

Probabilistic representations of solutions to the heat equation

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Abstract. In this paper we provide a new (probabilistic) proof of a classical result in partial differential equations, viz. if ϕ is a tempered distribution, then the solution of the heat equation for the Laplacian, with initial condition ϕ , is given by the convolution of ϕ with the heat kernel (Gaussian density). Our results also extend the probabilistic representation of solutions of the heat equation to initial conditions that are arbitrary tempered distributions.

Keywords. Brownian motion; heat equation; translation operators; infinite dimensional stochastic differential equations.

1. Introduction

Let $(X_t)_{t \geq 0}$ be a d -dimensional Brownian motion, with $X_0 \equiv 0$. Let $\varphi \in \mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions. Let φ_t represent the unique solution to the heat equation with initial value φ , viz.

$$\partial_t \varphi_t = \frac{1}{2} \Delta \varphi_t \quad 0 \leq t \leq T; \quad \varphi_0 = \varphi.$$

It is well-known that $\varphi_t = \varphi * p_t$, where $p_t(x) = \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/2t}$ and $*$ denotes convolution. When φ is smooth, say $\varphi \in \mathcal{S}$, the space of rapidly decreasing smooth functions, then the probabilistic representation of the solution is given by the equality $\varphi(t, x) = E\varphi(X_t + x)$ and is obtained by taking expectations in the Ito formula

$$\varphi(X_t + x) = \varphi(x) + \int_0^t \nabla \varphi(X_s + x) \cdot dX_s + \frac{1}{2} \int_0^t \Delta \varphi(X_s + x) ds.$$

Such representations are well-known (see [1–4]) and extend to a large class of initial value problems, with the Laplacian Δ replaced by a suitable (elliptic) differential operator L and (X_t) being replaced by the diffusion generated by L . A basic problem here is to extend the representation to situations where φ is not smooth.

The main contribution of this paper is to give a probabilistic representation of solutions to the initial value problem for the Laplacian with an arbitrary initial value $\varphi \in \mathcal{S}'$. This representation follows from the Ito formula developed in [9], for the \mathcal{S}' -valued process $(\tau_{X_t} \varphi)$, where $\tau_x \varphi$ is the translation of φ by $x \in \mathbb{R}^d$. Our representation (Theorem 2.4) then reads, $\varphi_t = E \tau_{X_t} \varphi$ where of course φ_t is the solution of the initial value problem for the Laplacian, with initial value $\varphi \in \mathcal{S}'$. In particular, the fundamental solution $p_t(x - \cdot)$

has the representation, $p_t(x \cdot) = E\tau_{X_t}\delta_x$. However, the results of [9] only show that if $\varphi \in \mathcal{S}'_p$, then there exists $q > p$ such that the process $(\tau_{X_t}\varphi)$ takes values in \mathcal{S}'_q . Here for each real p , the \mathcal{S}_p 's are the 'Sobolev spaces' associated with the spectral decomposition of the operator $|x|^2 - \Delta$ or equivalently they are the Hilbert spaces defining the countable Hilbertian structure of \mathcal{S}' (see [6]). \mathcal{S}'_p , the dual of \mathcal{S}_p , is the same as \mathcal{S}_{-p} . Clearly it would be desirable to have the process $(\tau_{X_t}\varphi)$ take values in \mathcal{S}'_p , whenever $\varphi \in \mathcal{S}'_p$. Such a result also has implications for the semi-martingale structure of the process (τ_{X_t}) – it is a semi-martingale in \mathcal{S}'_{p+1} (Corollary 2.2) and fails to have this property in \mathcal{S}'_q for $q < p + 1$ (see Remark 5.2 of [5]).

Given the above remarks and the results of [9], the properties of the translation operators become significant. We show in Theorem 2.1 that the operators $\tau_x : \mathcal{S}_p \rightarrow \mathcal{S}_p$ for $x \in \mathbb{R}^d$, are indeed bounded operators, for any real p , with the operator norms being bounded above by a polynomial in $|x|$. The proof uses interpolation techniques well-known to analysts. Theorem 2.4 then gives a comprehensive treatment of the initial value problem for the Laplacian from a probabilistic point of view.

2. Statements of the main results

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space with a filtration (\mathcal{F}_t) satisfying usual conditions: $\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s$ and \mathcal{F}_0 contains all P -null sets. Let $(X_t)_{t \geq 0}$ be a d -dimensional, (\mathcal{F}_t) -Brownian motion with $X_0 \equiv 0$.

\mathcal{S} denotes the space of rapidly decreasing smooth functions on \mathbb{R}^d (real valued) and \mathcal{S}' its dual, the space of tempered distributions. We refer to [11] for formal definitions. For $x \in \mathbb{R}^d$, $\delta_x \in \mathcal{S}'$ will denote the Dirac distribution at x . Let $\{\tau_x : x \in \mathbb{R}^d\}$ denote the translation operators defined on functions by the formula $\tau_x f(y) = f(y - x)$ and let $\tau_x : \mathcal{S}' \rightarrow \mathcal{S}'$ act on distributions by

$$\langle \tau_x \varphi, f \rangle = \langle \varphi, \tau_{-x} f \rangle.$$

The nuclear space structure of \mathcal{S}' is given by the family of Hilbert spaces \mathcal{S}_p , $p \in \mathbb{R}$, obtained as the completion of \mathcal{S} under the Hilbertian norms $\{\|\cdot\|_p\}_{p \in \mathbb{R}}$ defined by

$$\|\varphi\|_p^2 = \sum_k (2|k| + d)^{2p} \langle \varphi, h_k \rangle^2,$$

where $\varphi \in \mathcal{S}$, and the sum is taken over $k = (k_1, \dots, k_d) \in \mathbb{Z}_+^d$, $|k| = (k_1 + \dots + k_d)$, $\langle \varphi, h_k \rangle$ denotes the inner product in $L^2(\mathbb{R}^d)$ and $\{h_k, k \in \mathbb{Z}_+^d\}$ is the ONB in $L^2(\mathbb{R}^d)$, constructed as follows: for $x = (x_1, \dots, x_d)$, $h_k(x) = h_{k_1}(x_1) \dots h_{k_d}(x_d)$. The one-dimensional Hermite functions are given by $h_\ell(s) = \frac{1}{(\sqrt{\pi}2^\ell \ell!)^{1/2}} e^{-(s^2/2)} H_\ell(s)$, where $H_\ell(s) = (-1)^\ell e^{s^2} \frac{d^\ell}{ds^\ell} e^{-s^2}$ are the Hermite polynomials. While we mainly deal with real valued functions, at times we need to use complex valued functions. In such cases, the spaces \mathcal{S}_p are defined in a similar fashion as above, i.e. as the completion of \mathcal{S} with respect to $\|\cdot\|_p$. However, in the definition of $\|\varphi\|_p^2$ above we need to replace the real L^2 inner product $\langle \varphi, h_k \rangle$ by the one for complex valued functions, viz. $\langle \varphi, \psi \rangle = \int_{\mathbb{R}^d} \varphi(x) \bar{\psi}(x) dx$ and $\langle \varphi, h_k \rangle^2$ is replaced by $|\langle \varphi, h_k \rangle|^2$. It is well-known (see [6, 7]) that $\mathcal{S} = \bigcap_p \mathcal{S}_p$, $\mathcal{S}' = \bigcup_p \mathcal{S}_p$ and $\mathcal{S}'_p =$: dual of $\mathcal{S}_p = \mathcal{S}_{-p}$. We will denote by $\langle \cdot, \cdot \rangle_p$, the inner product corresponding to the norm $\|\cdot\|_p$.

Let $(Y_t)_{t \geq 0}$ be an \mathcal{S}_p -valued, locally bounded, previsible process, for some $p \in \mathbb{R}$. Let $\partial_i : \mathcal{S}_p \rightarrow \mathcal{S}_{p-1/2}$ be the partial derivatives, $1 \leq i \leq d$, in the sense of distributions. Then since $\partial_i, 1 \leq i \leq d$ are bounded linear operators it follows that $(\partial_i Y_t)_{t \geq 0}$ is an $\mathcal{S}_{p-1/2}$ -valued, locally bounded, previsible process. From the theory of stochastic integration in Hilbert spaces [8], it follows that the processes

$$\left(\int_0^t Y_s dX_s^i \right)_{t \geq 0}, \left(\int_0^t \partial_i Y_s dX_s^i \right)_{t \geq 0}$$

are continuous \mathcal{F}_t local martingales for $1 \leq i \leq d$, with values in \mathcal{S}_p and $\mathcal{S}_{p-1/2}$ respectively. If $X_t = (X_t^1, \dots, X_t^d)$ is a continuous \mathbb{R}^d -valued, \mathcal{F}_t -semi-martingale, it follows from the general theory that the above processes too are continuous \mathcal{F}_t -semi-martingales with values in \mathcal{S}_p and $\mathcal{S}_{p-1/2}$ respectively.

Theorem 2.1. *Let $p \in \mathbb{R}$. There exists a polynomial $P_k(\cdot)$ of degree $k = 2(\lfloor p \rfloor + 1)$ such that the following holds: For $x \in \mathbb{R}^d, \tau_x : \mathcal{S}_p \rightarrow \mathcal{S}_p$ is a bounded linear map and we have*

$$\|\tau_x \varphi\|_p \leq P_k(|x|) \|\varphi\|_p$$

for all $\varphi \in \mathcal{S}_p$.

In ([9], Theorem 2.3) we showed that if $(X_t)_{t \geq 0}$ is a continuous, d -dimensional, \mathcal{F}_t -semi-martingale and $\varphi \in \mathcal{S}_p \subset \mathcal{S}'$, then the process $(\tau_{X_t} \varphi)_{t \geq 0}$ is an \mathcal{S}_q -valued continuous semi-martingale for some $q < p$. Corollary 2.2 below says that we can take $q = p - 1$.

COROLLARY 2.2

Let $(X_t)_{t \geq 0}$ be a continuous d -dimensional, \mathcal{F}_t -semi-martingale. Let $\varphi \in \mathcal{S}_p, p \in \mathbb{R}$. Then $(\tau_{X_t} \varphi)_{t \geq 0}$ is an \mathcal{S}_p -valued, continuous adapted process. Moreover it is an \mathcal{S}_{p-1} -valued, continuous \mathcal{F}_t -semi-martingale and the following Ito formula holds in \mathcal{S}_{p-1} : a.s., $\forall t \geq 0$,

$$\begin{aligned} \tau_{X_t} \varphi &= \tau_{X_0} \varphi - \sum_{i=1}^d \int_0^t \partial_i (\tau_{X_s} \varphi) dX_s^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \partial_{ij}^2 (\tau_{X_s} \varphi) d\langle X^i, X^j \rangle_s, \end{aligned} \tag{2.1}$$

where $X_t = (X_t^1, \dots, X_t^d)$ and $(\langle X^i, X^j \rangle_t)$ is the quadratic variation process between (X_t^i) and $(X_t^j), 1 \leq i, j \leq d$.

Proof. From Theorem 2.1, it follows that $(\tau_{X_t} \varphi)$ is an \mathcal{S}_p -valued continuous adapted process. By Theorem 2.3 of [9], $\exists q < p$, such that $(\tau_{X_t} \varphi)$ is an \mathcal{S}_q semi-martingale and the above equation holds in \mathcal{S}_q . Clearly each of the terms in the above equation is in \mathcal{S}_{p-1} and the result follows. \square

The next corollary pertains to the case when $(X_t) = (X_t^1, \dots, X_t^d)$ is a d -dimensional Brownian motion, $X_0 \equiv 0$. In ([5], Definition 3.1), we introduced the notion of an $\mathcal{S}'_p (= \mathcal{S}_{-p}, p > 0)$ -valued strong solution of the SDE

$$\begin{aligned} dY_t &= \frac{1}{2} \Delta(Y_t) dt + \nabla Y_t \cdot dX_t, \\ Y_0 &= \varphi, \end{aligned} \tag{2.2}$$

where $\nabla = (\partial_1, \dots, \partial_d)$ and $\Delta = \sum_{i=1}^d \partial_i^2$. There we showed that if $\varphi \in \mathcal{S}'_p$, then the above equation has a unique \mathcal{S}'_q -valued strong solution, $q \geq p + 2$. Theorem 2.1 implies that we indeed have an (unique) \mathcal{S}'_p -valued strong solution.

COROLLARY 2.3

Let $\varphi \in \mathcal{S}'_p$. Then, eq. (2.2) has a unique \mathcal{S}'_p -valued strong solution on $0 \leq t \leq T$.

Proof. By Corollary 2.2, the process $(\tau_{X_t}\varphi)$, where (X_t) is a d -dimensional Brownian motion, $X_0 \equiv 0$, satisfies eq. (2.1). Further,

$$E \int_0^T \|\tau_{X_t}\varphi\|_{-p}^2 dt = \int_0^T \int_{\mathbb{R}^d} \|\tau_x\varphi\|_{-p}^2 \frac{e^{-(|x|^2/2t)}}{(2\pi t)^{d/2}} dx dt < \infty.$$

Uniqueness follows as in Theorem 3.3 of [5]. □

We now consider the heat equation for the Laplacian with initial condition $\varphi \in \mathcal{S}_p$, for some $p \in \mathbb{R}$.

$$\begin{aligned} \partial_t \varphi_t &= \frac{1}{2} \Delta \varphi_t \quad 0 < t \leq T, \\ \varphi_0 &= \varphi. \end{aligned} \tag{2.3}$$

By an \mathcal{S}_p -valued solution of (2.3), we mean a continuous map $t \rightarrow \varphi_t : [0, T] \rightarrow \mathcal{S}_p$ such that the following equation holds in \mathcal{S}_{p-1} :

$$\varphi_t = \varphi + \int_0^t \frac{1}{2} \Delta \varphi_s ds. \tag{2.4}$$

Let $\{h_k^{p-1}\}$ be the ONB in \mathcal{S}_{p-1} given by $h_k^{p-1} = (2|k| + d)^{-(p-1)} h_k$. We then have for $p < 0$ and $t \leq T$:

$$\begin{aligned} \|\varphi_t\|_{p-1}^2 &= \sum_{|k|=0}^{\infty} \langle \varphi_t, h_k^{p-1} \rangle_{p-1}^2 \\ &= \sum_{|k|=0}^{\infty} \left\{ \langle \varphi, h_k^{p-1} \rangle_{p-1}^2 + 2 \int_0^t \langle \varphi_s, h_k^{p-1} \rangle_{p-1} d \langle \varphi_s, h_k^{p-1} \rangle_{p-1} \right\} \\ &= \|\varphi\|_{p-1}^2 + \sum_{|k|=0}^{\infty} 2 \int_0^t \langle \varphi_s, h_k^{p-1} \rangle_{p-1} \left\langle \frac{1}{2} \Delta \varphi_s, h_k^{p-1} \right\rangle_{p-1} ds \\ &= \|\varphi\|_{p-1}^2 + 2 \int_0^t \left\langle \frac{1}{2} \Delta \varphi_s, \varphi_s \right\rangle_{p-1} ds. \end{aligned}$$

It follows from the results of [5] (the monotonicity condition) that for $p < 0$,

$$2 \left\langle \frac{1}{2} \Delta \varphi, \varphi \right\rangle_{p-1} + \sum_{i=1}^d \|\partial_i \varphi\|_{p-1}^2 \leq C \|\varphi\|_{p-1}^2$$

for some constant $C > 0$ for all $\varphi \in \mathcal{S}_p$. We then get

$$\|\varphi_t\|_{p-1}^2 \leq \|\varphi\|_{p-1}^2 + C \int_0^t \|\varphi_s\|_{p-1}^2 ds.$$

Hence for the case $p < 0$, uniqueness follows from the Gronwall lemma. Uniqueness for the case $p \geq 0$, follows from uniqueness for the case $p < 0$ and the inclusion $\mathcal{S}_p \subset \mathcal{S}_q$ for $q < p$. It is well-known that the solutions of the initial value problem (2.3) in $\mathcal{S}'(\mathbb{R}^d)$ are given by convolution of φ and $p_t(x)$, the heat kernel. That these coincide (as they should) with the \mathcal{S}_p -valued solutions follows from the ‘probabilistic representation’ given by Theorem 2.4 below. Define the Brownian semi-group $(T_t)_{t \geq 0}$ on \mathcal{S} in the usual manner:

$$T_t \varphi(x) = \varphi * p_t(x) \quad t > 0, \quad T_0 \varphi = \varphi$$

where $p_t(x) = \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/2t}$, $t > 0$ and ‘*’ denotes convolution: $f * g(x) = \int_{\mathbb{R}^d} f(y)g(x - y)dy$. In the next theorem we consider standard Brownian motion (X_t) .

Theorem 2.4. (a) Let $\varphi \in \mathcal{S}_p$. Then for $t \geq 0$, the \mathcal{S}_p -valued random variable $\tau_{X_t} \varphi$ is Bochner integrable and we have

$$E \tau_{X_t} \varphi = \varphi * p_t = T_t \varphi.$$

In particular, for every $p \in \mathbb{R}$, and $T > 0$, $\sup_{t \leq T} \|T_t\| < \infty$ where $\|T_t\|$ is the operator norm of $T_t : \mathcal{S}_p \rightarrow \mathcal{S}_p$.

(b) For $\varphi \in \mathcal{S}_p$, the initial value problem (2.3) has a unique \mathcal{S}_p -valued solution φ_t given by

$$\varphi_t = E \tau_{X_t} \varphi.$$

Further $\varphi_t \rightarrow \varphi$ strongly in \mathcal{S}_p as $t \rightarrow 0$.

3. Proofs of Theorems 2.1 and 2.4

The spaces \mathcal{S}_p can be described in terms of the spectral properties of the operator H defined as follows:

$$Hf = (|x|^2 - \Delta)f, \quad f \in \mathcal{S}.$$

If $\{h_k\}$ is the ONB in $L^2(\mathbb{R}^d)$ consisting of Hermite functions (defined in §2), then it is well-known (see [10]) that

$$Hh_k = (2|k| + d)h_k.$$

For $f \in \mathcal{S}$, define the operator H^p as follows:

$$H^p f = \sum_k (2|k| + d)^p \langle f, h_k \rangle h_k.$$

Here p is any real number. For $f \in \mathcal{S}$ and $z = x + iy \in \mathbb{C}$ define $H^z f = \sum_k (2|k| + d)^z \langle f, h_k \rangle h_k$ and note that, $H^z f = H^x(H^{iy} f) = H^{iy}(H^x f)$ and $H^{iy} : L^2 \rightarrow L^2$ is an isometry. Further,

$$\begin{aligned} \|H^z f\|_0^2 &= \sum_k (2|k| + d)^{2x} \langle f, h_k \rangle^2 \\ &= \|f\|_x^2. \end{aligned}$$

The following propositions (3.1, 3.2 and 3.3) may be well-known. We include the proofs for completeness.

PROPOSITION 3.1

For any p and q , $\|H^p \varphi\|_{q-p} = \|\varphi\|_q$ for $\varphi \in \mathcal{S}$. Consequently, $H^p : \mathcal{S}_q \rightarrow \mathcal{S}_{q-p}$ extends as a linear isometry. Moreover, this isometry is onto.

Proof. Let $h_k^p = (2|k| + d)^{-p} h_k$. Then from the relation $\langle \varphi, h_k \rangle_p = (2|k| + d)^{2p} \langle \varphi, h_k \rangle$ it follows that $\{h_k^p\}$ is an ONB for \mathcal{S}_p . Let $\varphi \in \mathcal{S}$. Since

$$\begin{aligned} H^p \varphi &= \sum_k \langle \varphi, h_k \rangle (2|k| + d)^p h_k \\ &= \sum_k \langle \varphi, h_k \rangle (2|k| + d)^q h_k^{q-p}, \end{aligned}$$

we get $\|H^p \varphi\|_{q-p}^2 = \|\varphi\|_q^2$.

To show that H^p is onto, consider $\psi \in \mathcal{S}_{q-p}$,

$$\psi = \sum_k \langle \psi, h_k^{q-p} \rangle_{q-p} h_k^{q-p}.$$

Defining $\varphi =: \sum_k \langle \psi, h_k^{q-p} \rangle_{q-p} h_k^q$, we see that $\varphi \in \mathcal{S}_q$. Also,

$$H^p \varphi = \sum_k \langle \varphi, h_k^q \rangle_q h_k^{q-p} = \sum_k \langle \psi, h_k^{q-p} \rangle_{q-p} h_k^{q-p} = \psi. \quad \square$$

Let $A_j = x_j + \partial_j$ and $A_j^+ = x_j - \partial_j$, $1 \leq j \leq d$. Then it is easy to see that

$$H = \frac{1}{2} \sum_{j=1}^d (A_j A_j^+ + A_j^+ A_j).$$

For multi-indices $\alpha = (\alpha_1, \dots, \alpha_d)$ and $\beta = (\beta_1, \dots, \beta_d)$ we define

$$A^\alpha =: A_1^{\alpha_1} \dots A_d^{\alpha_d}, \quad (A^+)^{\beta} =: (A_1^+)^{\beta_1} \dots (A_d^+)^{\beta_d}.$$

For an integer $\ell \geq 0$ and $x \in \mathbb{R}$, recall that

$$h_\ell(x) = \frac{1}{(\sqrt{\pi} 2^\ell \ell!)^{1/2}} e^{-(x^2/2)} H_\ell(x),$$

where H_ℓ is the Hermite polynomial defined by

$$H_\ell(x) = (-1)^\ell e^{x^2} \frac{d^\ell}{dx^\ell} e^{-x^2}.$$

It is easily verified that

$$\begin{aligned} \left(x + \frac{d}{dx}\right) \left(e^{-(x^2/2)} H_\ell(x)\right) &= 2\ell \left(e^{-(x^2/2)} H_{\ell-1}(x)\right), \\ \left(x - \frac{d}{dx}\right) \left(e^{-(x^2/2)} H_\ell(x)\right) &= e^{-(x^2/2)} H_{\ell+1}(x). \end{aligned}$$

It then follows that

$$\begin{aligned} A_j^+ h_{k_j}(x_j) &= \sqrt{2(k_j + 1)} h_{k_j+1}(x_j), \\ A_j h_{k_j}(x_j) &= \sqrt{2k_j} h_{k_j-1}(x_j). \end{aligned}$$

Iterating these two formulas we get the following:

PROPOSITION 3.2

Let k, β and α be multi-indices such that $k_j \geq \alpha_j, j = 1, \dots, d$. Then

$$(A^+)^{\beta} h_k(x) = 2^{|\beta|/2} \left(\frac{(k + \beta)!}{k!} \right)^{1/2} h_{k+\beta}(x),$$

$$A^{\alpha} h_k(x) = 2^{|\alpha|/2} \left(\frac{k!}{(k - \alpha)!} \right)^{1/2} h_{k-\alpha}(x),$$

where $k! = k_1! \dots k_d!$.

PROPOSITION 3.3

For all $m \geq 0, \exists$ constants $C_1 = C_1(m)$ and $C_2 = C_2(m)$ such that the following hold:

(a) For all $f \in \mathcal{S}$,

$$\|f\|_m \leq C_1 \sum_{|\alpha|+|\beta| \leq 2m} \|A^{\alpha} (A^+)^{\beta} f\|_0 \leq C_2 \|f\|_m.$$

(b) For all $f \in \mathcal{S}$,

$$\|f\|_m \leq C_1 \sum_{|\alpha|+|\beta| \leq 2m} \|x^{\alpha} \partial^{\beta} f\|_0 \leq C_2 \|f\|_m.$$

Proof. (a) We can write

$$H^m = \sum_{|\alpha|+|\beta| \leq 2m} C_{\alpha\beta} A^{\alpha} (A^+)^{\beta},$$

where $C_{\alpha\beta}$ are constants. Since $\|f\|_m = \|H^m f\|_0$, the first part of the inequality follows. To show the second half of the inequality it is sufficient to show that for $f \in \mathcal{S}$ and $|\alpha| + |\beta| \leq 2m, \|A^{\alpha} (A^+)^{\beta} H^{-m} f\|_0 \leq C_{\alpha\beta} \|f\|_0$. Now,

$$\begin{aligned} \|A^{\alpha} (A^+)^{\beta} H^{-m} f\|_0^2 &= \sum_{\ell} \langle A^{\alpha} (A^+)^{\beta} H^{-m} f, h_{\ell} \rangle^2 \\ &= \sum_{\ell} \left[\sum_k (2|k| + d)^{-m} \langle f, h_k \rangle \langle A^{\alpha} (A^+)^{\beta} h_k, h_{\ell} \rangle \right]^2 \\ &= \sum_{\ell} \left[\sum_k (2|k| + d)^{-m} \langle f, h_k \rangle C_{k,\beta,\alpha} \langle h_{k+\beta-\alpha}, h_{\ell} \rangle \right]^2 \\ &= \sum_{\ell} (2|\ell + \alpha - \beta| + d)^{-2m} C_{\ell+\alpha-\beta,\beta,\alpha}^2 \langle f, h_{\ell+\alpha-\beta} \rangle^2, \end{aligned}$$

where the sum is taken over $\ell = (\ell_1, \dots, \ell_d)$ such that $\ell_j + \alpha_j - \beta_j \geq 0$ for $1 \leq j \leq d$ and where we have used Proposition 3.2 in the last but one equality above. From the same proposition, it follows that

$$(2|\alpha + \ell - \beta| + d)^{-2m} C_{\ell+\alpha-\beta,\beta,\alpha}^2$$

are uniformly bounded in ℓ for $|\alpha| + |\beta| \leq 2m$ and the second inequality in (a) follows.

(b) Since $\|f\|_m = \|H^m f\|_0$ and clearly $H^m = \sum_{|\alpha|+|\beta|\leq 2m} C_{\alpha\beta} x^\alpha \partial^\beta$, the first inequality follows. To prove the second inequality, note that

$$x_j = \frac{1}{2}(A_j + A_j^+), \quad \partial_j = \frac{1}{2}(A_j - A_j^+).$$

Hence, using $[A_j, A_k^+] = \delta_{jk} I$,

$$x^\alpha \partial^\beta = \sum_{|k|+|\ell|\leq|\alpha|+|\beta|} C_{k,\ell} A^k (A^+)^{\ell}$$

and hence by part (a) we get

$$\sum_{|\alpha|+|\beta|\leq 2m} \|x^\alpha \partial^\beta f\|_0 \leq C_1 \sum_{|k|+|\ell|\leq 2m} \|A^k (A^+)^{\ell} f\|_0 \leq C_2 \|H^m f\|_0.$$

□

Proof of Theorem 2.1. We first show that for an integer $m \geq 0$,

$$\|\tau_x \varphi\|_m \leq P_{2m}(|x|) \|\varphi\|_m,$$

where $P_{2m}(t)$ is a polynomial in $t \in \mathbb{R}$ of degree $2m$ with non-negative coefficients. This follows from Proposition 3.3:

$$\begin{aligned} \|\tau_x f\|_m &\leq C_1 \sum_{|\alpha|+|\beta|\leq 2m} \|y^\alpha \partial^\beta \tau_x f\|_0 \\ &\leq C_1 \sum_{|\alpha|+|\beta|\leq 2m} \|(y+x)^\alpha \partial^\beta f\|_0. \end{aligned}$$

The last sum is clearly dominated by $P_{2m}(|x|) \|f\|_m$ for some polynomial P_{2m} . If $m < p < m+1$, where $m \geq 0$ is an integer, we prove the result using the 3-line lemma: for $f, g \in \mathcal{S}$, let

$$F(z) = \langle H^{\bar{z}} \tau_x H^{-z} f, g \rangle_0.$$

Then from the expansion in L^2 for the RHS it is verified that $F(z)$ is analytic in $m < \operatorname{Re} z < m+1$ and continuous in $m \leq \operatorname{Re} z \leq m+1$. We will show that

$$\begin{aligned} |F(m+iy)| &\leq P_{2m}(|x|) \|f\|_0 \|g\|_0, \\ |F(m+1+iy)| &\leq P_{2(m+1)}(|x|) \|f\|_0 \|g\|_0 \end{aligned} \tag{3.1}$$

for $-\infty < y < \infty$. Hence from the 3-line lemma [12], it follows that

$$\begin{aligned} |F(p+iy)| &\leq (P_{2m}(|x|) \|f\|_0 \|g\|_0)^{m+1-p} (P_{2(m+1)}(|x|) \|f\|_0 \|g\|_0)^{p-m} \\ &\leq P_k(|x|) \|f\|_0 \|g\|_0, \end{aligned}$$

where $P_k(t)$ is a polynomial in t of degree $k = 2([p] + 1)$. It follows that

$$\|\tau_x f\|_p \leq P_k(|x|) \|f\|_p.$$

Using the fact that $\mathcal{S}_{-p} = \mathcal{S}'_p$ we get $\|\tau_x f\|_{-p} \leq P_k(|x|)\|f\|_{-p}$ for $m \leq p \leq m + 1$.

The following chain of inequalities establish the inequalities (3.1):

$$\begin{aligned} |F(m + iy)| &\leq \|H^{m-iy} \tau_x H^{-(m+iy)} f\|_0 \|g\|_0 \\ &\leq \|H^m \tau_x H^{-(m+iy)} f\|_0 \|g\|_0 \\ &= \|\tau_x H^{-(m+iy)} f\|_m \|g\|_0 \\ &\leq P_{2m}(|x|) \|H^{-(m+iy)} f\|_m \|g\|_0 \\ &= P_{2m}(|x|) \|H^{-iy} f\|_0 \|g\|_0 \\ &= P_{2m}(|x|) \|f\|_0 \|g\|_0. \end{aligned}$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.4. (a) Let $\varphi \in \mathcal{S}_p$, $p \in \mathbb{R}$. From Theorem 2.1 we have

$$\|\tau_{X_t} \varphi\|_p \leq P_k(|X_t|) \|\varphi\|_p,$$

where P_k is a polynomial. Since $E P_k(|X_t|) < \infty$, Bochner integrability follows. For $\psi \in \mathcal{S}$, $\varphi \in \mathcal{S}$,

$$\begin{aligned} \left\langle \psi, \int \tau_x \varphi p_t(x) dx \right\rangle &= \int \langle \psi, \tau_x \varphi \rangle p_t(x) dx \\ &= \int p_t(x) dx \int \psi(y) \varphi(y - x) dy \\ &= \int \psi(y) dy \int \varphi(y - x) p_t(x) dx \\ &= \int \psi(y) \varphi * p_t(y) dy \\ &= \langle \psi, \varphi * p_t \rangle. \end{aligned}$$

The result for $\varphi \in \mathcal{S}_p$ follows by a continuity argument: Let $\varphi_n \in \mathcal{S}$, $\varphi_n \rightarrow \varphi$ in \mathcal{S}_p . Hence $\varphi_n * p_t \rightarrow \varphi * p_t$ weakly in \mathcal{S}' . Hence,

$$\begin{aligned} \langle \psi, \varphi * p_t \rangle &= \lim_{n \rightarrow \infty} \langle \psi, \varphi_n * p_t \rangle \\ &= \lim_{n \rightarrow \infty} \int \psi(y) \varphi_n * p_t(y) dy \\ &= \lim_{n \rightarrow \infty} \int \langle \psi, \tau_x \varphi_n \rangle p_t(x) dx \\ &= \int \langle \psi, \tau_x \varphi \rangle p_t(x) dx \\ &= \left\langle \psi, \int \tau_x \varphi p_t(x) dx \right\rangle, \end{aligned}$$

where we have used DCT in the last but one equality. That $T_t : \mathcal{S}_p \rightarrow \mathcal{S}_p$ is a (uniformly) bounded operator follows:

$$\begin{aligned} \|T_t\varphi\|_p &= \|\varphi * p_t\|_p = \|E\tau_{X_t}\varphi\|_p \\ &= \left\| \int \tau_x\varphi p_t(x)dx \right\|_p \leq \int \|\tau_x\varphi\|_p p_t(x)dx \\ &\leq \|\varphi\|_p \int P_k(|x|)p_t(x)dx \leq C \|\varphi\|_p, \end{aligned}$$

where $C = \sup_{s \leq T} \int P_k(|x|)p_s(x)dx < \infty$.

(b) Let (X_t) be the standard Brownian motion so that $\langle X^i, X^j \rangle \equiv 0$ for $i \neq j$. Equation (2.1) then reads, for $\varphi \in \mathcal{S}_p$, $p \in \mathbb{R}$,

$$\tau_{X_t}\varphi = \varphi - \int_0^t \nabla(\tau_{X_s}\varphi) \cdot dX_s + \frac{1}{2} \int_0^t \Delta(\tau_{X_s}\varphi)ds. \quad (3.2)$$

The stochastic integral is a martingale in \mathcal{S}_{p-1} :

$$\begin{aligned} E \left\| \int_0^t \partial_i(\tau_{X_s}\varphi)dX_s^i \right\|_{p-1}^2 &\leq C_1 E \int_0^t \|\partial_i(\tau_{X_s}\varphi)\|_{p-1}^2 ds \\ &= C_1 \int_0^t \left(\int \|\partial_i(\tau_x\varphi)\|_{p-1}^2 p_s(x)dx \right) ds \\ &\leq C_2 \int_0^t \left(\int \|\tau_x\varphi\|_p^2 p_s(x)dx \right) ds \\ &\leq C_3 \|\varphi\|_p \int_0^t \left(\int P_k(|x|)p_s(x)dx \right) ds \\ &< \infty. \end{aligned}$$

Let $\varphi_t = E\tau_{X_t}\varphi$. Taking expected values in (3.2) we get eq. (2.4). Hence φ_t is the solution to the heat equation with initial value $\varphi \in \mathcal{S}_p$. The uniqueness of the solution is well-known and also follows from the remarks preceding the statement of Theorem 2.4.

To complete the proof of the theorem, we need to show that $\varphi_t \rightarrow \varphi$ in \mathcal{S}_p as $t \downarrow 0$. Let \mathcal{F} denote the Fourier transform, i.e. $\mathcal{F}f(\xi) = \int e^{-i(x \cdot \xi)} f(x)dx$ for $f \in \mathcal{S}$. Then \mathcal{F} extends to \mathcal{S}' by duality, where we consider \mathcal{S}' as a complex vector space. Since $\mathcal{F}(h_n) = (-\sqrt{-1})^n h_n$ ([10], p. 5, Lemma 1.1.3), \mathcal{F} acts as a bounded operator from \mathcal{S}_p to \mathcal{S}_p , for all p . Let $\varphi \in \mathcal{S}_p$.

$$\varphi_t - \varphi = T_t\varphi - \varphi = \mathcal{F}^{-1}(S_t(\mathcal{F}\varphi)),$$

where

$$S_t\varphi(x) = \mathcal{F}(T_t - I)\mathcal{F}^{-1}\varphi(x) = (e^{-(t/2)|x|^2} - 1)\varphi(x).$$

Clearly, $S_t : \mathcal{S}_p \rightarrow \mathcal{S}_p$ is a bounded operator and

$$\|\varphi_t - \varphi\|_p = \|S_t(\mathcal{F}\varphi)\|_p.$$

The following proposition completes the proof of the theorem.

PROPOSITION 3.4

Let $\varphi \in \mathcal{S}_p$, $p \in \mathbb{R}$. Then $\|S_t\varphi\|_p \rightarrow 0$ as $t \rightarrow 0$.

Proof. We prove the proposition by showing that (i) $S_t : \mathcal{S}_p \rightarrow \mathcal{S}_p$ are uniformly bounded, $0 < t \leq T$ and (ii) $\|S_t \varphi\|_p \rightarrow 0$ for every $\varphi \in \mathcal{S}$, as $t \rightarrow 0$. Let us assume these results for a moment and complete the proof.

Let $\epsilon > 0$ be given. By (i), there is a constant $C > 0$ such that

$$\sup_{0 \leq t \leq T} \|S_t f\|_p \leq C \|f\|_p, f \in \mathcal{S}_p.$$

Choose $\varphi \in \mathcal{S}$, so that $\|f - \varphi\|_p \leq (\frac{\epsilon}{2C})$. Then,

$$\begin{aligned} \|S_t f\|_p &\leq \|S_t(f - \varphi)\|_p + \|S_t \varphi\|_p \\ &\leq \epsilon/2 + \|S_t \varphi\|_p. \end{aligned}$$

Now choose $\delta > 0$ such that $\|S_t \varphi\|_p \leq \epsilon/2$ for all $0 \leq t < \delta$, to get $\|S_t f\|_p < \epsilon$ for all $0 \leq t < \delta$.

Since $S_t = \mathcal{F}(T_t - I)\mathcal{F}^{-1}$, (i) follows from the fact that $T_t : \mathcal{S}_p \rightarrow \mathcal{S}_p$ are uniformly bounded (Theorem 2.4a) and $\mathcal{F} : \mathcal{S}_p \rightarrow \mathcal{S}_p$ is a unitary operator. The proof of (ii) is by a direct calculation when $p = m$ is a non-negative integer.

$$\|S_t \varphi\|_m = \|H^m S_t \varphi\|_0 \leq C_1 \sum_{|\alpha|+|\beta| \leq 2m} \|x^\alpha \partial^\beta S_t \varphi\|_0.$$

Since $S_t \varphi(x) = (e^{-(t/2)|x|^2} - 1)\varphi(x)$, by Leibniz rule

$$\|x^\alpha \partial^\beta S_t \varphi\|_0 \leq \sum_{|\mu|+|\gamma| = |\beta|} C_{\mu\gamma} \|x^\alpha \partial^\mu (e^{-(t/2)|x|^2} - 1) \partial^\gamma \varphi\|_0.$$

When $\mu \neq 0$, we have

$$\|x^\alpha \partial^\mu (e^{-(t/2)|x|^2} - 1) \partial^\gamma \varphi\|_0 \leq C_2 t^{|\mu|} \|\varphi\|_m$$

and when $\mu = 0$, using the elementary inequality $|1 - e^{-u}| \leq C_3 u$, $u > 0$ we get

$$\|x^\alpha (e^{-(t/2)|x|^2} - 1) \partial^\gamma \varphi\|_0 \leq C_4 t \|\varphi\|_{m+1}.$$

Therefore, $\|S_t \varphi\|_m \leq C t \|\varphi\|_{m+1}$ for some constant C , which shows that $\|S_t \varphi\|_m \rightarrow 0$ as $t \rightarrow 0$. If p is real and m is a non-negative integer such that $p \leq m$, we have

$$\|S_t \varphi\|_p \leq \|S_t \varphi\|_m \leq C t \|\varphi\|_{m+1}$$

and so $\|S_t \varphi\|_p \rightarrow 0$ as $t \rightarrow 0$ in this case as well. □

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