ijkl} = \lambda \delta^{ij}\delta^{kl} + \mu(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}) \tag{4.10}$$

and

$$A^{ijkl} M_{kl} M_{ij} \ge C_4 M_{ij} M_{ij} \tag{4.11}$$

for $0 < \epsilon \le \epsilon_0$ *and for all symmetric tensors* (M_{ij}).

Proof. The proof is based on Lemma 4.1 of [2]. ■

From relation (2.6) and definition (3.2), it follows that there exists a constant C_5 such that for any vector $(t_i) \in \mathbb{R}^3$,

$$\mathcal{E}^{ij}(\epsilon) t_i t_j \ge C_5 \sum_{j=1}^{3} t_j^2. \tag{4.12}$$

We assume that there exists functions P^{kij} and $\mathcal{E}^{ij}$ such that

$$\sup_{0<\epsilon\leq\epsilon_0} \max_{x\in\Omega} |P^{kij}(\epsilon) - P^{kij}| \leq C_6\epsilon, \tag{4.13}$$

$$\sup_{0<\epsilon\leq\epsilon_0} \max_{x\in\Omega} |\mathcal{E}^{ij}(\epsilon) - \mathcal{E}^{ij}| \leq C_7\epsilon. \tag{4.14}$$

Lemma 4.2. *Let $\theta \in C^3(\omega)$ be a given function and let the functions $\tilde{e}_{ij}$ be defined as in* (4.4)–(4.6). *Then there exists a constant C_8 such that the following generalised Korn's inequality holds*:

$$\|v\|_{1,\Omega} \leq C_8 \left\{ \sum_{i,j} \|\tilde{e}_{ij}(v)\|^2_{0,\Omega} \right\}^{1/2} \tag{4.15}$$

for all $v \in V(\Omega)$ where $V(\Omega)$ is the space defined in (3.9).

Proof. The proof is based on Lemma 4.2 of [2]. ■

5. *A priori* estimates

In this section, we show that for each positive integer m, the scaled eigenvalues $\{\xi^m(\epsilon)\}$ are bounded uniformly with respect to ϵ.

Let $\varphi \in H_0^2(\omega)$. Then

$$v_\varphi := (-x_3\partial_1\varphi, -x_3\partial_2\varphi, \varphi) \in V(\Omega) \tag{5.1}$$

and

$$\tilde{e}_{\alpha\beta}(v_\varphi) = -x_3\partial_{\alpha\beta}\varphi - \varphi\partial_{\alpha\beta}\theta, \quad \tilde{e}_{i3}(v_\varphi) = 0. \tag{5.2}$$

Hence

$$e_{\alpha\|\beta}(\epsilon, v_\varphi) = -x_3\partial_{\alpha\beta}\varphi - \varphi\partial_{\alpha\beta}\theta + O(\epsilon^2), \tag{5.3}$$

$$e_{\alpha\|3}(\epsilon, v_\varphi) = O(\epsilon), \tag{5.4}$$

$$e_{3\|3}(\epsilon, v_\varphi) = 0. \tag{5.5}$$

We need the following lemma to prove the boundedness of the scaled eigenvalues.

Lemma 5.1. *There exists a constant $C_9 > 0$ such that*

$$|\partial_3(T(\epsilon)(v_\varphi))|_{0,\Omega} \leq C_9|\varphi|_{2,\omega}, \tag{5.6}$$

$$|\epsilon\partial_\alpha(T(\epsilon)(v_\varphi))|_{0,\Omega} \leq C_9|\varphi|_{2,\omega}. \tag{5.7}$$

Proof. With the scalings (3.3)–(3.7), the variational equation (eq. (2.33)) posed on the domain Ω reads as follows:

For each $h \in (H^1(\Omega))^3$, there exists a unique solution $T(\epsilon)(h) \in (H^1(\Omega))^3$ such that

$$\begin{aligned}
&\int_\Omega \mathcal{E}^{33}(\epsilon)\partial_3 T(\epsilon)(h)\partial_3\psi\sqrt{g(\epsilon)}\mathrm{d}x \\
&\quad + \epsilon\int_\Omega [\mathcal{E}^{\alpha 3}(\epsilon)(\partial_\alpha T(\epsilon)(h)\partial_3\psi + \partial_3 T(\epsilon)(h)\partial_\alpha\psi)]\sqrt{g(\epsilon)}\mathrm{d}x \\
&\quad + \epsilon^2\int_\Omega \mathcal{E}^{\alpha\beta}(\epsilon)\partial_\alpha T(\epsilon)(h)\partial_\beta\psi\sqrt{g(\epsilon)}\mathrm{d}x \\
&= \int_\Omega P^{3kl}(\epsilon)\partial_3\psi e_{k\|l}(\epsilon, h)\sqrt{g(\epsilon)}\mathrm{d}x \\
&\quad + \epsilon\int_\Omega P^{\alpha kl}(\epsilon)\partial_\alpha\psi e_{k\|l}(\epsilon, h)\sqrt{g(\epsilon)}\mathrm{d}x \quad \forall\psi \in \Psi.
\end{aligned} \tag{5.8}$$

Taking $h = v_\varphi$ and $\psi = T(\epsilon)(v_\varphi)$ in the above equation, we have

$$\begin{aligned}
&\int_\Omega \mathcal{E}^{33}(\epsilon)\partial_3 T(\epsilon)(v_\varphi)\partial_3 T(\epsilon)(v_\varphi)\sqrt{g(\epsilon)}\mathrm{d}x \\
&\quad + \epsilon\int_\Omega [\mathcal{E}^{\alpha 3}(\epsilon)(\partial_\alpha T(\epsilon)(v_\varphi)\partial_3 T(\epsilon)(v_\varphi) \\
&\quad + \partial_3 T(\epsilon)(v_\varphi)\partial_\alpha T(\epsilon)(v_\varphi))]\sqrt{g(\epsilon)}\mathrm{d}x \\
&\quad + \epsilon^2\int_\Omega \mathcal{E}^{\alpha\beta}(\epsilon)\partial_\alpha T(\epsilon)(v_\varphi)\partial_\beta T(\epsilon)(v_\varphi)\sqrt{g(\epsilon)}\mathrm{d}x \\
&= \int_\Omega P^{3kl}(\epsilon)\partial_3 T(\epsilon)(v_\varphi) e_{k\|l}(\epsilon, v_\varphi)\sqrt{g(\epsilon)}\mathrm{d}x \\
&\quad + \epsilon\int_\Omega P^{\alpha kl}(\epsilon)\partial_\alpha T(\epsilon)(v_\varphi) e_{k\|l}(\epsilon, v_\varphi)\sqrt{g(\epsilon)}\mathrm{d}x.
\end{aligned} \tag{5.9}$$

Using the relations (4.12) and (5.2)–(5.5), it follows that there exists a constant $C_9 > 0$ such that

$$\begin{aligned}
&|\partial_3(T(\epsilon)(v_\varphi))|_{0,\Omega}^2 + |\epsilon\partial_\alpha(T(\epsilon)(v_\varphi))|_{0,\Omega}^2 \\
&\qquad \le C_9\{|\partial_3 T(\epsilon)(v_\varphi)|_{0,\Omega}|\varphi|_{2,\omega} + |\epsilon\partial_\alpha T(\epsilon)(v_\varphi)|_{0,\Omega}|\varphi|_{2,\omega}\}
\end{aligned} \tag{5.10}$$

and hence the result follows. ■

Theorem 5.2. *For each positive integer* m, *there exists a constant* $C(m) > 0$ *such that*

$$\xi^m(\epsilon) \le C(m). \tag{5.11}$$

Proof. Since problem (3.11) was derived from (2.43) after a change of scale, we still have the variational characterization of the scaled eigenvalues $\xi^m(\epsilon)$. Let V_m denote the collection of all m-dimensional subspaces of $V(\Omega)$. Then

$$\xi^m(\epsilon) = \min_{W\in V_m}\max_{v\in W}\frac{N(\epsilon)(v, v)}{D(\epsilon)(v, v)}, \tag{5.12}$$

where

$$N(\epsilon)(v, v) = \int_\Omega A^{ijkl} e_{k\|l}(\epsilon, v) e_{i\|j}(\epsilon, v)\sqrt{g(\epsilon)}\mathrm{d}x$$
$$+ \int_\Omega P^{3kl}\partial_3 T(\epsilon)(v) e_{k\|l}(\epsilon, v)\sqrt{g(\epsilon)}\mathrm{d}x$$
$$+ \epsilon \int_\Omega P^{\alpha kl}\partial_\alpha T(\epsilon)(v) e_{k\|l}(\epsilon, v)\sqrt{g(\epsilon)}\mathrm{d}x, \tag{5.13}$$

$$D(\epsilon)(v, v) = \int_\Omega [\epsilon^2 v_\alpha v_\alpha + v_3 v_3]\sqrt{g(\epsilon)}\mathrm{d}x. \tag{5.14}$$

Let W_m be the collection of all m-dimensional subspaces of $H_0^2(\omega)$. Let $W \in W_m$. Define

$$\mathbf{W} = \{v_\varphi | \varphi \in W\}. \tag{5.15}$$

It follows that $\mathbf{W} \in V_m$. Hence, it follows from (5.12) that

$$\xi^m(\epsilon) \leq \min_{W\in W_m} \max_{\varphi\in W} \frac{N(\epsilon)(v_\varphi, v_\varphi)}{D(\epsilon)(v_\varphi, v_\varphi)}. \tag{5.16}$$

Now,

$$D(\epsilon)(v_\varphi, v_\varphi) = \int_\Omega [\epsilon^2 x_3^2 |\partial_\alpha \varphi|^2 + |\varphi|^2]\sqrt{g(\epsilon)}\mathrm{d}x.$$
$$\geq \int_\omega \varphi^2 \mathrm{d}\omega. \tag{5.17}$$

Using the relations (5.3)–(5.5) and Lemma 5.1, it follows that

$$\int_\Omega A^{ijkl} e_{k\|l}(\epsilon, v_\varphi) e_{i\|j}(\epsilon, v_\varphi)\sqrt{g(\epsilon)}\mathrm{d}x \leq C\int_\omega |\triangle\varphi|^2 \mathrm{d}\omega, \tag{5.18}$$

$$\int_\Omega P^{3kl}\partial_3 T(\epsilon)(v_\varphi) e_{k\|l}(\epsilon, v_\varphi)\sqrt{g(\epsilon)}\mathrm{d}x \leq C\int_\omega |\triangle\varphi|^2 \mathrm{d}\omega, \tag{5.19}$$

$$\epsilon\int_\Omega P^{\alpha kl}\partial_\alpha T(\epsilon)(v_\varphi) e_{k\|l}(\epsilon, v_\varphi)\sqrt{g(\epsilon)}\mathrm{d}x \leq C\int_\omega |\triangle\varphi|^2 \mathrm{d}\omega. \tag{5.20}$$

Hence

$$\xi^m(\epsilon) \leq C \min_{W\in W_m} \max_{\varphi\in W} \frac{\int_\omega |\triangle\varphi|^2 \mathrm{d}\omega}{\int_\omega \varphi^2 \mathrm{d}\omega}$$
$$\leq C\lambda^m, \tag{5.21}$$

where λ^m is the mth eigenvalue of the two-dimensional elliptic eigenvalue problem

$$\triangle^2 u = \lambda u \quad \text{in } \omega$$
$$u = \partial_\nu u = 0 \quad \text{on } \partial\omega. \tag{5.22}$$

This completes the proof of the theorem on setting $C(m) = C\lambda^m$. ■

6. The limit problem

Theorem 6.1. (a) *For each positive integer m, there exists* $u^m \in H^1(\Omega), \varphi^m \in L^2(\Omega)$ *and* $\xi^m \in \mathbb{R}$ *such that*

$$u^m(\epsilon) \to u^m \text{ in } H^1(\Omega), \quad \varphi^m(\epsilon) \to \varphi^m \text{ in } L^2(\Omega), \tag{6.1}$$

$$(\epsilon\partial_1\varphi^m(\epsilon), \epsilon\partial_2\varphi^m(\epsilon), \partial_3\varphi^m(\epsilon)) \to (0, 0, \partial_3\varphi^m) \text{ in } L^2(\Omega), \tag{6.2}$$

$$\xi^m(\epsilon) \to \xi^m. \tag{6.3}$$

(b) *Define the spaces*

$$V_H(\omega) = \{(\eta_\alpha) \in (H^1(\omega))^2; \eta_\alpha = 0 \text{ on } \gamma_0\}, \tag{6.4}$$

$$V_3(\omega) = \{\eta_3 \in H^2(\omega); \eta_3 = \partial_\nu\eta_3 = 0 \text{ on } \gamma_0\}, \tag{6.5}$$

$$V_{KL} = \{v \in H^1(\Omega) | v = \eta_\alpha - x_3\partial_\alpha\eta_3, (\eta_i) \in V_H(\omega) \times V_3(\omega)\}, \tag{6.6}$$

$$\Psi_l = \{\psi \in L^2(\Omega), \partial_3\psi \in L^2(\Omega)\}, \tag{6.7}$$

$$\Psi_{l0} = \{\psi \in L^2(\Omega), \partial_3\psi \in L^2(\Omega), \psi|\Gamma^\pm = 0\}. \tag{6.8}$$

Then there exists $(\zeta_\alpha^m, \zeta_3^m) \in V_H \times V_3(\omega)$ *such that*

$$u_\alpha^m = \zeta_\alpha^m - x_3\partial_\alpha\zeta_3^m \quad \text{and} \quad u_3^m = \zeta_3^m, \tag{6.9}$$

$$\varphi^m = (1 - x_3^2)\frac{p^{3\alpha\beta}}{p^{33}}\partial_{\alpha\beta}\xi_3^m \tag{6.10}$$

and $(\zeta^m, \xi^m) \in V_H \times V_3 \times \mathbb{R}$ *satisfies*

$$-\int_\omega m_{\alpha\beta}(\zeta^m)\partial_{\alpha\beta}\eta_3 \mathrm{d}\omega + \int_\omega n_{\alpha\beta}^\theta(\zeta^m)\partial_{\alpha\beta}\theta\eta_3 \mathrm{d}\omega + \frac{2}{3}\int_\omega \frac{p^{3\alpha\beta}p^{3\rho\tau}}{p^{33}}\partial_{\rho\tau}\zeta_3^m\partial_{\alpha\beta}\eta_3 \mathrm{d}\omega$$

$$= \xi^m \int_\omega \zeta_3^m\eta_3 \mathrm{d}\omega \quad \forall\eta_3 \in V_3(\omega), \tag{6.11}$$

$$\int_\omega n_{\alpha\beta}^\theta\partial_\beta\eta_\alpha \mathrm{d}\omega = 0 \quad \forall\eta_\alpha \in V_H(\omega), \tag{6.12}$$

where

$$m_{\alpha\beta}(\zeta) = -\left\{\frac{4\lambda\mu}{3(\lambda + 4\mu)}\triangle\zeta_3\delta_{\alpha\beta} + \frac{4\mu}{3}\partial_{\alpha\beta}\zeta_3\right\} \tag{6.13}$$

$$n_{\alpha\beta}^\theta(\zeta) = \frac{4\lambda\mu}{\lambda + 2\mu}\tilde{e}_{\sigma\sigma}(\zeta)\delta_{\alpha\beta} + 4\mu\tilde{e}_{\alpha\beta}(\zeta) \tag{6.14}$$

$$p^{33} = \frac{1}{\mu}P^{3\alpha3}P^{3\alpha3} + \frac{1}{\lambda + 2\mu}P^{333}P^{333} + \mathcal{E}^{33} \tag{6.15}$$

$$p^{3\alpha\beta} = P^{3\alpha\beta} - \frac{\lambda}{\lambda + 2\mu}P^{333}\delta^{\alpha\beta}. \tag{6.16}$$

Proof. For the sake of clarity, the proof is divided into several steps.

Step (i). Define the vector $\tilde{\varphi}_i^m(\epsilon)$ and the tensor $\tilde{K}^m(\epsilon) = (\tilde{K}_{ij}^m(\epsilon))$ by

$$\tilde{\varphi}_i^m(\epsilon) = (\epsilon\partial_1\varphi^m(\epsilon), \epsilon\partial_2\varphi^m(\epsilon), \partial_3\varphi^m(\epsilon)), \tag{6.17}$$

$$\tilde{K}_{\alpha\beta}^m(\epsilon) = \tilde{e}_{\alpha\beta}(u^m(\epsilon)), \quad \tilde{K}_{\alpha 3}^m(\epsilon) = \frac{1}{\epsilon}\tilde{e}_{\alpha 3}(u^m(\epsilon)), \quad \tilde{K}_{33}^m(\epsilon) = \frac{1}{\epsilon^2}\tilde{e}_{33}(u^m(\epsilon)). \tag{6.18}$$

Then there exists a constant $C_{10} > 0$ such that

$$\|u^m(\epsilon)\|_{1,\Omega} \le C_{10}, \quad |\tilde{K}_{ij}^m(\epsilon)|_{0,\Omega} \le C_{10}, \quad |\tilde{\varphi}_i^m(\epsilon)|_{0,\Omega} \le C_{10} \tag{6.19}$$

for all $0 < \epsilon \le \epsilon_0$.

Letting $v = u^m(\epsilon)$ in (3.11), we have

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\epsilon)e_{k\|l}(\epsilon)(u^m(\epsilon))e_{i\|j}(\epsilon)(u^m(\epsilon))\sqrt{g(\epsilon)}\mathrm{d}x \\
&\quad + \int_\Omega P^{3kl}(\epsilon)\partial_3\varphi^m(\epsilon)e_{k\|l}(\epsilon)(u^m(\epsilon))\sqrt{g(\epsilon)}\mathrm{d}x \\
&\quad + \epsilon\int_\Omega P^{\alpha kl}(\epsilon)\partial_\alpha\varphi^m(\epsilon)e_{k\|l}(\epsilon)(u^m(\epsilon))\sqrt{g(\epsilon)}\mathrm{d}x \\
&= \xi^m(\epsilon)\int_\Omega [\epsilon^2 u_\alpha^m(\epsilon)u_\alpha^m(\epsilon) + u_3^m(\epsilon)u_3^m(\epsilon)]\sqrt{g(\epsilon)}\mathrm{d}x.
\end{aligned} \tag{6.20}$$

Letting $\psi = \varphi^m(\epsilon)$ in (3.12) and using it in the above equation, we get

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\epsilon)e_{k\|l}(\epsilon, u^m(\epsilon))e_{i\|j}(\epsilon, u^m(\epsilon))\sqrt{g(\epsilon)}\mathrm{d}x \\
&\quad + \int_\Omega \mathcal{E}^{ij}(\epsilon)\tilde{\varphi}_i^m(\epsilon)\tilde{\varphi}_j^m(\epsilon)\sqrt{g(\epsilon)}\mathrm{d}x \\
&= \xi^m(\epsilon)\int_\Omega [\epsilon^2 u_\alpha^m(\epsilon)\cdot u_\alpha^m(\epsilon) + u_3^m(\epsilon)u_3^m(\epsilon)]\sqrt{g(\epsilon)}\mathrm{d}x.
\end{aligned} \tag{6.21}$$

Using the coerciveness properties (4.11) and (4.12), the inequality $(a-b)^2 \ge a^2/2 - b^2$ and the generalized Korn's inequality (4.15), we have for $\epsilon \le \min\{\epsilon_0, 1\}$,

$$\begin{aligned}
&\int_\Omega A^{ijkl}(\epsilon)e_{k\|l}(\epsilon, u^m(\epsilon))e_{i\|j}(\epsilon, u^m(\epsilon))\sqrt{g(\epsilon)}\mathrm{d}x \\
&\quad + \int_\Omega \mathcal{E}^{ij}(\epsilon)\tilde{\varphi}_i^m(\epsilon)\tilde{\varphi}_j^m(\epsilon)\sqrt{g(\epsilon)}\mathrm{d}x \\
&\ge C_{11}\sum_{i,j}\|e_{i\|j}(\epsilon, u^m(\epsilon))\|_{0,\Omega}^2 + C_{11}\sum_i\|\tilde{\varphi}_i^m(\epsilon)\|_{0,\Omega}^2
\end{aligned}$$

$$= C_{11} \sum_{\alpha,\beta} \|\tilde{e}_{\alpha\beta}(u^m(\epsilon)) + \epsilon^2 e^\sharp_{\alpha\beta}(\epsilon, u^m(\epsilon))\|^2_{0,\Omega}$$

$$+ 2C_{11} \sum_{\alpha} \left\| \frac{1}{\epsilon}\tilde{e}_{\alpha 3}(u^m(\epsilon)) + \epsilon e^\sharp_{\alpha 3}(\epsilon, u^m(\epsilon)) \right\|^2_{0,\Omega}$$

$$+ C_{11} \left\| \frac{1}{\epsilon^2}\tilde{e}_{33}(u^m(\epsilon)) \right\|^2_{0,\Omega} + C_{11} \sum_i \|\tilde{\varphi}^m_i(\epsilon)\|^2_{0,\Omega}$$

$$\geq C_{11} \left\{ \frac{1}{2} \sum_{i,j} |\tilde{K}^m_{ij}(\epsilon)|^2_{0,\Omega} - C_1^2(2\epsilon^2 + \epsilon^4)\|u^m(\epsilon)\|^2_{1,\Omega} \right\}$$

$$+ C_{11} \sum_i \|\tilde{\varphi}^m_i(\epsilon)\|^2_{0,\Omega}$$

$$\geq C_{11} \left\{ \frac{1}{2} \sum_{i,j} \|\tilde{e}_{ij}(u^m(\epsilon))\|^2_{0,\Omega} - 3\epsilon^2 C_1^2 \|u^m(\epsilon)\|^2_{1,\Omega} \right\}$$

$$+ C_{11} \sum_i \|\tilde{\varphi}^m_i(\epsilon)\|^2_{0,\Omega}$$

$$\geq C_{11} \left\{ \frac{1}{2}(C_8)^{-2} - 3\epsilon^2 C_1^2 \right\} \|u^m(\epsilon)\|^2_{1,\Omega} + C_{11} \sum_i \|\tilde{\varphi}^m_i(\epsilon)\|^2_{0,\Omega}. \tag{6.22}$$

Combining eqs (6.21) and (6.22) with relations (3.13) and (5.11), we get the relation (6.19).

Step (ii). From Step (i) it follows that there exists a subsequence $(\tilde{\varphi}^m_i(\epsilon))$ and $(\tilde{\varphi}^m_i) \in L^2(\Omega)$ such that

$$(\epsilon\partial_1\varphi^m(\epsilon), \epsilon\partial_2\varphi^m(\epsilon), \partial_3\varphi^m(\epsilon)) \rightharpoonup (\tilde{\varphi}^m_1, \tilde{\varphi}^m_2, \tilde{\varphi}^m_3) \quad \text{in } (L^2(\Omega))^3. \tag{6.23}$$

Since Γ_{eD} contains Γ^-, we have

$$\varphi^m(\epsilon)(x_1, x_2, x_3) = \int_{-1}^{x_3} \partial_3\varphi^m(\epsilon)(x_1, x_2, s)\mathrm{d}s \tag{6.24}$$

and it follows that $\|\varphi^m(\epsilon)\|_{0,\Omega} \leq \sqrt{2}\|\partial_3\varphi^m(\epsilon)\|_{0,\Omega}$. This implies that $\varphi^m(\epsilon)$ is bounded in $L^2(\Omega)$. Therefore there exists a φ^m in $L^2(\Omega)$ and a subsequence, still indexed by ϵ, such that $\varphi^m(\epsilon)$ converges weakly to φ^m. Hence it follows from (6.23) that

$$(\epsilon\partial_1\varphi^m(\epsilon), \epsilon\partial_2\varphi^m(\epsilon), \partial_3\varphi^m(\epsilon)) \rightharpoonup (0, 0, \partial_3\varphi^m). \tag{6.25}$$

Step (iii). From Step (i) it follows that there exists a subsequence, indexed by ϵ for notational convenience, and functions $u^m \in V(\Omega)$ and $\tilde{K}^m_{ij} \in (L^2(\Omega))^9$ such that

$$u^m(\epsilon) \rightharpoonup u^m \quad \text{in } H^1(\Omega), \quad \tilde{K}^m(\epsilon) \rightharpoonup \tilde{K}^m \quad \text{in } L^2(\Omega) \text{ as } \epsilon \to 0. \tag{6.26}$$

Then there exist functions $(\zeta^m_\alpha) \in H^1(\omega)$ and $\zeta^m_3 \in H^2(\omega)$ satisfying $\zeta^m_i = \partial_\nu\zeta^m_3 = 0$ on γ_0 such that

$$u^m_\alpha = \zeta^m_\alpha - x_3\partial_\alpha\zeta^m_3 \quad \text{and} \quad u^m_3 = \zeta^m_3 \tag{6.27}$$

and

$$\tilde{K}^m_{\alpha\beta} = \tilde{e}_{\alpha\beta}(u^m), \quad \tilde{K}^m_{\alpha 3} = -\frac{1}{\mu} P^{3\alpha 3}\partial_3\varphi^m,$$
$$\tilde{K}^m_{33} = -\frac{1}{\lambda + 2\mu}(P^{333}\partial_3\varphi^m + \lambda \tilde{K}^m_{\beta\beta}). \tag{6.28}$$

From definition (6.18) and the boundedness of $(\tilde{K}^m_{ij}(\epsilon))$, we deduce that

$$\|e_{\alpha 3}(u^m(\epsilon))\|_{0,\Omega} \le \epsilon C_{13} \quad \text{and} \quad \|e_{33}(u^m(\epsilon))\|_{0,\Omega} \le \epsilon^2 C_{13},$$

where $e_{ij}(v) = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$. Since norm is a weakly lower semicontinuous function

$$\|e_{i3}(u^m)\|_{0,\Omega} \le \liminf_{\epsilon\to 0} \|e_{i3}(u^m(\epsilon)\|_{0,\Omega} = 0, \tag{6.29}$$

we obtain $e_{i3}(u^m) = 0$. Then it is a standard argument that the components u^m_i of the limit u^m are of the form (6.27).

Since $u^m(\epsilon) \rightharpoonup u^m$ in $H^1(\Omega)$, definition (4.4) of the functions $\tilde{e}_{\alpha\beta}(v)$ shows that the function $\tilde{K}^m_{\alpha\beta}(\epsilon) = \tilde{e}_{\alpha\beta}(u^m(\epsilon))$ converges weakly in $L^2(\Omega)$ to the function $\tilde{e}_{\alpha\beta}(u^m)$.

We next note the following result. Let $w \in L^2(\Omega)$ be given; then

$$\int_\Omega w\partial_3 v \mathrm{d}x = 0 \quad \text{for all } v \in H^1(\Omega) \text{ with } v = 0 \text{ on } \Gamma_0, \text{ then } w = 0. \tag{6.30}$$

Multiplying (3.11) by ϵ^2, taking $(v_\alpha) = 0$ and letting $\epsilon \to 0$, we get

$$\int_\Omega (\lambda\tilde{K}^m_{\sigma\sigma} + (\lambda + 2\mu)\tilde{K}_{33} + P^{333}\partial_3\varphi^m)\partial_3 v_3 \mathrm{d}x = 0 \tag{6.31}$$

which implies $(\lambda\tilde{K}^m_{\sigma\sigma} + (\lambda + 2\mu)\tilde{K}_{33} + P^{333}\partial_3\varphi^m) = 0$ and hence the third relation in (6.28) follows.

Again, multiplying (3.11) by ϵ, taking $v_3 = 0$ and letting $\epsilon \to 0$, we get

$$\int_\Omega (\mu\tilde{K}^m_{\alpha 3} + P^{3\alpha 3}\partial_3\varphi^m)\partial_3 v_\alpha \mathrm{d}x = 0 \tag{6.32}$$

which implies $(\mu\tilde{K}^m_{\alpha 3} + P^{3\alpha 3}\partial_3\varphi^m) = 0$ and hence the second relation in (6.28) follows.

Step (iv). The function φ^m is of the form (6.10).

Letting $\epsilon \to 0$ in eq. (3.12), we get

$$\int_\Omega (P^{3\alpha\beta}\tilde{K}^m_{\alpha\beta} - \mathcal{E}^{33}\partial_3\varphi^m)\partial_3\psi \mathrm{d}x = 0 \quad \forall\psi \in \Psi(\Omega). \tag{6.33}$$

Since $D(\Omega)$ is dense in Ψ_{l0} (and hence in $\Psi(\Omega)$) for the norm $\|.\|_{\Psi_l}$, eq. (6.33) is equivalent to

$$\partial_3(P^{3\alpha\beta}\tilde{K}^m_{\alpha\beta} - \mathcal{E}^{33}\partial_3\varphi^m) = 0 \quad \text{in } D'(\Omega) \tag{6.34}$$

which implies that $(P^{3\alpha\beta}\tilde{K}^m_{\alpha\beta} - \mathcal{E}^{33}\partial_3\varphi^m) = d^1$, with $d^1 \in D(\omega)$. Then

$$\partial_3\varphi^m = \frac{p^{3\alpha\beta}}{p^{33}}[\tilde{e}_{\alpha\beta}(\zeta^m) - x_3\partial_{\alpha\beta}\zeta^m_3] - \frac{1}{p^{33}}d^1 \tag{6.35}$$

which gives

$$\varphi^m = \frac{p^{3\alpha\beta}}{p^{33}}[x_3\tilde{e}_{\alpha\beta}(\zeta^m) - x_3^2\partial_{\alpha\beta}\zeta_3^m] - \frac{x_3}{p^{33}}d^1 + d^0. \tag{6.36}$$

Since φ^m satisfies the boundary conditions $\varphi^m_{|\Gamma^+} = \varphi^m_{|\Gamma^-} = 0$, we have

$$d^0 = \frac{p^{3\alpha\beta}}{2p^{33}}\partial_{\alpha\beta}\zeta_3^m, \quad d^1 = p^{3\alpha\beta}\tilde{e}_{\alpha\beta}(\zeta^m). \tag{6.37}$$

Thus the conclusion follows.

Step (v). The function (ζ_i^m) satisfies (6.11) and (6.12).

Taking $v \in V_{KL}$ and letting $\epsilon \to 0$ in (3.11) we get

$$\int_\Omega A^{\alpha\beta kl}\tilde{K}_{kl}^m\tilde{K}_{\alpha\beta}(v)\mathrm{d}x + \int_\Omega P^{3\alpha\beta}\partial_3\varphi^m\tilde{K}_{\alpha\beta}(v)\mathrm{d}x = \xi^m\int_\Omega u_3^m \cdot v_3\mathrm{d}x. \tag{6.38}$$

Replacing u^m and $\tilde{K}_{ij}^m$ by the expressions obtained in (6.27) and (6.28), and taking v of the form

$$v_\alpha = \eta_\alpha - x_3\partial_\alpha\eta_3 \quad \text{and} \quad v_3 = \eta_3$$

with $(\eta_i) \in V_H(\omega) \times V_3(\omega)$, it is verified that (6.38) coincides with eqs (6.11) and (6.12).

Step (vi). The convergences $u^m(\epsilon) \rightharpoonup u^m$ in $H^1(\Omega)$ and $\varphi^m(\epsilon) \rightharpoonup \varphi^m$ in $L^2(\Omega)$ are strong.

To show that the family $(u^m(\epsilon))$ converges strongly to u^m in $H^1(\Omega)$, by Lemma 4.2, it is enough to show that

$$\tilde{e}_{ij}(u^m(\epsilon)) \to \tilde{e}_{ij}(u^m) \quad \text{in } L^2(\Omega). \tag{6.39}$$

Since $\tilde{e}_{i3}(u^m) = 0$ and

$$\begin{aligned}&\sum_{i,j}\|\tilde{e}_{ij}(u^m(\epsilon)) - \tilde{e}_{ij}(u^m)\|_{0,\Omega}^2\\ &= \sum_{\alpha,\beta}\|\tilde{K}_{\alpha\beta}^m(\epsilon) - \tilde{K}_{\alpha\beta}^m\|_{0,\Omega}^2 + 2\epsilon^2\sum_\alpha\|\tilde{K}_{\alpha 3}^m(\epsilon)\|_{0,\Omega}^2 + \epsilon^4\|\tilde{K}_{33}^m(\epsilon)\|_{0,\Omega}^2,\end{aligned} \tag{6.40}$$

convergence (6.39) is equivalent to showing that

$$\tilde{K}^m(\epsilon) \to \tilde{K}^m \quad \text{in } L^2(\Omega). \tag{6.41}$$

We define a norm on $(L^2(\Omega))^9 \times (L^2(\Omega))^3$ by letting for any matrix $M \in (L^2(\Omega))^9$ and any vector $\chi \in (L^2(\Omega))^3$,

$$\|(M,\chi)\| = \left\{\int_\Omega A^{ijkl}M : M\sqrt{g(\epsilon)}\mathrm{d}x + \int_\Omega \mathcal{E}^{ij}\chi_i\chi_j\sqrt{g(\epsilon)}\mathrm{d}x\right\}^{1/2}. \tag{6.42}$$

Let $X^m(\epsilon)$ be the norm of $(\tilde{K}^m(\epsilon), \epsilon\partial_1\varphi^m(\epsilon), \epsilon\partial_2\varphi^m(\epsilon), \partial_3\varphi^m(\epsilon))$ in $(L^2(\Omega))^{12}$. Using the weak convergence equation (eqs (6.25) and (6.26)) and the relation (6.28), it can be shown that

$$\lim_{\epsilon\to 0} X^m(\epsilon) = X^m = \left(\int_\Omega A^{ijkl}\tilde{K}^m : \tilde{K}^m \mathrm{d}x + \int_\Omega \mathcal{E}^{33}(\partial_3\varphi^m)^2 \mathrm{d}x\right)^{1/2} \tag{6.43}$$

which is the norm of $(\tilde{K}^m, 0, 0, \partial_3\varphi^m)$. Since we have already proved that $(\tilde{K}^m(\epsilon), \epsilon\partial_1\varphi^m(\epsilon), \epsilon\partial_2\varphi^m(\epsilon), \partial_3\varphi^m(\epsilon))$ converges weakly to $(\tilde{K}, 0, 0, \partial_3\varphi^m)$ in $(L^2(\Omega))^{12}$, we have the following strong convergences:

$$\tilde{K}^m(\epsilon) \to \tilde{K}^m \text{ strongly in } (L^2(\Omega))^9, \tag{6.44}$$

$$(\epsilon\partial_1\varphi^m(\epsilon), \epsilon\partial_2\varphi^m(\epsilon), \partial_3\varphi^m(\epsilon)) \to (0, 0, \partial_3\varphi^m) \text{ strongly in } (L^2(\Omega))^3. \tag{6.45}$$

Hence $u^m(\epsilon)$ converges strongly to u^m in $H^1(\Omega)$ and since $\varphi^m(\epsilon) - \varphi^m$ is in Ψ_{l0}, the equivalence of norms $\|\psi\|_{\Psi_l}$ and $\psi \to |\partial_3\psi|_\Omega$ in Ψ_{l0} proves that $\varphi^m(\epsilon)$ converges strongly to φ^m in $L^2(\Omega)$. ■

Equation (6.12) can be written as

$$\begin{aligned}&\int_\omega \left[\frac{2\lambda\mu}{\lambda+2\mu} e_{\rho\rho}(\zeta)\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}(\zeta)\right]\partial_\beta\eta_\alpha \mathrm{d}\omega \\ &= \int_\omega \left[\frac{2\lambda\mu}{\lambda+2\mu}(\partial_\sigma\theta\partial_\sigma\zeta_3)\delta_{\alpha\beta} + \mu(\partial_\alpha\theta\partial_\beta\zeta_3 + \partial_\beta\theta\partial_\alpha\zeta_3)\right]\partial_\beta\eta_\alpha \mathrm{d}\omega.\end{aligned} \tag{6.46}$$

Clearly, the bilinear form

$$\begin{aligned}\tilde{b}(\zeta_\alpha, \eta_\alpha) &= \int_\omega \left[\frac{2\lambda\mu}{\lambda+2\mu} e_{\rho\rho}(\zeta)\delta_{\alpha\beta} + 2\mu e_{\alpha\beta}(\zeta)\right]\partial_\beta\eta_\alpha \mathrm{d}\omega \\ &= \int_\omega \left[\frac{2\lambda\mu}{\lambda+2\mu} e_\rho\rho(\zeta)e_{\sigma\sigma}(\eta) + 2\mu e_{\alpha\beta}(\zeta)e_{\alpha\beta}(\eta)\right]\mathrm{d}\omega\end{aligned} \tag{6.47}$$

is $V_H(\omega)$ elliptic. Also for a given $\zeta_3 \in V_3(\omega)$, the functional

$$\langle\zeta_3, \eta_\alpha\rangle = \int_\omega \left[\frac{2\lambda\mu}{\lambda+2\mu}(\partial_\sigma\theta\partial_\sigma\zeta_3)\delta_{\alpha\beta} + \mu(\partial_\alpha\theta\partial_\beta\zeta_3 + \partial_\beta\theta\partial_\alpha\zeta_3)\right]\partial_\beta\eta_\alpha \mathrm{d}\omega \tag{6.48}$$

is continous on $V_H(\omega)$. Thus, given $\zeta_3 \in V_3(\omega)$, there exists a unique vector $(\zeta_\alpha) \in V_H(\omega)$ such that

$$\tilde{b}(\zeta_\alpha, \eta_\alpha) = \langle\zeta_3, \eta_\alpha\rangle. \tag{6.49}$$

We denote by $T\zeta_3 \in V_H(\omega) \times V_3(\omega)$ the vector (ζ_α, ζ_3). In particular, $T\zeta_3^m = (\zeta_\alpha^m, \zeta_3^m)$.

Substituting this in (6.11), we get

$$b(\zeta_3^m, \eta_3) = \xi^m \int_\omega \zeta^m\eta_3 \mathrm{d}\omega \quad \text{for all } \eta_3 \in V_3(\omega), \tag{6.50}$$

where

$$b(\zeta_3, \eta_3) = -\int_\omega m_{\alpha\beta}\partial_{\alpha\beta}\eta_3 \mathrm{d}\omega + \int_\omega n^\theta_{\alpha\beta}(T\zeta_3)\partial_{\alpha\beta}\theta\eta_3 \mathrm{d}\omega + \frac{2}{3}\int_\omega \frac{p^{3\alpha\beta}p^{3\rho\tau}}{p^{33}}\partial_{\rho\tau}\zeta_3\partial_{\alpha\beta}\eta_3 \mathrm{d}\omega. \tag{6.51}$$

Lemma 6.2. *The bilinear form* $b(\cdots)$ *defined by* (6.51) *is* $V_H(\omega)$*-elliptic and symmetric.*

Proof. It follows from Lemma 6.2 in [8] that the bilinear form $\tilde{b}(\cdots)$ defined by

$$\tilde{b}(\zeta_3, \eta_3) = -\int_\omega m_{\alpha\beta}(\zeta_3)\partial_{\alpha\beta}\eta_3 \mathrm{d}\omega + \int_\omega n^\theta_{\alpha\beta}(T\zeta_3)\partial_{\alpha\beta}\theta\eta_3 \mathrm{d}\omega \tag{6.52}$$

is $V_H(\omega)$-elliptic and symmetric. Hence it is clear that $b(\cdots)$ is also $V_H(\omega)$-elliptic and symmetric. ∎

Lemma 6.3. *Let* (ζ_3^m, ξ^m), $m \geq 1$, *be the eigensolutions of problem* (6.51) *found as limits of the subsequence* $(u^m(\epsilon), \xi^m(\epsilon))$, $m \geq 1$ *of eigensolutions of the problem* (3.11). *Then the sequence* $(\xi^m)_{m=1}^\infty$ *comprises all the eigenvalues, counting multiplicities, of problem* (6.51) *and the associated sequence* $(\zeta_3^m)_{m=1}^\infty$ *of eigenfunctions forms a complete orthonormal set in the space* $V_3(\omega)$.

Proof. The proof is similar to the proof of Lemma 5.4 in [3]. ∎

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