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Subanalytic bundles and tubular neighbourhoods of zero-loci

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Abstract. We introduce the natural and fairly general notion of a subanalytic bundle (with a finite dimensional vector space P of sections) on a subanalytic subset X of a real analytic manifold M, and prove that when M is compact, there is a Baire subset U of sections in P whose zero-loci in X have tubular neighbourhoods, homeomorphic to the restriction of the given bundle to these zero-loci.

Keywords. Subanalytic set; subanalytic bundle; Strong Whitney stratification; Verdier stratification; tubular neighbourhood; zero-locus of subanalytic bundle; stratified transversality.

1. Introduction

In this paper, we introduce the notion of a subanalytic bundle E (generated by a finite dimensional space P of global sections) on a (not necessarily closed) subanalytic set X inside a real analytic manifold M, as a natural generalisation of real analytic bundles on real analytic spaces to the subanalytic setting. We prove (in Theorem 6.6 below) that for M compact, there exists a Baire subset U of sections in P, such that for $s \in U$, there exist tubular neighbourhoods of the zero-locus $Z = s^{-1}(0_E)$ of s in X, i.e. which are homeomorphic to the restriction of the given bundle to Z. To keep the account self-contained we recall basic facts about subanalytic sets in §2 and Strong Whitney (SW) stratifications (defined by Verdier) in §4.

We remark here that the main Theorem 6.6 would follow from Theorem 1.11 on p. 48 of [G-M]. However, the proof ('deformation to the normal bundle') sketched in [G-M] is incomplete, at least in the generality that it is stated. In this generality, the stratified submersion they construct is not proper (as was pointed out by V Srinivas), and hence Thom's First Isotopy Lemma is inapplicable. To circumvent this, we have imposed the hypothesis of compactness on the ambient real analytic manifold M containing the subanalytic set X, but no compactness assumption on X. Our hypotheses are general enough to cover most situations arising in real or complex algebraic geometry (see Example 2.2 and Remark 6.7).

2. Subanalytic sets and maps

Let M be a real-analytic manifold. We will always assume M to be connected, Hausdorff, second countable and paracompact.

DEFINITION 2.1

We say $X \subset M$ is a *subanalytic set* of M if there exists an open covering $\mathcal{U}$ of M (*not* just of X) such that for each $U \in \mathcal{U}$,

$$X \cap U = \bigcup_{i=1}^{p} (f_{i1}(A_{i1}) - f_{i2}(A_{i2})),$$

where $f_{ij} : N_{ij} \to U$ for $1 \leq i \leq p$ and $j = 1, 2$, are real analytic maps of real analytic manifolds N_{ij}, A_{ij} are closed analytic subsets of N_{ij} and $f_{ij|A_{ij}}$ are proper maps (see Proposition 3.13 in [B-M] and Definition 3.1 in [Hi]).

Example 2.2. All real (resp. complex) analytic subsets of a real (resp. complex) analytic manifold are subanalytic sets. In particular, (real or complex) algebraic subsets of a (real or complex) algebraic manifold (such as projective space, or Grassmannians) are subanalytic sets. Also since subanalytic subsets of a real analytic manifold form a Boolean algebra (see (i) of Proposition 2.7 below) all real (resp. complex) analytically (or algebraically) constructible sets in a real (resp. complex) analytic (or algebraic) manifold are subanalytic sets. In particular, all (real or complex) affine algebraic varieties are subanalytic in both affine space, and projective space. Real or complex quasiprojective varieties are subanalytic sets in the corresponding projective spaces.

Remark 2.3. A real analytic subset X (subspace) of a real analytic manifold M is a closed subset of M by definition. In particular if M is compact, so is X. By contrast, a subanalytic set X of a real analytic manifold M need not be closed, and need not be compact even if M is compact.

DEFINITION 2.4

Let $X \subset M$ and $Y \subset N$ be subanalytic sets in the real analytic manifolds M, N respectively. We say that a map $f : (X, M) \to (Y, N)$ is a *subanalytic map* if $f : X \to Y$ is a continuous map, and the graph

$$\Gamma_f := \{(x, y) \in M \times N : x \in X,\ \ y = f(x)\}$$

is a subanalytic set in $M \times N$ (see [Ha], 4.1, or Definition 3.2 in [B-M]). Note that although the map f is defined only on X, its subanalyticity depends on the ambient M, N, as we shall see in Remark 2.6 below.

Notation 2.5. If X, M, N are as above, and $f : (X, M) \to (N, N)$ is a subanalytic map, we shall write $f : (X, M) \to N$ is a subanalytic map, for notational convenience.

Remark 2.6. The subanalyticity (or analyticity) of a set, or of a map *depends on the ambient spaces* M, N. For example, $X = \{\frac{1}{n}\}_{n \in \mathbb{N}}$ is a subanalytic set in $(0, \infty)$, but not in $\mathbb{R}$. In the former, it is the zero set of the analytic function $\sin \frac{\pi}{x}$, so analytic and hence subanalytic in $(0, \infty)$. It is not subanalytic in $\mathbb{R}$ because the connected components of its germ at 0 in $\mathbb{R}$ do not form a locally finite collection (see (viii) of Proposition 2.7 below).

Similarly, the map $((0, 1), (0, 1)) \to (\mathbb{R}, \mathbb{R})$ defined by $x \mapsto \sin \frac{\pi}{x}$ is clearly subanalytic, because its graph $\Gamma := \{(x, \sin \frac{\pi}{x}) : x \in (0, 1)\}$ is an analytic (hence subanalytic) subset of $(0, 1) \times \mathbb{R}$. On the other hand, the same mapping regarded as a map $((0, 1), \mathbb{R}) \to (\mathbb{R}, \mathbb{R})$ is *not* subanalytic, since Γ is not a subanalytic subset in $\mathbb{R} \times \mathbb{R}$.

(By (i) of Proposition 2.7, if Γ were subanalytic in $\mathbb{R} \times \mathbb{R}$, its intersection with the x-axis would have to be subanalytic in $\mathbb{R} \times \mathbb{R}$. But this intersection is the set $\{(\frac{1}{n}, 0)\}_{n\in\mathbb{N}}$, which is not subanalytic because the connected components of its germ at (0, 0) in $\mathbb{R} \times \mathbb{R}$ is not a locally finite collection (see (viii) of Proposition 2.7 below.).

PROPOSITION 2.7 (Facts on subanalytic sets and maps)

We collect some well-known facts on subanalytic sets and maps:

(i) *The collection of subanalytic sets of a real analytic manifold M forms a Boolean algebra.*

(ii) *If $f : M \to N$ is a proper real analytic map of real analytic manifolds, and $X \subset M$ a subanalytic set, then $f(X)$ is subanalytic in N. In particular, if M is compact, the image $f(X)$ is subanalytic. If $g : (X, M) \to N$ is a subanalytic map and X is relatively compact in M, then $g(X) \subset N$ is subanalytic.*

(iii) *If $X \subset M$ and $Y \subset N$ are subanalytic, then $X \times Y$ is subanalytic in $M \times N$.*

(iv) *If $X \subset M$ is subanalytic, then the diagonal*

$$\Delta_X := \{(x, x) \in M \times M : x \in X\}$$

is subanalytic in $M \times M$. Thus the inclusion map $i : (X, M) \to M$ is always a subanalytic map.

(v) *Let $f : M \to N$ be a real analytic map of real analytic manifolds. If $X \subset M$ is a subanalytic set, then the restricted map $f_{|X} : (X, M) \to (N, N)$ is a subanalytic map.*

(vi) *Let $f : M \to N$ be a real analytic map of real analytic manifolds. If $Y \subset N$ is subanalytic, then $f^{-1}(Y) \subset M$ is subanalytic.*

(vii) *The closure $\overline{X}$ of a subanalytic set $X \subset M$ is also subanalytic.*

(viii) *Let $X \subset M$ be a subanalytic set in a real analytic manifold M. Then each connected component of X is also subanalytic. The collection of connected components of X is a locally finite collection in M.*

Proof. For a proof of (i), see Proposition 3.2 of [Hi] or §3 of [B-M].

For a proof of the first statement of (ii), see [Hi], Proposition 3.8. For a proof of the second statement of (ii), see the remark after Definition 3.2 in [B-M]. Easy examples can be constructed to show that the properness condition cannot be dropped from the hypothesis.

To see (iii), first note that $X \subset M$ subanalytic implies $X \times N \subset M \times N$ is also subanalytic. For, in the Definition 2.1 above, one merely takes the open covering $\mathcal{U} \times N := \{U \times N : U \in \mathcal{U}\}$, the closed sets $A_{ij} \times N \subset N_{ij} \times N$ and the maps $f_{ij} \times \mathrm{id}_N : N_{ij} \times N \to U \times N$, which are proper on $A_{ij} \times N$ since f_{ij} are proper on A_{ij}. Similarly $M \times Y \subset M \times N$ is also subanalytic. By (i) above, the intersection $X \times Y = (X \times N) \cap (M \times Y)$ is also subanalytic. This proves (iii).

To see (iv), note that by (iii) $X \times X \subset M \times M$ is subanalytic. The diagonal $\Delta_M \subset M \times M$ is subanalytic since it is analytic in $M \times M$. The intersection $\Delta_X = \Delta_M \cap (X \times X)$ is therefore subanalytic by (i). Since Δ_X is the graph of the inclusion $i : (X, M) \to M$ inside $M \times M$, it follows that i is a subanalytic map.

To see (v), we note that the graph of $f_{|X}$ in $M \times N$ is just the intersection of the graph Γ_f of f and $X \times N$ inside $M \times N$. Since Γ_f is an analytic set in $M \times N$, it is subanalytic, and since by (iii) $X \times N$ is subanalytic, their intersection is subanalytic by (i).

For (vi), let $Y \subset N$ be subanalytic and let $\mathcal{U}$ be an open covering of N such that for each $U \in \mathcal{U}$ we have

$$Y \cap U = \bigcup_{i=1}^{p} (f_{i1}(A_{i1}) - f_{i2}(A_{i2})),$$

where $f_{ij} : N_{ij} \to U$ are real analytic maps of real analytic manifolds N_{ij}, A_{ij} are closed analytic subsets of N_{ij} and $f_{ij|A_{ij}}$ are proper maps. Now take the open covering

$$f^{-1}(\mathcal{U}) := \{f^{-1}(U) : U \in \mathcal{U}\}$$

of M, and set $\widetilde{N}_{ij} := N_{ij} \times f^{-1}(U)$, with $\widetilde{f}_{ij} : \widetilde{N}_{ij} \to f^{-1}(U)$ being the second projection. Let $\widetilde{A}_{ij} = A_{ij} \times_U f^{-1}(U)$ (the fibre product, a closed analytic subset of $\widetilde{N}_{ij}$). Observe that the restriction to $\widetilde{A}_{ij}$ of the natural real analytic projection $\widetilde{f}_{ij} : \widetilde{N}_{ij} \to f^{-1}(U)$, is the 'base change' to $f^{-1}(U)$ of the restriction $f_{ij|A_{ij}} : A_{ij} \to U$, and this last map is given to be proper. Hence this restriction $\widetilde{f}_{ij|\widetilde{A}_{ij}}$ is proper. It is easily verified that

$$f^{-1}(Y) \cap f^{-1}(U) = \bigcup_{i=1}^{p} \left(\widetilde{f}_{i1}(\widetilde{A}_{i1}) - \widetilde{f}_{i2}(\widetilde{A}_{i2})\right)$$

which shows that $f^{-1}(Y)$ is subanalytic.

For (vii), see the immediate consequences following Definition 3.1 in [B-M], and also Corollary 3.2.9 in [Hi].

For (viii), see the immediate consequences following Definition 3.1 in [B-M]. (Also see Proposition 3.6 and Corollary 3.7.10 in [Hi].) □

For an analytic subset X inside a real analytic manifold M, there is a structure sheaf, making it a locally ringed space. Thus mappings (= morphisms) of real analytic spaces are easy to define, and obey the usual functorial properties. For subanalytic sets inside a real analytic manifold, we note that there is no such structure sheaf, and the definition of a subanalytic map is dependent on the ambient manifold M. Thus the notion of 'subanalytic equivalence' of subanalytic sets $X \subset M$ and $Y \subset N$ requires some care. We propose one such below, which may not be the most general, but is good enough for our purposes. We are unaware if this notion exists in the literature.

Lemma 2.8. *Let M, M_1 be real analytic manifolds. Let $X \subset M$ be a subanalytic set, and suppose $j : (X, M) \to M_1$ is a subanalytic map. Suppose there exists a* proper *real analytic map $p : M_1 \to M$ such that $p \circ j = id_X$. Then*

(i) *$j : X \to j(X)$ is a homeomorphism, and its image $X_1 := j(X) \subset M_1$ is subanalytic in M_1. Further $X \subset M$ is relatively compact if and only if $X_1 = j(X) \subset M_1$ is relatively compact.*

(ii) *For each subanalytic map $f : (X_1, M_1) \to N$, N a real analytic manifold, the composite map*:

$$f \circ j : (X, M) \to N$$

is a subanalytic map.

(iii) *For each subanalytic map $g : (X, M) \to N$, N a real analytic manifold, the composite:*

$$g \circ p_{|X_1} : (X_1, M_1) \to N$$

is a subanalytic map.

Proof. It is clear that j is a homeomorphism, with inverse $p_{|j(X)}$. Consider the real analytic map

$$\begin{aligned} \theta : M_1 &\to M \times M_1 \\ z &\mapsto (p(z), z). \end{aligned}$$

Also let

$$\Gamma_j = \{(x, j(x)) \in M \times M_1 : x \in X\}$$

be the graph of j. Since j is a subanalytic map, $\Gamma_j \subset M \times M_1$ is subanalytic.

We claim that the inverse image $\theta^{-1}(\Gamma_j) \subset M_1$ is precisely X_1. For $z \in \theta^{-1}(\Gamma_j) \Rightarrow (p(z), z) \in \Gamma_j \Rightarrow p(z) \in X$ and $z = jp(z) \Rightarrow z \in j(X)$. Conversely, $z = j(x)$ for $x \in X \Rightarrow \theta(z) = (p \circ j(x), j(x)) = (x, j(x))$ which is clearly in Γ_j. Hence the claim. Now, since θ is real analytic, and Γ_j is subanalytic, we have by (vi) of Proposition 2.7 that $X_1 = \theta^{-1}(\Gamma_j) \subset M_1$ is subanalytic. This proves the first assertion of (i). For the second assertion, note that the continuity of p implies $p(\overline{X}_1) \subset \overline{p(X_1)} = \overline{X}$. Since p is proper, it is a closed map, and so $p(\overline{X}_1)$ is a closed set containing $p(X_1) = X$. Hence $\overline{X} = p(\overline{X}_1)$. Thus if X_1 is relatively compact in M_1, X is relatively compact in M. Conversely if X is relatively compact in M, $\overline{X}_1 \subset p^{-1}(\overline{X})$ and p is proper implies that $\overline{X}_1$ is a closed subset of the compact set $p^{-1}(\overline{X})$, and hence also compact. That is, X_1 is relatively compact in M_1. This proves (i).

To see (ii), let $f : (X_1, M_1) \to N$ be a subanalytic map. Thus the graph

$$\Gamma_f = \{(x, f(x)) \in M_1 \times N : x \in X_1\}$$

is a subanalytic set. The real analytic map:

$$(p \times \text{id}) : M_1 \times N \to M \times N$$

is proper, since p is proper. But the image

$$\begin{aligned} (p \times \text{id})\left(\Gamma_f\right) &= \{(p(x), f(x)) \in M \times N : x \in X_1\} \\ &= \{(pj(y), fj(y)) \in M \times N : y \in X\} \\ &= \{(y, fj(y)) \in M \times N : y \in X\} \\ &= \Gamma_{f \circ j}, \end{aligned}$$

the graph of the composite $f \circ j$. Since $p \times \text{id}$ is a proper real analytic map, and Γ_f is subanalytic, it follows by (ii) of Proposition 2.7 that this image, the graph $\Gamma_{f \circ j}$ is subanalytic in $M \times N$, so that $f \circ j : (X, M) \to N$ is a subanalytic map. This proves (ii).

To see (iii), let $g : (X, M) \to N$ be a subanalytic map. This means that the graph $\Gamma_g \in M \times N$ is a subanalytic set. Consider the set

$$\begin{aligned}(p \times \mathrm{id})^{-1}(\Gamma_g) \cap (X_1 \times N) &= \{(m, n) \in M_1 \times N : \ (p(m), n) \in \Gamma_g, \ m \in X_1\} \\ &= \{(m, n) \in M_1 \times N : \ n = gp(m), \ m \in X_1\} \\ &= \Gamma_{g \circ p_{|X_1}}.\end{aligned}$$

Since $X_1 \times N$ is subanalytic in $M_1 \times N$ by (iii) of Proposition 2.7, Γ_g is subanalytic in $M \times N$ by definition, and $p \times \mathrm{id} : M_1 \times N \to M \times N$ is real analytic, (iii) follows from (i) and (vi) of Proposition 2.7. □

The above Lemma 2.8 shows that under the hypotheses stated there, the subanalytic sets (X, M) and (X_1, M_1) are 'equivalent' in some sense. More precisely, we make the following definition:

DEFINITION 2.9 (Pseudoequivalence of subanalytic sets)

Let $M, \ M_1$ be real analytic manifolds, with $X \subset M$ a subanalytic set and $j : (X, M) \to M_1$ a subanalytic map. If there exists a *proper real analytic map* $p : M_1 \to M$ such that $p \circ j = \mathrm{id}_X$, then we say that the subanalytic sets (X, M) and $(j(X), M_1)$ are *subanalytically pseudoequivalent*. The map j is called a *subanalytic pseudoequivalence*. We note that X and $X_1 := j(X)$ are therefore *a fortiori* homeomorphic, and also (i) of the Lemma 2.8 implies that X is relatively compact in M iff X_1 is relatively compact in M_1.

The prototypical example of such a subanalytic pseudoequivalence of interest to us in the sequel is the following.

Example 2.10 (Graph embeddings). Let $X \subset M$ be a subanalytic set, and $f : (X, M) \to N$ a subanalytic map. *Assume that N is compact*. Set $M_1 := M \times N$, and let $j : (X, M) \to M_1$ be the graph embedding defined by $j(x) = (x, f(x))$ for $x \in X$. j is a subanalytic map because its graph in $M \times M_1$ is the set $\{(x, x, f(x)) : x \in X\}$, which is precisely the intersection of the two subanalytic sets $\Delta_X \times N$ and $M \times \Gamma_f$ in $M \times M_1$, and therefore subanalytic (by Proposition 2.7(i), (iii), and (iv)). The projection $p : M \times N \to M$ is proper since N is compact, and thus we have the requirements of Definition 2.9. That is, $j : (X, M) \to (\Gamma_f, M \times N)$ is a subanalytic pseudoequivalence.

3. Analytic bundle theory

We review some basic notions of bundles from the real-analytic set-up, with a view to generalising them to the subanalytic set-up.

Suppose that $X \subset M$ is a real analytic subset (= subspace) in a real analytic manifold M. By definition, its germ at each point of M (not just X) is given by the vanishing of some ideal, so by definition X is closed in M. Then X comes equipped with a structure sheaf $\mathcal{O}_X$, whose stalk $\mathcal{O}_{X,x}$ at $x \in X$ consists of germs of real analytic functions on X at x. It is, by definition, the local ring $\mathcal{O}_{M,x}/\mathcal{I}_{X,x}$, where $\mathcal{O}_{M,x}$ is the local ring of M at x consisting of convergent power series at $x \in M$, and $\mathcal{I}_{X,x}$ is the ideal of functions in $\mathcal{O}_{M,x}$ vanishing on the germ of X at x.

Let $\pi : E \to X$ be a *real analytic vector bundle of rank k* on X. That is, its transition cocycles $g_{ij} : U_i \cap U_j \to GL(k, \mathbb{R})$ are real-analytic functions for all i, j. The sheaf (of germs of analytic sections) of an analytic vector bundle E on X is a locally free sheaf $\mathcal{E}$ of modules over the structure sheaf $\mathcal{O}_X$. Global sections of this sheaf are called global sections of E. E is said to be *generated by a vector space* $P \subset \mathcal{E}(X)$ *of global sections* if the natural sheaf map:

$$P \otimes \mathcal{O}_X \to \mathcal{E}$$

is a surjective map of $\mathcal{O}_X$-modules. This is equivalent to demanding that the evaluation map:

$$\begin{aligned} \epsilon_x : P &\to E_x \\ s &\mapsto s(x) \end{aligned}$$

is surjective for all $x \in X$, where E_x is the fibre of E at x.

It is clear from the cocycle formulation of analytic vector bundles that the natural bundle operations, such as direct sums, tensor products, homs, duals and pullbacks under analytic maps of analytic vector bundles are again analytic. Real analytic sections of the real analytic bundle $\hom(E, F)$ are defined to be real analytic bundle morphisms.

DEFINITION 3.1 (Universal exact sequence)

Let $G(n-k, P)$ denote the Grassmannian of $(n-k)$-dimensional subspaces of P, where P is a finite dimensional real vector space of dimension n. On $G(n-k, P)$, there is the short exact sequence of real analytic vector bundles and real analytic morphisms:

$$0 \to \gamma^{n-k} \to G(n-k, P) \times P \stackrel{\phi}{\to} \nu^k \to 0, \tag{1}$$

where $G(n-k, P) \times P$ is the trivial rank-n bundle on $G(n-k, P)$, γ^{n-k} is the tautological rank-$(n-k)$ real analytic bundle on $G(n-k, P)$ (having fibre V over the point $V \in G(n-k, P)$). The bundle ν^k is the universal quotient bundle of rank k on $G(n-k, P)$. P gets identified with the constant sections of $G(n-k, P) \times P$, and the bundle ν^k is generated by the global sections $(\phi \circ s)$, $s \in P$.

Indeed, the bundles and morphisms defined above are all real algebraic, and hence real analytic.

Lemma 3.2 (*Classifying maps*). *Let M be a real analytic manifold, and $X \subset M$ be a real analytic subset* (= *subspace*). *Let $\pi : E \to X$ be a real analytic vector bundle of rank k with corresponding sheaf $\mathcal{E}$. Let P be an n-dimensional real vector subspace of $\mathcal{E}(X)$. Then the following are equivalent*:

(i) *$E \to X$ is generated by the global sections P.*
(ii) *There exists a real analytic map $f : X \to G(n-k, P)$ called the* classifying map *such that the pullback of the universal short exact sequence* (1) *under f yields the short exact sequence*

$$0 \to f^*\gamma^{n-k} \to X \times P \stackrel{\epsilon}{\to} E \to 0 \tag{2}$$

on X. Here ϵ is the evaluation map $(x, s) \mapsto s(x)$.
(iii) *There is a real analytic manifold M_1, a real analytic map $j : X \to M_1$, and a proper real analytic map $p : M_1 \to M$ such that*:

(a) *$p \circ j = \mathrm{id}_X$ and $X_1 := j(X)$ is isomorphic to X as a real analytic space via j.*
(b) *There is an exact sequence of real analytic bundles on M_1:*

$$0 \to K \to M_1 \times P \stackrel{\epsilon_1}{\to} C \to 0$$

with C generated by a space of global sections equal to P, and such that the pullback of the last two terms to X via the analytic isomorphism j is the morphism of bundles $X \times P \stackrel{\epsilon}{\to} E \to 0$, defined by evaluation (i.e. $\epsilon(x,s) = \epsilon_x(s) = s(x)$).
(*In keeping with Definition* 2.9, *one may want to call j an* analytic pseudoequivalence.)

Proof.
(i) ⇒ (ii). First we note that by choosing a basis of P of real analytic sections $\{e_i\}_{i=1}^n$, that the maps:

$$\begin{aligned} \epsilon_i : X &\to E \\ x &\mapsto e_i(x) \end{aligned}$$

are all real analytic, so that the map:

$$\begin{aligned} \epsilon : X \times P &\to E \\ \left(x, s = \sum_i a_i e_i\right) &\mapsto \sum_i a_i \epsilon_i(x) = s(x) \end{aligned}$$

is also real analytic. In particular, the map $\epsilon_x : P \to E$ defined by $\epsilon_x(s) = \epsilon(x,s)$ is also real analytic.

The classifying map f is now defined as the map $x \mapsto \ker \epsilon_x$, where $\epsilon_x : P \to E_x$ is the evaluation map. To show that this map f is real analytic, it is enough to do it locally. Since E is analytically locally trivial, we may assume without loss of generality that E is trivial. In this case, the map f is just the composite:

$$X \to S\hom_{\mathbb{R}}(P, \mathbb{R}^k) \to G(n-k, P),$$

where $S\hom_{\mathbb{R}}(P, \mathbb{R}^k)$ is the open subset of $\hom_{\mathbb{R}}(P, \mathbb{R}^k)$ consisting of surjective maps, the first arrow is $x \mapsto \epsilon_x$, and the second arrow is the map $L \mapsto \ker L$. Since both these maps are real analytic, the composite is real analytic. Thus the classifying map $f : X \to G(n-k, P)$ is real analytic. So the bundle $f^*(\gamma^{n-k})$ is a real analytic bundle, and it is clear that its fibre is

$$\left(f^*\gamma^{n-k}\right)_x = \ker(\epsilon_x : P \to E_x).$$

Finally, since $f^*(G(n-k,P) \times P) = X \times P$, and $f^*\phi = \epsilon$, we have $f^*\nu^k = E$, and (ii) follows.
(ii) ⇒ (iii). This follows by using the graph of the classifying map $f : X \to G(n-k, P)$. More precisely, let us define $M_1 = M \times G(n-k, P)$, a real analytic manifold, and set

$$X_1 := \{(x,y) \in M_1 : \; y = f(x)\},$$

where f is the classifying map of (ii). Define $j : X \to X_1$ to be the graph embedding $j(x) = (x, f(x))$. Let $p : M_1 \to M$ be the first projection. Clearly $p \circ j = \mathrm{id}_X$.

That X_1 is a real analytic space in M_1 is clear from the corresponding local fact, i.e. that the germ of the graph X_1, at any point $(x, y) \in M_1$ is defined by the real analytic ideal $\mathcal{I}_{X_1,(x,y)}$ generated by $\mathcal{I}_{X,x} \otimes 1$ and (the component functions of) $(1 \otimes v) - (f(u) \otimes 1)$ in the completed tensor product

$$\mathcal{O}_{M_1,(x,y)} = \mathcal{O}_{M,x} \widehat{\otimes} \mathcal{O}_{G(n-k,P),y},$$

where u and v are local coordinates around $x \in M$ and $y \in G(n-k, P)$. The graph embedding:

$$\begin{aligned} j : X &\to M_1 \\ x &\mapsto (x, f(x)) \end{aligned}$$

is a real analytic map, with image X_1. Since the first projection $p : M_1 \to M$ provides the analytic inverse to $j : X \to X_1$, the map j is an analytic isomorphism of the real analytic spaces X and X_1. This proves (a).

If we let $p_2 : M_1 \to G(n-k, P)$ denote the real analytic map defined by the second projection, and set $K := p_2^* \gamma^{n-k}$, and $C := p_2^* \nu^k$, we have the p_2-pullback of the universal exact sequence of (1):

$$0 \to K \to M_1 \times P \overset{\epsilon_1}{\to} C \to 0,$$

where $\epsilon_1 := p_2^* \phi$. This is a short exact sequence of real analytic vector bundles on M_1. If we pullback this sequence via the real analytic isomorphism j, the fact that $p_2 \circ j = f$ implies the short exact sequence (2) on X. Thus we have the assertion (b).

(iii) $\Rightarrow$ (i). We pull back the given exact sequence of real analytic bundles on M_1 of (iii) to X via j. It continues to be an exact sequence of real analytic bundles on X. If we let ϵ denote the pullback morphism $j^* \epsilon_1$, and set $j^* C = E$, the last two terms of this pulled back exact sequence read:

$$X \times P \overset{\epsilon}{\to} E \to 0$$

which shows that the ϵ images of the constant sections $x \mapsto (x, s)$ generate E, and hence (i) follows. □

Remark 3.3. The equivalence of (i) and (iii) of the above lemma shows that in considering analytic bundles generated by global sections on an analytic subset $X \subset M$, we lose no generality in assuming (up to analytic pseudoequivalence) that such bundles are *restrictions* of similar (viz. generated by global sections) real analytic bundles on the *ambient smooth* M to X. We will generalise all this to a subanalytic setting in the next section.

4. Subanalytic bundles and sections

Let $X \subset M$ a subanalytic set in M, with M a real analytic manifold. Note that unlike analytic subsets, X need not be closed anymore. We need to define 'subanalytic bundles' in some reasonable fashion, but are hindered by the fact that there is no structure sheaf for a subanalytic space.

Motivated by the equivalence of (i) and (ii) of Lemma 3.2 of the last section, we *propose* the following:

DEFINITION 4.1

A *subanalytic real vector bundle E of rank k generated by an n-dimensional vector space P of global sections* on a subanalytic set $X \subset M$ is the pullback $f^*\nu^k$, where ν^k is the universal rank k analytic quotient bundle on $G(n-k, P)$ defined in Definition 3.1, and $f : (X, M) \to G(n-k, P)$ is a *subanalytic* map (see Definition 2.4). Then, by pulling back the last two terms of the exact sequence (1), there is, by definition, an exact sequence:

$$X \times P \stackrel{\epsilon}{\to} E \to 0$$

of continuous vector bundles on X.

Remark 4.2. If $X \subset M$ is an analytic subset, and E is a real analytic vector bundle of rank k, generated by an n-dimensional vector space of real analytic global sections P, we have by (i)$\Rightarrow$ (ii) of Lemma 3.2 that there is a real analytic classifying map $f : X \to G(n-k, P)$. Since $X \subset M$ is an analytic subset, it is subanalytic in M, and since the graph Γ_f is an analytic space in $M \times G(n-k, P)$, it is subanalytic, so that f is subanalytic. Thus (ii) of Lemma 3.2 shows that the definition above is a generalisation of the real analytic setup to the subanalytic setting.

Example 4.3. (A typical subanalytic bundle). Let $X \subset M$ be a subanalytic set, M a real-analytic manifold, and let $\pi : E \to M$ be a *real analytic vector bundle* generated by a vector space P of global real analytic sections. Thus by definition, there is an exact sequence of analytic bundles and morphisms:

$$M \times P \to E \to 0.$$

By the remark above, there is a real analytic classifying map $f : M \to G(n-k, P)$ such that the exact sequence above is the pullback via f of the universal sequence

$$G(n-k, P) \times P \to \nu^k \to 0.$$

If one restricts f to X, the restricted map $f_{|X} : (X, M) \to G(n-k, P)$ is a subanalytic map, by (v) Proposition 2.7. Thus the restricted bundle $E_{|X}$ (which is the pullback via $f_{|X}$ of the above sequence on $G(n-k, P)$) is by definition a subanalytic bundle generated by global sections P.

Again, motivated by the equivalence of (ii) and (iii) of Lemma 3.2, we would like to assert that *up to a subanalytic pseudoequivalence* of X (see Definition 2.9), the above Example 4.3 of a subanalytic bundle generated by global sections P is the *only* one. More precisely, in complete analogy with Lemma 3.2 of the analytic setting in §3, we have the following:

Lemma 4.4. *The following are equivalent*:

(i) *E is a rank k subanalytic vector bundle on X generated by an n-dimensional vector space P of global sections in the sense of Definition* 4.1 *above.*
(ii) *There exists a real analytic manifold M_1, a subanalytic map $j : (X, M) \to M_1$ and a proper map $p : M_1 \to M$ such that*:

   (a) *$p \circ j = \mathrm{id}_X$, and hence $j : (X, M) \to (j(X), M_1)$ is a* subanalytic pseudoequivalence *in the sense of Definition* 2.9.

(b) *There is an exact sequence of* real analytic bundles *on the real analytic manifold* M_1:

$$0 \to K \to M_1 \times P \stackrel{\epsilon_1}{\to} C \to 0$$

with C *generated by a space of global sections equal to* P, *and such that the pullback of the last two terms via* j *to* X, *is a surjective morphism of bundles* $X \times P \stackrel{\epsilon}{\to} E \to 0$.

Proof.
(i) $\Rightarrow$ (ii). As in Lemma 3.2, set $M_1 = M \times G(n-k, P)$. By definition of a subanalytic map, the graph of the subanalytic classifying map f:

$$X_1 := \{(x, y) \in M_1 : \ y = f(x)\}$$

is a subanalytic space in M_1. By Example 2.10, the graph embedding:

$$\begin{aligned} j : (X, M) &\to M_1 \\ x &\mapsto (x, f(x)) \end{aligned}$$

is a subanalytic map. $G(n-k, P)$ is compact, so by Example 2.10, the map $j : (X, M) \to (X_1, M_1)$ is a subanalytic pseudoequivalence in the sense of Definition 2.9.

The proof of (b) of (ii) is exactly as in the proof of part (b) of (iii) in Lemma 3.2, and therefore omitted.
(ii) $\Rightarrow$ (i). From the given exact sequence of real analytic bundles on M_1:

$$0 \to K \longrightarrow M_1 \times P \stackrel{\epsilon_1}{\longrightarrow} C \to 0 \tag{3}$$

it follows from (i) and (ii) of Lemma 3.2 that there is a real analytic classifying map $g : M_1 \to G(n-k, P)$ (defined by $g(x) = K_x \subset P$) such that the above exact sequence is the g-pullback of the universal exact sequence (1) on $G(n-k, P)$. If one composes this classifying map with the the subanalytic map $j : (X, M) \to M_1$, which is given to be a subanalytic pseudoequivalence of (X, M) with (X_1, M_1), we have:

1. $X_1 = j(X)$ is a subanalytic set in M_1 by (i) of Lemma 2.8, and
2. The composite $g \circ j : (X, M) \to G(n-k, P)$ is a subanalytic map, by (ii) of Lemma 2.8 (note that $g : M_1 \to G(n-k, P)$ a real analytic map implies that $g : (X_1, M_1) \to G(n-k, P)$ is subanalytic by (v) of Proposition 2.7).

Clearly this subanalytic composite map:

$$g \circ j : (X, M) \to G(n-k, P)$$

is the classifying map for the bundle j^*K. Thus, letting $\epsilon := j^*\epsilon_1 = j^*g^*\phi$, and taking the j^* of the last two terms of the given exact sequence (3) of real analytic bundles on M_1, we have the exact sequence

$$X \times P \stackrel{\epsilon}{\to} E \to 0 \tag{4}$$

of bundles on X, where $E := j^*C$. Since the exact sequence (3) is the g-pullback of the universal exact sequence (1), the above exact sequence (4) is the pullback of the last two terms of universal sequence (1) via the above subanalytic map $f := g \circ j : (X, M) \to G(n-k, P)$. In particular, E is a subanalytic bundle of rank k generated by the global sections P, by Definition 4.1. This implies (i). □

Remark 4.5. In complete analogy with Remark 3.3, we observe that (ii) of the above Lemma says that in considering subanalytic bundles on a subanalytic set $X \subset M$, generated by a vector space P global sections, we lose no generality (up to subanalytic pseudoequivalence) in assuming that they are *restrictions of real analytic bundles* (generated by the vector space of analytic sections $= P$) on the *ambient smooth* M.

5. Strong Whitney stratifications and transversality

In this section, we recall some known definitions and results from the theory of stratifications of subanalytic sets. The general references for this section are the papers by Verdier [Ve] and Hironaka [Hi].

DEFINITION 5.1 (The Verdier condition)

Let M and M' be two locally closed C^∞ submanifolds of some real finite dimensional inner product space E, and such that $M \cap M' = \phi$, with $M' \subset \overline{M}$.

The *property (w)* or *Verdier condition* (see §1.4 in [Ve]) for the pair (M, M') is the following:

For each $y \in \overline{M} \cap M'$, there exists a neighbourhood U of y in E and a constant $C \in \mathbb{R}_+^*$ such that, for all $y' \in U \cap M'$ and $x \in U \cap M$, we have

$$\delta(T_{y'}(M'), T_x(M)) \leq C\|x - y'\|,$$

where δ is the distance between two vector subspaces F and G of E (see [Ve], §1.1) defined by

$$\delta(F, G) = \sup_{x \in F, \|x\|=1} \operatorname{dist}(x, G),$$

$\operatorname{dist}(x, G)$ being the Euclidean distance (in the given norm $\| \ \|$ on E) between x and G in E (viz., $d(x, G) = \|\pi_{G^\perp}(x)\|$). This property is invariant under smooth local diffeomorphisms of E, and hence makes sense for M, M' contained in a smooth manifold E.

Now we can define stratifications and *Verdier* (or *Strong Whitney* stratifications).

DEFINITION 5.2 (Stratification and Strong Whitney stratification) (see [Ve], §2.1)

Let M be a real analytic manifold, countable at ∞ (i.e. M is the countable union of compact sets).

A *stratification* $\mathcal{S}$ of M is a partition of M as $M = \cup_\alpha M_\alpha$ satisfying:

(SW1) $M_\alpha \cap M_\beta = \phi$ for $\alpha \neq \beta$. Each M_α is a locally closed real analytic submanifold of M, smooth, connected, subanalytic in M.
(SW2) The family M_α is locally finite.
(SW3) The family M_α has the boundary property: i.e. $\overline{M}_\alpha \cap M_\beta \neq \phi$ implies $M_\beta \subset \overline{M}_\alpha$.

The M_α are called the *strata* of the stratification $\mathcal{S}$.

A *Strong Whitney stratification* (or *SW-stratification*) $\mathcal{S}$ of M is a stratification $M = \cup_\alpha M_\alpha$ with the following additional property:

(SW4) If $\overline{M}_\alpha \supset M_\beta$ and if $\alpha \neq \beta$, then the pair (M_α, M_β) has the property (w) of 5.1 above.

More generally, if X is a subset of M, we may define a *stratification* $\mathcal{S}$ of X to be a partition of X into $X = \cup_\alpha X_\alpha$ where X_α satisfy (SW1) through (SW3) above. Similarly, a Strong Whitney or SW-stratification of X to be a stratification of X having the additional property (SW4). Note that all the conditions (such as subanalyticity, real analyticity, or the Verdier condition) *are required to hold inside the ambient real analytic manifold* M.

Finally, we say a stratification $\mathcal{S}'$ of X is *finer* or a *refinement* of the stratification $\mathcal{S}$ if each stratum X_α of the stratification $\mathcal{S}$ is a union of strata X'_β of the stratification $\mathcal{S}'$.

Remark 5.3.

(i) The condition (SW4) above is stronger than Whitney's condition (b). See Theorem 1.5 (due to Kuo) in [Ve].
(ii) Since strata are disjoint sets, and constitute a locally finite collection, a compact set $K \subset M$ can intersect only finitely many strata. Otherwise, choose a point in $x_i \in K \cap M_i$ for each α in some countably infinite index set of strata M_i, and observe that it will have a limit point $x \in K$ since K is compact. Every neighbourhood of this x will meet infinitely many M_i, contradicting local finiteness. In particular, if M itself is compact, M is automatically countable at infinity, and the number of strata in *any* stratification of M is finite.
(iii) If M is SW-stratified as above by strata $\{M_\alpha\}$, and $X \subset M$ is any subset which is a union of strata, then X is also SW-stratified (by the strata of which it is a union).

The following is a key proposition due to Verdier ([Ve], Theorem 2.2).

Theorem 5.4 (**Existence of arbitrarily fine SW-stratifications**). *Let M be a real analytic manifold, and Y_β a locally finite family of subanalytic subsets of M. Then there exists a Strong Whitney* (*or SW*)*-stratification of M such that each Y_β is a union of strata. If M is compact, then the stratification is* finite, *i.e. the number of strata is finite.*

In particular, for any subanalytic set $X \subset M$, there is a SW-stratification of M such that X becomes a union of strata, and hence SW-stratified with the induced stratification. Note that the assertion for M compact follows from (ii) of Remark 5.3 above. The analogous statement proving the existence of a Whitney-(b) stratification for the above setting is Theorem 4.8 of [Hi].

DEFINITION 5.5 (Stratified transversality) (see 1.3.1 on p. 38 in [G-M])

Let $f : M \to N$ be a smooth map of real analytic manifolds. Let $X \subset M$ and $Y \subset N$ be SW-stratified subsets, with strata $\{X_\alpha\}$ and $\{Y_\beta\}$ respectively. We say that $f_{|X} : X \to N$ is *transverse to* Y (denoted by $f_{|X} \pitchfork Y$) if:

(i) for each stratum X_α of X, $f_{|X_\alpha} : X_\alpha \to N$ is a smooth map, and
(ii) for each stratum X_α of X, and each stratum Y_β of Y, the map $f_{|X_\alpha} : X_\alpha \to N$ is transverse to Y_β.

Note that the above notion does not depend on the analytic, but just the underlying smooth structures.

Lemma 5.6. *Let M be a real analytic manifold with a SW-stratification $\mathcal{S}$, and let $X \subset M$ be a subanalytic set which is a union of strata. Let X_α be the strata of X, and let $\mathcal{S}_X$*

denote this induced stratification of X. Let $N \subset M$ be a smooth real analytic submanifold, subanalytic in M. Then if the inclusion map $i : N \to M$ is transverse to the stratification $\mathcal{S}_X$, the intersected stratification $\mathcal{S}_X \cap N$ of $X \cap N$ is defined by

$$\{M_{\alpha\beta} :\ M_{\alpha\beta} \textit{ is a connected component of } X_\alpha \cap N \textit{ for some } \alpha\}.$$

This stratification $\mathcal{S}_X \cap N$ defines a SW-stratification of $X \cap N$. In particular, each $X_\alpha \cap N$ is a union of strata $M_{\alpha\beta}$ of $\mathcal{S}_X \cap N$ of the same dimension.

Proof. Since each X_α is subanalytic and locally closed, and N being a real analytic submanifold, is also locally closed and given to be subanalytic, the sets $X_\alpha \cap N$ are subanalytic and locally closed, by (i) of Proposition 2.7. Thus the connected components $M_{\alpha\beta}$ of $X_\alpha \cap N$ are locally closed, and subanalytic by (viii) of Proposition 2.7. Since X_α meets N transversely for each α, the $M_{\alpha\beta}$ are real analytic submanifolds of M. Hence (SW1) follows. (SW2) (local finiteness) also follows from the second statement of (viii) in 2.7.

To see (SW3), we need to show that if $M_{\alpha\beta} \cap \overline{M}_{\delta\gamma} \neq \phi$, then $M_{\alpha\beta} \subset \overline{M}_{\delta\gamma}$. Clearly, $\overline{M}_{\delta\gamma} \cap M_{\alpha\beta}$ is closed in $M_{\alpha\beta}$. We claim that it is also open in $M_{\alpha\beta}$.

For, let $x \in \overline{M}_{\delta\gamma} \cap M_{\alpha\beta}$. This last intersection being non-empty implies by (SW3) applied to X_α and $\overline{X}_\delta$, that $x \in X_\alpha \subset \overline{X}_\delta$. By the topological local triviality of Whitney (b) stratifications (see [G-M], §1.4 or [Ma], Corollary 8.4), and the remark in Definition 5.2 that SW-stratifications are Whitney (b) stratifications, there exists a connected neighbourhood U of x in X_α such that a neighbourhood W of x in $\overline{X}_\delta$ is of the form $W = U \times N(x)$, where $N(x)$ is the *normal slice to x* inside $\overline{X}_\delta$. Note that $N(x)$, being the cone on the link $L(x)$, is connected.

Since the real analytic submanifold N intersects each stratum transversely, it follows by intersecting everything with N, that there is a neighbourhood $U \cap N$ of x in $X_\alpha \cap N$ such that the neighbourhood $W \cap N$ of x in $\overline{X}_\delta \cap N$ is of the form $W \cap N = (U \cap N) \times N(x)$. If we choose U' to be the connected component of $U \cap N$ containing x, and W' to be the connected component of $W \cap N \cap \overline{M}_{\delta\gamma}$ containing x, it follows that W' is of the form $W' = U' \times N'(x)$, where $N'(x)$ is the intersection of $N(x)$ and $\overline{M}_{\delta\gamma}$. Thus $U' \subset \overline{M}_{\delta\gamma} \cap M_{\alpha\beta}$, and so this last intersection is open in $M_{\alpha\beta}$, and our claim is proven.

The connectedness of $M_{\alpha\beta}$ implies that the open and closed subset $\overline{M}_{\delta\gamma} \cap M_{\alpha\beta}$ is equal to $M_{\alpha\beta}$, and (SW3) follows for our stratification $\mathcal{S} \cap N$.

To prove the Verdier condition (SW4) for $\mathcal{S} \cap N$, we first need the following linear algebraic claim.

Claim. Let $(E, \langle -, - \rangle)$ be a finite dimensional inner product space, with a fixed linear subspace L. For every subspace G of E intersecting L transversely (viz. $G + L = E$) there exists a positive constant $C(G)$, depending continuously on G, such that for all linear subspaces $F \subset E$, we have

$$\delta(F \cap L, G \cap L) \leq C(G)\delta(F, G), \tag{5}$$

where δ is the Verdier distance with respect to $\langle -, - \rangle$ introduced in Definition 5.1.

First note that $L + G = E$ implies $G^\perp \cap L^\perp = \{0\}$, where $\perp$ denotes orthogonal complement with respect to $\langle -, - \rangle$. Thus the orthogonal projection π_G onto G is an isomorphism when restricted to $L^\perp$. Hence the subspace $\pi_G(L^\perp)$ has dimension $= \dim\ L^\perp = \operatorname{codim} L$ which, by transversality of G and L, is precisely the codimension of $G \cap L$ in G. Also, for $y \in G \cap L,\ z \in L^\perp$, we have $\langle y, \pi_G(z) \rangle = \langle \pi_G(y), z \rangle = \langle y, z \rangle = 0$, thus implying that

$\pi_G(L^\perp)$ is orthogonal to $G \cap L$. Hence $L_1 := \pi_G(L^\perp)$ is the orthogonal complement of $G \cap L$ in G.

Define a new inner product $\langle -, - \rangle'$ on E by setting $\langle y, z \rangle' = \langle y, z \rangle$ for $y, z \in L$, $\langle y, z \rangle' = \langle y, z \rangle$ for $y, z \in L_1$, and $\langle L_1, L \rangle' = 0$. The orthogonal complement of a subspace W with respect to $\langle -, - \rangle'$ will be denoted $W^\top$. Then, with respect to this new inner product $\langle -, - \rangle'$, we have an orthogonal direct sum decomposition

$$E = L_1 \oplus' L,$$

where '$\oplus'$' signifies that the decomposition is orthogonal with respect to $\langle -, - \rangle'$. Also, $L^\top = L_1 \subset G$. Thus $G^\top \subset L$. Hence, by counting dimensions, and noting that $\left\langle G \cap L, G^\top \right\rangle' = 0$, it follows that

$$L = (G \cap L) \oplus' G^\top.$$

Hence for any $x \in F \cap L$, we will have $\pi_{G^\top}(x) = \pi_{(G \cap L)^\top}(x)$. Thus, denoting the Verdier distance with respect to $\langle -, - \rangle'$ by δ', we have

$$\begin{aligned} \delta'(F \cap L, G \cap L) &= \sup_{\|x\|'=1, x \in F \cap L} \|\pi_{(G \cap L)^\top}(x)\|' \\ &= \sup_{\|x\|'=1, x \in F \cap L} \|\pi_{G^\top}(x)\|' \\ &\leq \sup_{\|x\|'=1, x \in F} \|\pi_{G^\top}(x)\|' \\ &= \delta'(F, G). \end{aligned} \tag{6}$$

(All orthogonal projections in the last six lines are with respect to $\langle -, - \rangle'$.)

Now, by the above definition of $\langle -, - \rangle'$, it follows that there is a positive constant $A(G)$ depending continuously on G such that for every $x \in E$:

$$\frac{1}{A(G)} \|x\| \leq \|x\|' \leq A(G) \|x\|$$

from which it is easy to deduce that there is an inequality

$$\delta'(F, G) \leq C(G) \delta(F, G),$$

where $C(G)$ is a positive constant depending continuously on G. Also since $\langle -, - \rangle$ and $\langle -, - \rangle'$ agree on L, it follows that $\delta'(F \cap L, G \cap L) = \delta(F \cap L, G \cap L)$. Hence the inequality (5) follows by applying (6), and hence the Claim.

If $y \in M_{\alpha\beta} \subset \overline{M}_{\delta\gamma}$, then $y \in X_\alpha$ and we have by (SW4) applied to $y \in X_\alpha \subset \overline{X}_\delta$, that there is some neighbourhood U of y such that for all $y' \in U \cap X_\alpha$ and $x \in X_\delta \cap U$ we have

$$\delta(T_{y'}(X_\alpha), T_x(X_\delta)) \leq C \|x - y'\|.$$

Let U' be the connected component of $U \cap M_{\alpha\beta}$ containing y, and simultaneously locally trivialise on U the manifold pair (M, N) so that the bundle pair $(TM, TN)_{|U}$ is isomorphic to $U \times (E, L)$. Set $F := T_{y'}(X_\alpha)$ and $G := T_x(X_\delta)$, so that $T_{y'}(M_{\alpha\beta}) = F \cap L$ and $T_x(M_{\delta\gamma}) = G \cap L$, by the fact that N meets X_α and X_δ transversely. For $x \in U$, one can find a bound C for the constant $C(G)$ in (5) by the continuity of $C(G)$ in G. Now we apply the inequality (5) above to get the Verdier condition (w) on U'. Thus (SW4) is verified for the stratification $\mathcal{S} \cap N$. This proves the lemma. ☐

Remark 5.7. We also need a straighforward extension of Proposition 5.6. Namely, in the above situation let N be a closed connected subset, which is a real analytic submanifold with real analytic boundary ∂N. This can be regarded as a SW-stratified subset of M with two strata, viz. $N^\circ := N \setminus \partial N$ and ∂N. For this stratification, (SW1), (SW2) and (SW3) are straightforward, and (SW4) holds because at any point $x \in \partial N$, the germ of the triple $(N, \partial N, M)$ is (by definition) real analytically isomorphic to $(\mathbb{R}^{k-1} \times \mathbb{R}_+, \mathbb{R}^{k-1}, \mathbb{R}^n)$, which obviously satisfies (SW4). Let $X \subset M$ be a SW-stratified subset, such that $X_\alpha \pitchfork N^\circ$ and $X_\alpha \pitchfork \partial N$ for each stratum X_α of X. Then the intersection stratification on $X \cap N$ defined by taking the connected components of all the intersections $X_\alpha \cap N^\circ$ and $X_\alpha \cap \partial N$ is an SW-stratification of $X \cap N$. The proof is exactly the same as in the case of 5.6 above.

DEFINITION 5.8 (Stratified submersion) (see [G-M], §1.5 on p. 41 and [Ve], 3.2)

Let $f : M \to N$ be a smooth map, and let $X \subset M$ be a SW-stratified subset, with strata $\{X_\alpha\}$. We say f is a *stratified submersion* if for each stratum X_α of X, the restriction $f_{|X_\alpha} : X_\alpha \to N$ is smooth and a submersion.

Again, the above notion depends only on the underlying smooth structures. In [Ve], Definition 3.2, such a map is referred to as a map *'transverse to the stratification'* (on X). The following is a key proposition due to Verdier.

Theorem 5.9 (First Isotopy Lemma). *(See* [Ve], *Theorem* 4.14*). Suppose that X is a* closed *SW-stratified subset in M, with stratification $\mathcal{S}$. Let N be a smooth real analytic manifold, $f : M \to N$ be a real analytic map*, proper on X, *and a stratified submersion. Let $y_0 \in N$, and M_0 and X_0 be the fibres over y_0 of M and X respectively.* (*Note that by the hypothesis, M_0 meets all the strata of X transversely, so that $X_0 = X \cap M_0$ acquires an induced stratification $\mathcal{S}_0$ via the connected components of $X \cap M_0$, as in* 5.6 *above*). *Then there exists an open neighbourhood V of y_0 in N, and a homeomorphism of the stratified spaces $(X \cap f^{-1}(V), \mathcal{S})$ onto $(X_0 \times V, \mathcal{S}_0 \times V)$ preserving the stratifications and compatible with the projections to V.* (*Again note that $X \cap f^{-1}(V)$, being an open subset of X, has a natural induced stratification, also denoted by $\mathcal{S}$, from X, and $X_0 \times V$ has a natural product stratification $\mathcal{S}_0 \times V$.*)

In fact, in Theorem 4.14 of [Ve], Verdier proves that the homeomorphism above is a 'rugeux' (coarse) homeomorphism, which is slightly stronger than saying that it is a homeomorphism. We do not need this stronger statement. The corresponding statement for Whitney (b) stratifications is Thom's First Isotopy Lemma, and due to Thom and Mather (see [7] in [Ve]).

6. A tubular neighbourhood theorem for subanalytic bundles

Let X be a topological space, and E be a continuous real vector bundle of rank k on X. Let $\| \ \|$ be some continuous bundle metric, and for $\epsilon > 0$ let $E(\epsilon)$ denote the ϵ-disc bundle of E with respect to this bundle metric. Denote the zero-section of this bundle by 0_E.

DEFINITION 6.1 (Tubular neighbourhoods)

Let $\pi : E \to X$ be as above and let $s : X \to E$ be a continuous section. We will say that *s has a tubular neighbourhood in X* if there exists a neighbourhood V of $s^{-1}(0_E)$ in X, and $\epsilon > 0$, and a homeomorphism

$$\Phi : V \to E(\epsilon)_{|s^{-1}(0_E)}$$

such that the composite

$$s^{-1}(0_E) \hookrightarrow V \xrightarrow{\Phi} E(\epsilon)_{|s^{-1}(0_E)}$$

is the map $x \to s(x) = 0_x$ for all $x \in s^{-1}(0_E)$.

Remark 6.2.

(i) If $E \to X$ and $F \to X$ are isomorphic real vector bundles, via a continuous bundle isomorphism $\tau : E \to F$, then clearly a section s of E will have a tubular neighbourhood in X iff the section $\tau \circ s$ of F has a tubular neighbourhood in X. We just need to observe that $s^{-1}(0_E) = (\tau \circ s)^{-1}(0_F)$, and that τ will induce a homeomorphism of the disc bundle $F(\epsilon)$ with the disc bundle $E(\epsilon)$ where E is given the τ-pullback bundle metric from F.

(ii) More generally, if there is a bundle diagram:

$$\begin{array}{ccc} E_1 & \xrightarrow{\tau} & E_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$

with f a *homeomorphism*, and τ a continuous vector bundle equivalence, then the section s_2 of E_2 has a tubular neighbourhood in X_2 iff the section $f^*s_2 := \tau^{-1} \circ s_2 \circ f$ of E_1 has a tubular neighbourhood in X_1. This is clear because we will have a continuous bundle equivalence $E_1 \to f^*E_2$ on X, the statement (i) above, and the fact that the homeomorphism f induces a homeomorphism of any neighbourhood V of $s_2^{-1}(0_{E_2})$ with the neighbourhood $f^{-1}(V)$ of $s_1^{-1}(0_{E_1})$.

Before proving the main result, we need some lemmas.

Lemma 6.3 (*Transversality to the* 0*-section in smooth case*) (*see Theorem* 1.3.6 *on p.* 39 *in* [G-M]). *Let M be a real analytic manifold, and $\pi : E \to M$ be a real analytic vector bundle of rank k on M, generated by an n-dimensional vector space P of real analytic global sections. Let $Y \subset M$ be a real analytic submanifold* (*in particular, locally closed in M*). *Then there exists a Baire subset* (*i.e. whose complement is of measure* 0, *in particular, dense*) *$U \subset P$ such that $s \in U$ implies that $s_{|Y} : Y \to E$, is transverse to the zero-section 0_E of E.*

Proof. By definition, and (ii) of Lemma 3.2, we have the exact sequence of real analytic bundles on M:

$$M \times P \xrightarrow{\epsilon} E \to 0$$

which is the pullback via the analytic classifying map $f : M \to G(n-k, P)$ of the last two terms of the universal exact sequence (1) on $G(n-k, P)$. In particular $\epsilon = f^*\phi$ above is real analytic. The section $s \in P$ is viewed as an analytic section of E by taking the map $\epsilon_s = \epsilon(-, s) : M \to E$ of E. Since ϵ is a smooth epimorphism of bundles, it is a smooth submersion.

Restricting the above real analytic map ϵ to Y, we have a real analytic map:

$$Y \times P \xrightarrow{\theta} E.$$

We claim that this map θ is transverse to the zero-section 0_E of E.

For, if $(y,s) \in Y \times P$ such that $\theta(y,s) = 0_y \in E_y$, the fact that $\theta_y = \theta(y,-)$ is a linear surjection shows that the partial derivative $\partial_s \theta_y = \theta_y$ takes the linear subspace $0 \oplus T_s P = 0 \oplus P$ of $T_{(y,s)}(Y \times P)$ surjectively onto the tangent space $T_{0_y}(E_y) = E_y$ of the fibre E_y. Hence the image of the total derivative $D\theta(y,s)$ contains $T_{0_y}(E_y) = E_y$. Since $T_{0_y}(E) = T_{0_y}(0_E) \oplus E_y$, it follows that $T_{0_y}(0_E)$ and $\mathrm{Im}\, D\theta(y,s)$ together span $T_{0_y}(E)$. Hence θ is transverse to the zero-section 0_E of E.

Since 0_E is a codimension k smooth submanifold of E, the inverse image $W := \theta^{-1}(0_E)$ is a smooth real-analytic submanifold of $Y \times P$, of dimension $\dim W = \dim Y + n - k$. Let $p : Y \times P \to P$ denote the second projection, and consider the following diagram:

$$\begin{array}{ccccc} W & \hookrightarrow & Y \times P & \stackrel{\theta}{\to} & E_{|Y} \\ p_{|W} & \searrow & \downarrow p & & \\ & & P & & \end{array}.$$

Note that the fibre $W_s = p^{-1}(s) \cap W = p_{|W}^{-1}(s)$ is precisely the set of zeroes of the section $\theta_s = \epsilon_{s|Y} = s_{|Y}$ of the restricted bundle $E_{|Y} \to Y$. We make the following:

Claim. The section $\theta_s = \theta(-,s) = s : Y \to E$ is transverse to the zero-section 0_E of E iff $s \in P$ is a regular value of $p_{|W} : W \to P$.

Proof of Claim. Let

$$q : T_{0_y}(E) \to T_{0_y}(E)/T_{0_y}(0_E) = E_y$$

denote the natural quotient map.

Since $\theta : Y \times P \to E$ is transverse to 0_E, for each $(y,s) \in W$ we have the diagram:

$$\begin{array}{ccccccc} & \ker Dp_{|W}(y,s) & \longrightarrow & T_y(Y) & \stackrel{q \circ D\theta_s(y)}{\longrightarrow} & T_{0_y}(E)/T_{0_y}(0_E) & \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & T_{(y,s)}(W) & \longrightarrow & T_{(y,s)}(Y \times P) & \stackrel{q \circ D\theta(y,s)}{\longrightarrow} & T_{0_y}(E)/T_{0_y}(0_E) & \to 0 \\ & \downarrow Dp_{|W(y,s)} & & \downarrow Dp & & \downarrow & \\ 0 \to & P & \longrightarrow & P & \longrightarrow & 0 \to 0 & \end{array}$$

in which the columns and last two rows are exact. By the Snake Lemma, and the fact that $\mathrm{Coker}\, Dp = 0$, we have an exact sequence:

$$0 \to \ker Dp_{|W} \to T_y(Y) \stackrel{q \circ D\theta_s(y)}{\longrightarrow} E_y \to \mathrm{Coker}\, Dp_{|W} \to 0.$$

Consequently, the map $q \circ D\theta_s(y)$ is surjective iff $Dp_{|W}(y,s)$ is surjective. Since $p_{|W}^{-1}(s) = \theta_s^{-1}(0_E) \times \{s\}$, s is a regular value of $p_{|W}$ iff $q \circ D\theta_s(y)$ is surjective for all $y \in \theta_s^{-1}(0_E)$, that is, iff $\theta_s : Y \to E$ is transverse to 0_E. Hence the Claim.

By Sard's theorem, there is a Baire subset $U \subset P$ of regular values for $p_{|W} : W \to P$. Hence the lemma follows, for this choice of U. □

We have the following immediate corollary for subanalytic sets.

COROLLARY 6.4 (*Stratified transversality to the* 0-*section*)

Let M be a real analytic manifold which is countable at ∞. Let $\mathcal{S}$ be a SW-stratification of M, and let $\{M_\alpha\}_{\alpha\in\Lambda}$ denote the strata (*see Definition* 5.2). *Let E be a real analytic vector bundle of rank k on M, generated by an n-dimensional real vector space P of real-analytic global sections. Then there exists a Baire* (*in particular dense*) *subset $U \subset P$ such that for $s \in U$, we have*

(i) $s : M \to E$ *is transverse to* 0_E, *and*
(ii) $s_{|M_\alpha} \pitchfork 0_E$ (*see Definition* 5.5) *for each* $\alpha \in \Lambda$.
(iii) *For any $X \subset M$ which is a union of strata* (*and hence itself SW-stratified*), $s_{|X} \pitchfork 0_E$.

Proof. Since the manifold M is a countable union of compact sets, and a compact set can intersect only finitely many members of locally finite collection, and SW-stratifications are locally finite, it follows that the SW-stratification above consists of only countably many strata. We thus assume that $\Lambda = \mathbb{N}$. In particular, M is the union:

$$M = \cup_{i=1}^{\infty} M_i$$

of countably many strata. Set $M_0 = M$, as a notational convention.

By (i) of Definition 5.2, each M_i is a locally closed real analytic submanifold of M. Applying the previous Lemma 6.3 to $Y = M_i$ for $i = 0, 1, ...$, we find a Baire subset $U_i \subset P$ such that for $s \in U_i$, $s_{|M_i} \pitchfork 0_E$. The set $U = \cap_{i=0}^{\infty} U_i$, being the countable intersection of Baire sets, is a Baire set (the countable union of measure zero sets is measure zero). Clearly for $s \in U$, (i) and (ii) follow. (iii) is obvious from (ii) by Definition 5.5. □

Theorem 6.5 (**Stratified Tubular Neighbourhood Theorem 1**). *Let M be a* compact *real analytic manifold, and $X \subset M$ a subanalytic set* (*not necessarily closed*) *in M. In accordance with Theorem* 5.4 *above, equip M with a SW-stratification $\mathcal{S}$ by strata $\{M_i\}_{i=1}^m$ such that the subanalytic set X is a union of strata.* (*Note that the strata form a finite collection by* (ii) *of Remark* 5.3). *Let $\pi : E \to M$ be a real analytic vector bundle of rank k, generated by an n-dimensional vector space P of real analytic global sections. Then, there exists a Baire subset $U \subset P$ such that for $s \in P$, the section $s_{|X}$ of the restricted subanalytic bundle $E_{|X} \to X$ has a tubular neighbourhood in the sense of Definition* 6.1.

Proof. The proof is rather involved, though the essential idea is contained in the sketch of the proof of Theorem 1.11, p. 47 in [G-M].

The main steps are as follows: Equip the compact manifold M with a finite (SW)-stratification by M_i so that the subanalytic set X is a union of strata, by Theorem 5.4. By Corollary 6.4, for a Baire subset $U \subset P$ and $s \in U$, the restriction of s to *each stratum* is transverse to 0_E. Let Z denote the zero locus of s in M, and $Z_i = Z \cap M_i$. Next we choose a thin enough ϵ-disc bundle in E which intersects only those strata $s(M_i)$ of $s(M)$ that meet 0_E (i.e. for which $Z_i \neq \phi$). Now, shrink down this ϵ-disc bundle to the zero-section 0_E via the scaling map $e \mapsto te$ of E, $t \in \mathbb{R}$. The final step is to prove, using the First Isotopy Lemma 5.9 that the t-family of intersections of these shrinking neighbourhoods with $s(M)$ is a stratified product $Y \times I$, where I is some interval containing 0. For this family, the fibre over $0 \in I$ is the bundle restricted to Z, and the fibre over $a \neq 0 \in I$, is a homeomorphic image of the above neighbourhood. Thus these fibres are both stratified homeomorphic to Y. Because it is a stratified product, and X is a union of strata, one can restrict the (stratification preserving) homeomorphisms above to X. We carry out the details below.

Define $M_0 = M$ for notational convenience. We know by Corollary 6.4 that for the given stratification of $M = \cup_{i=1}^m M_i$ above, there exists a Baire subset $U \subset P$ such that $s \in U$ implies that $s_i := s_{|M_i}$ is transverse to the zero section 0_E of E for each $i = 0, 1, ..., m$. (Note that $s_i : M_i \to E_{|M_i}$ is transverse to the zero section of $E_{|M_i}$ iff $s_i : M_i \to E$ is transverse to the zero section 0_E of E.) In particular $s_{|X}$ is transverse to the zero section 0_E of E (or, in the notation of Definition 5.5, $s_{|X} \pitchfork 0_E$, for the real analytic map $s : M \to E$). Thus the zero-locus $Z := Z_0 := s^{-1}(0_E)$ is a real analytic submanifold of M, and the restriction of Z to M_i, viz. the zero-locus $Z_i = Z \cap M_i = s_i^{-1}(0_E)$ is a real analytic submanifold of M_i, and hence M, for each $i = 1, ..., m$. Denote the restricted bundle $E_{|M_i}$ by E_i.

We shall prove that for such an $s \in U$, the section $s_{|X} : X \to E_{|X}$ has a tubular neighbourhood in X, in the sense of Definition 6.1.

Since $\pi : E \to M$ is isomorphic as a real analytic bundle to $f^*\nu^k$, and since $\nu^k \to G(n-k, P)$ is a real analytic subbundle of $G(n-k, P) \times P$ via the real analytic splitting coming from a constant (hence real analytic) bundle metric on $G(n-k, P) \times P$, it follows that $\pi : E \to M$ has a real analytic pullback bundle metric $\| \|$. Let $p : M \times \mathbb{R} \to M$ denote the first projection. Then the bundle $\pi_1 := (\pi \times 1) : E \times \mathbb{R} \to M \times \mathbb{R}$, which is precisely p^*E, has the pulled back metric, also denoted $\| \|$, also real analytic. Note that $M \times \mathbb{R}$ has the product stratification denoted $\mathcal{S} \times \mathbb{R}$, and $X \times \mathbb{R}$ is a union of strata.

Similarly E has the SW-stratification $\mathcal{S}_E = \pi^{-1}(\mathcal{S})$, with strata $E_i = E_{|M_i} = \pi^{-1}(M_i)$. (It is easy to verify, using the local analytic triviality of E, that $\pi^{-1}(\mathcal{S})$ satisfies (SW1) through (SW4). Also E_i are subanalytic inside E because M_i are subanalytic in M and (vi) of Proposition 2.7.) Similarly, $E \times \mathbb{R}$ is SW-stratified by $\mathcal{S}_E \times \mathbb{R}$, which is the same as the SW-stratification $(\pi_1)^{-1}(\mathcal{S} \times \mathbb{R})$.

Consider the real analytic map:

$$\psi : E \times \mathbb{R} \to E$$
$$(e, t) \mapsto te$$

and denote $\psi(-, t) : E \to E$ by ψ_t. Denote the restricted bundle map $\psi_{|E_i \times \mathbb{R}} \to E_i$ by ψ_i.

Claim $(1)_0$. The map $\psi : E \times \mathbb{R} \to E$ is transverse to the smooth real analytic submanifold $s(M) \subset E$.

In fact we shall show that for each $t \in \mathbb{R}$, the map $\psi_t : E \to E$ is transverse to $s(M)$. Since for each $t \in \mathbb{R}$, the image $\operatorname{Im} D\psi(e, t)$ contains $\operatorname{Im} D\psi_t(e)$, this will prove the assertion. If $t \neq 0$, we have $\psi_t : E \to E$ is a real analytic diffeomorphism, and hence transverse to $s(M)$. If $t = 0$, we have $\psi_0 : E \to E$ equal to projection onto the zero-section 0_E, viz. the map $e \mapsto 0_{\pi(e)}$. Thus for $e \in E$, if we denote $\pi(e) = x \in M$, we have $\psi_0(e) = 0_x$ and $\operatorname{Im} D\psi_0(e) = T_{0_x}(0_E)$. If $\psi_0(e) \in s(M)$, it follows that $e = s(x) = 0_x$, so that $x \in s^{-1}(0_E)$. By the choice of s, s is transverse to 0_E, so that we have $\operatorname{Im} Ds(x) + T_{0_x}(0_E) = T_{0_x}(E)$. That is, $T_{s(x)}(s(M)) + \operatorname{Im} D\psi_0(e) = T_{0_x}(E)$. Thus ψ_0 is also transverse to $s(M)$, and our claim follows.

By the transversality above, it follows that $N := \psi^{-1}(s(M))$ is a real analytic submanifold of $E \times \mathbb{R}$, whose fibre over $t \in \mathbb{R}$ is the real analytic manifold $N(t) := \psi_t^{-1}(s(M))$.

Since $s : M \to E$ is a real analytic embedding, and sections of bundles are proper maps, it follows that $s(M)$ is a smooth real analytic subspace of E. In particular it is closed. Since ψ is real analytic, it follows that $N = \psi^{-1}(s(M))$ is a real analytic subspace in $E \times \mathbb{R}$ which is closed. By the transversality of ψ to $s(M)$, it follows that N is a smooth real analytic subspace of $E \times \mathbb{R}$.

Next we have

Claim $(2)_0$. The real analytic submanifold N is transverse to the zero section $0_{E\times\mathbb{R}}$, and the intersection $N \cap 0_{E\times\mathbb{R}}$ is $s(Z) \times \mathbb{R}$, where $Z := s^{-1}(0_E)$ is the zero locus of s.

Since $s : M \to E$ is transverse to 0_E by the choice of s above, it follows that for each $0_x \in s(M)$ ($\Leftrightarrow x \in Z = s^{-1}(0_E)$) we have

$$T_{0_x}(s(M)) + T_{0_x}(0_E) = T_{0_x}(E). \tag{7}$$

Now for $(0_x, t) \in N \cap 0_{E\times\mathbb{R}}$, we have the identifications $T_{0_x}(E) = T_{0_x}(0_E) + E_x$ and $T_{(0_x,t)}(E\times\mathbb{R}) = T_{0_x}(0_E) + E_x + \mathbb{R}$. With this identification, clearly $D\psi(0_x, t)(W, 0, 0) = W$ for $W \in T_{0_x}(0_E)$, since ψ restricted to $0_E \times \mathbb{R}$ is the first projection to 0_E. Let us denote $L := D\psi(0_x, t)$ for notational convenience. Then by the foregoing remark and (7) above, we have:

$$T_{0_x}(s(M)) + L(T_{(0_x,t)}(0_{E\times\mathbb{R}})) = T_{0_x}(E). \tag{8}$$

For any linear map L, any two linear subspaces A, B, we have the identity $L^{-1}A + B = L^{-1}(A + L(B))$. Substituting $A = T_{0_x}(s(M))$ and $B = T_{(0_x,t)}(0_{E\times\mathbb{R}})$, and using (8) above we find

$$\begin{aligned} L^{-1}T_{0_x}(s(M)) + T_{(0_x,t)}(0_{E\times\mathbb{R}}) &= L^{-1}(T_{0_x}(s(M))) + L(T_{(0_x,t)}(0_{E\times\mathbb{R}})) \\ &= L^{-1}(T_{0_x}(E)) \\ &= T_{(0_x,t)}(E \times \mathbb{R}). \end{aligned}$$

Since by Claim $(1)_0$, ψ is transverse to $s(M)$, we have $L^{-1}T_{0_x}(s(M)) = T_{(0_x,t)}(N)$. Thus we have

$$T_{(0_x,t)}(N) + T_{(0_x,t)}(0_{E\times\mathbb{R}}) = T_{(0_x,t)}(E \times \mathbb{R}).$$

That is, N meets $0_{E\times\mathbb{R}}$ transversely. The first part of Claim $(2)_0$ follows.

To see the second part of Claim $(2)_0$, note that $(0_x, t) \in N \cap 0_{E\times\mathbb{R}}$ iff $\psi(0_x, t) = 0_x$ is in $s(M)$. But $0_x \in s(M)$ iff $0_x = s(x)$ iff $x \in s^{-1}(0_E) = Z$. Thus $N \cap 0_{E\times\mathbb{R}} = s(Z) \times \mathbb{R}$, and Claim $(2)_0$ is proved.

In the proofs of the above two claims, the transversality of s to the zero section was the only fact used. By the choice of $s \in U$ we have $s_i : M_i \to E_i$ is transverse to the zero section 0_{E_i} for all i. Thus, in the proofs of the above two claims, we replace M by the stratum M_i, E by E_i, s by $s_i := s_{|M_i}$, ψ by $\psi_i := \psi_{|E_i\times\mathbb{R}}$, and repeat everything (noting that $s_i : M_i \to E_i$ is transverse to 0_{E_i} iff $s_i : M_i \to E$ is transverse to 0_E). By so doing, for $i = 0, 1, ..., m$, we obtain:

Claim $(1)_i$. $\psi_i : E_i \times \mathbb{R} \to E_i$ is transverse to $s(M_i)$, and thus $N_i := \psi_i^{-1}(s(M_i)) = \psi^{-1}(s(M_i))$ is a smooth real analytic subspace of E_i, and is equal to the intersection $N \cap (E_i \times \mathbb{R})$. (For notational consistency, we set $N_0 := N$.)

Claim $(2)_i$. $N_i := \psi^{-1}(s(M_i))$ intersects $0_{E\times\mathbb{R}}$ transversely, and this intersection is $s(Z_i)\times\mathbb{R} = s(Z\cap M_i)\times\mathbb{R}$.

From the given SW-stratification $\mathcal{S}$ in the hypothesis of the theorem, the smooth analytic subspace $s(M)$ gets an induced SW-stratification $\mathcal{S}_s$, defined by the strata $s(M_i)$, $i = 1, 2, ..., m$. (Note $s(M_i)$ are subanalytic in E by (ii) of Proposition 2.7 because s, being a continuous section of a bundle, is proper.)

Claim (3). The manifold $s(M)$ meets the strata E_i of $\mathcal{S}_E$ transversely (inside E), and N meets the strata $E_i\times\mathbb{R}$ of $\mathcal{S}_{E\times\mathbb{R}}$ transversely (inside $E\times\mathbb{R}$).

To prove the first assertion, let $s(x)\in s(M)\cap E_i = s(M_i)$. Since $\pi : E\to M$ satisfies $\pi\circ s = \mathrm{id}_M$, the image $\mathrm{Im}\, Ds(x) = T_{s(x)}(s(M))$ is a vector space complement to $\ker D\pi(s(x)) = T_{s(x)}(E_x) = E_x$ inside $T_x(E)$, and hence

$$T_{s(x)}(s(M)) + E_x = T_{s(x)}(E). \tag{9}$$

Since $E_x\subset T_{s(x)}(E_i)$ for all $x\in M_i$, it follows that

$$T_{s(x)}(s(M)) + T_{s(x)}(E_i) = T_{s(x)}(E)$$

and hence $s(M)$ is transverse to E_i for all i. The first assertion of Claim (3) follows.

Now we prove the second assertion of Claim (3), i.e. N meets $E_i\times\mathbb{R}$ transversally for all $i = 1, 2, ..., m$ inside $E\times\mathbb{R}$. We have already observed that $D\psi(e,t) : T_{(e,t)}(E\times\mathbb{R})\to T_{te}(E)$ is surjective for all $t\neq 0$ (because $\psi_t : E\to E$ is a diffeomorphism). Also for such a $t\neq 0$, $D\psi(e,t)$ maps $E_x\oplus\mathbb{R}$ onto E_x. Hence, from (9) we conclude that

$$T_{s(x)}(s(M)) + D\psi(e,t)(E_x\oplus\mathbb{R}) = T_{s(x)}(E) \tag{10}$$

for $(e,t)\in N\cap(E_i\times\mathbb{R})$ with $t\neq 0$, and $\pi(e) = x$. Applying the identity $L^{-1}(A+L(B)) = L^{-1}A + B$ to the linear map $L = D\psi(e,t)$, $A = T_{s(x)}(s(M))$, $B = E_x\oplus\mathbb{R}$, and noting that $L^{-1}(T_{s(x)}(s(M)) = T_{(e,t)}(N)$ by definition and Claim $(1)_0$, we obtain

$$T_{(e,t)}(N) + (E_x\oplus\mathbb{R}) = T_{(e,t)}(E\times\mathbb{R}).$$

Since $E_x\oplus\mathbb{R} = T_{(e,t)}(E_x\times\mathbb{R})\subset T_{(e,t)}(E_i\times\mathbb{R})$ for all $i = 0, 1, ..., m$, we obtain that the intersection of N with $E_i\times\mathbb{R}$ is transverse at all points (e,t) for $t\neq 0$.

Now we look at the situation where $t = 0$. If a point $(e,0)\in N\cap(E_i\times\mathbb{R})$, we have that $\psi(e,0) = 0_x\in s(M_i) = s(M)\cap E_i$ and $s(x) = 0_x$. From the paragraph preceding the proof of Claim $(1)_i$, we have that $s : M_i\to E_i$ is transverse to the zero-section 0_{E_i} in E_i, so that

$$T_{0_x}(s(M_i)) + T_{0_x}(0_{E_i}) = T_{0_x}(E_i). \tag{11}$$

Since $Ds(x)(T_x(M_i))$ is a complement to E_x in $T_x(E_i)$, for each $x\in M_i$, and $i = 1, 2, ..., m$, it follows that

$$T_{0_x}(E_i) = E_x + Ds(x)(T_x(M_i)). \tag{12}$$

Let ν_i denote the normal bundle to M_i in M, and $\nu_{i,x}$ its fibre at $x \in M_i$. Since $T_{0_x}(s(M)) = Ds(x)(T_x(M))$ and $T_{0_x}(s(M_i)) = Ds(x)(T_x(M_i))$, we have

$$\begin{aligned}
T_{s(x)}(s(M)) + T_{0_x}(0_{E_i}) &= Ds(x)(T_x(M)) + T_{0_x}(0_{E_i}) \\
&= Ds(x)(T_x(M_i)) + Ds(x)(\nu_{i,x}) + T_{0_x}(0_{E_i}) \\
&= T_{0_x}(E_i) + Ds(x)(\nu_{i,x}) \qquad \text{(by eq. (11))} \\
&= E_x + Ds(x)(T_x(M_i)) + Ds(x)(\nu_{i,x}) \quad \text{(by eq. (12))} \\
&= E_x + Ds(x)(T_x(M)) \\
&= T_{0_x}(E). \qquad (13)
\end{aligned}$$

$\psi_0 : E \to E$ is just the projection of E to its zero section 0_E, and hence $D\psi_0(e)(T_{(e,0)}(E_i)) = T_{0_x}(0_{E_i})$, so that $D\psi(e,0)(T_{(e,0)}(E_i \times \mathbb{R}))$ contains $T_{0_x}(0_{E_i})$. Combining with eq. (13), we have

$$T_{s(x)}(s(M)) + D\psi(e,0)(T_{(e,0)}(E_i \times \mathbb{R})) = T_{0_x}(E).$$

Now we apply our lemma $L^{-1}A + B = L^{-1}(A + L(B))$ to the linear map $L = D\psi(e,0)$, $A = T_{s(x)}(s(M))$, $B = T_{(e,0)}(E_i \times \mathbb{R})$, and noting that by definition and Claim $(1)_0$, $L^{-1}(T_{s(x)}(s(M)) = T_{(e,0)}(N)$, we obtain

$$T_{(e,0)}(N) + T_{(e,0)}(E_i \times \mathbb{R}) = T_{(e,0)}(E \times \mathbb{R})$$

which shows that N meets $E_i \times \mathbb{R}$ transversely at $(e,0)$ and our Claim (3) follows.

Since $s(M_i) = E_i \cap s(M)$, the first assertion of Claim (3) shows that the stratification $\mathcal{S}_s$ is just the intersection stratification $\mathcal{S}_E \cap s(M)$ as defined in Lemma 5.6. In particular, by that same lemma, $\mathcal{S}_s$ is a (SW)-stratification of $s(M)$.

By the second assertion of Claim (3) and Lemma 5.6, we can give the SW-stratification $\mathcal{S}_N := \mathcal{S}_{E\times\mathbb{R}} \cap N$ to the smooth real analytic subspace $N = \psi^{-1}(s(M))$. The strata are precisely the connected components of $N_i = N \cap (E_i \times \mathbb{R}) = \psi^{-1}(s(M_i))$.

Now we need another assertion:

Claim (4). For each $i = 0, 1, ..., m$, define $p_i : N_i \to \mathbb{R}$ to be the restriction of the second projection $p : E \times \mathbb{R} \to \mathbb{R}$ to N_i. The derivative

$$Dp_i(0_x, t) : T_{(0_x,t)}(N_i) \to \mathbb{R}$$

is surjective at all points $(0_x, t) \in N_i \cap 0_{E\times\mathbb{R}} = s(Z_i) \times \mathbb{R}$.

We now prove Claim (4). For the curve $s \mapsto (0_x, t+s)$ in $E \times \mathbb{R}$, starting at $(0_x, t)$ with initial velocity $(0,0,1) \in T_{(0_x,t)}(E \times \mathbb{R}) = T_{0_x}(0_E) + E_x + \mathbb{R}$, we have $\psi(0_x, s+t) = (s+t)0_x \equiv 0_x$ for all s so that

$$D\psi(0_x, t)(0,0,1) = 0$$

for all $t \in \mathbb{R}$. Thus the subspace $0 \oplus 0 \oplus \mathbb{R} \subset T_{(0_x,t)}(E \times \mathbb{R})$ lies in $\ker D\psi(0_x, t)$. Since (by Claim $(1)_i$ above), $T_{(0_x,t)}(N_i) = (D\psi(0_x,t))^{-1}(T_{0_x}s(M_i))$, it contains this kernel $\ker D\psi(0_x, t)$. Thus

$$T_{(0_x,t)}(N_i) \supset 0 \oplus 0 \oplus \mathbb{R}$$

for all $(0_x, t) \in N_i$. Since Dp_i maps the subspace $(0 \oplus 0 \oplus \mathbb{R})$ isomorphically onto $\mathbb{R} = T_t(\mathbb{R})$, it follows that

$$Dp_i(0_x, t) : T_{(0_x,t)}(N_i) \to \mathbb{R}$$

is surjective at all points $(0_x, t) \in N_i \cap 0_{E\times\mathbb{R}}$ and all $i = 0, 1, ..., m$. Hence Claim (4) is proved.

Recall that $Z_i := Z \cap M_i$, and by the second part of Claim $(2)_i$, we have $N_i \cap 0_{E\times\mathbb{R}} = s(Z_i) \times \mathbb{R}$.

From Claim (4), and by continuity of Dp_i, we have the following claim.

Claim (5). For each i such that $Z_i \neq \phi$, and for $(0_x, t) \in N_i \cap 0_{E\times\mathbb{R}} = s(Z_i) \times \mathbb{R}$, there exists a neighbourhood $U_{x,t}$ of $(0_x, t)$ in $E \times \mathbb{R}$ such that the derivative

$$Dp_i(e, s) : T_{(e,s)}(N_i) \to \mathbb{R}$$

is surjective for all $(e, s) \in U_{x,t} \cap N_i$.

Renumber the strata so that $Z_i := Z \cap M_i \neq \phi$ for $i = 1, 2, ..., r$ and $Z_i = \phi$ for $i = r + 1, ..., m$. Again, for notational convenience, set $Z_0 := Z = s^{-1}(0_E)$. For $\delta > 0$, let $E(\delta)$ and $S(\delta)$ denote the open δ-disc bundle and the δ-sphere bundles of E respectively (with respect to the above real analytic bundle metric $\| \ \|$).

Claim (6). For each $i = 0, 1, ..., r$, and for $(0_x, t) \in N_i \cap 0_{E\times\mathbb{R}} = s(Z_i) \times \mathbb{R}$, there exists a neighbourhood $U_{x,t}$ of $(0_x, t)$ in $E \times \mathbb{R}$ such that $N_i \cap U_{x,t}$ meets $(S(\delta) \times \mathbb{R}) \cap U_{x,t}$ transversely for all $\delta > 0$. (Note that for δ very large, the intersection $(S(\delta) \times \mathbb{R}) \cap U_{x,t}$ might be empty, in which case this assertion is vacuously true.)

To see this, we first prove that for $i = 0, 1, ..., r$ and $0_x \in s(M_i) \cap 0_E = s(Z_i)$, there is a neighbourhood U_x of 0_x in E such that $s(M_i) \cap U_x$ meets $S(\delta) \cap U_x$ transversely for all $\delta > 0$. Since this assertion is local around 0_x, and E is locally trivial, we may assume without loss of generality that E is trivial.

By this triviality, there is a natural linear surjection:

$$\rho(e) : T_e(E) \to E_{\pi(e)} = (\pi^* E)_e$$

for each $e \in E$. The fact that $s(M_i)$ meets 0_E transversely at 0_x implies that the composite

$$T_{0_x}(s(M_i)) \hookrightarrow T_{0_x}(E) \overset{\rho(0_x)}{\to} (\pi^* E)_{0_x} = E_x$$

is surjective. By continuity, there exists a neighbourhood U_x of 0_x in E such that the composite

$$T_e(s(M_i)) \hookrightarrow T_e(E) \overset{\rho(e)}{\to} (\pi^* E)_e$$

is surjective for all $e \in s(M_i) \cap U_x$.

At a point $e \in S(\delta)$, it is obvious that the one-dimensional quotient space $T_e(E)/T_e(S(\delta))$ is a quotient of $E_{\pi(e)} = (\pi^* E)_e$ (by the tangent space to the sphere fibre $S(\delta)_{\pi(e)}$). Thus the composite map

$$T_e(s(M_i)) \hookrightarrow T_e(E) \overset{\rho(e)}{\to} (\pi^* E)_e \to T_e(E)/T_e(S(\delta))$$

is also surjective for all $e \in s(M_i) \cap S(\delta) \cap U_x$. But this is precisely the statement that $s(M_i) \cap U_x$ meets $S(\delta) \cap U_x$ transversely inside E.

For the sake of convention, we define $S(\delta) = 0_E$ for $\delta = 0$. Then, by the fact that $s(M_i)$ is transverse to 0_E, we have that $s(M_i) \cap U_x$ meets $S(\delta) \cap U_x$ transversely in E for all $\delta \geq 0$.

Now let $t \in \mathbb{R}$, and let $U_{x,t}$ be a neighbourhood of $(0_x, t)$ such that $\psi(U_{x,t}) \subset U_x$, U_x as above. Hence $(v, \lambda) \in (S(\delta) \times \mathbb{R}) \cap N_i \cap U_{x,t}$ implies that $e := \psi(v, \lambda) = \lambda v \in (S(|\lambda|\delta)) \cap s(M_i) \cap U_x$.

By the above choice of the neighbourhood U_x, at the point $e = \lambda v \in U_x$, $\lambda \in \mathbb{R}$, we have

$$T_e(S(|\lambda|\delta)) + T_e(s(M_i)) = T_e(E). \tag{14}$$

If $(v, \lambda) \in U_{x,t}$ with $\lambda \neq 0$, then $\psi_\lambda : E \to E$ is a diffeomorphism. In this event, the derivative

$$D\psi(v, \lambda) : T_{(v,\lambda)}(E \times \mathbb{R}) \to T_{\lambda v}(E)$$

is surjective. Taking the inverse image under this surjective map $D\psi(v, \lambda)$ of both sides of the equality (14), noting that ψ is transverse to $s(M_i)$, and that $\psi_\lambda : S(\delta) \times \{\lambda\} \to S(|\lambda|\delta)$ is a diffeomorphism for $\lambda \neq 0$, we have

$$T_{(v,\lambda)}(S(\delta) \times \{\lambda\}) + T_{(v,\lambda)}(N_i) = T_{(v,\lambda)}(E \times \mathbb{R}).$$

Since $T_{(v,\lambda)}(S(\delta) \times \mathbb{R})$ contains the subspace $T_{(v,\lambda)}(S(\delta) \times \{\lambda\})$, we see that at all points (v, λ) with $\lambda \neq 0$, which lie in $U_{x,t}$ and in $(S(\delta) \times \mathbb{R}) \cap N_i$, the intersection of $S(\delta) \times \mathbb{R}$ and N_i is transverse.

To check the case $\lambda = 0$, let $(v, 0) \in (S(\delta) \times \mathbb{R}) \cap N_i \cap U_{x,t}$. Note $\psi(v, 0) = 0_y$ where $y = \pi(v)$ and $\pi : E \to M$ is the bundle projection. Since $(v, 0) \in N_i \cap U_{x,t}$, we have $0_y \in s(M_i) \cap U_x$. Thus, setting $L := D\psi(v, 0)$, we have $L(T_{(v,0)}(S(\delta) \times \{0\})) = T_{0_y}(0_E)$, since the projection $\pi : S(\delta) \to M$ is a submersion. Since $0_y \in s(M_i) \cap 0_E$, and $s(M_i)$ intersects 0_E transversely, we have

$$T_{0_y}(s(M_i)) + T_{0_y}(0_E) = T_{0_y}(E)$$

from which it follows that

$$T_{0_y}(s(M_i)) + L(T_{(v,0)}(S(\delta) \times \{0\})) = T_{0_y}(E)$$

which implies

$$T_{0_y}(s(M_i)) + L(T_{(v,0)}(S(\delta) \times \mathbb{R})) = T_{0_y}(E).$$

Again we use the identity $L^{-1}(A + L(B)) = L^{-1}A + B$ for any linear map L, and any subspaces A, B and obtain, by setting $A = T_{0_y}(s(M_i))$ and $B = T_{(v,0)}(S(\delta) \times \mathbb{R})$, that

$$L^{-1}T_{0_y}(s(M_i)) + T_{(v,0)}(S(\delta) \times \mathbb{R}) = T_{(v,0)}(E \times \mathbb{R})$$

which is to say,

$$T_{(v,0)}(N_i) + T_{(v,0)}(S(\delta) \times \mathbb{R}) = T_{(v,0)}(E \times \mathbb{R}).$$

Thus the intersection of $N_i \cap U_{x,t}$ and $(S(\delta) \times \mathbb{R}) \cap U_{x,t}$ is transverse. This proves Claim (6).

Since M is compact, 0_E is a compact subset of E. Since, by Claim $(2)_i$ above, N_i intersects $0_{E\times\mathbb{R}}$ transversely for each $i = 0, 1, ..., m$, each connected component of each N_i intersects $0_{E\times\mathbb{R}}$ transversely inside $E \times \mathbb{R}$. Thus no such connected component can be contained in $0_{E\times\mathbb{R}}$. That is, no stratum of the stratification $\mathcal{S}_N$ of the smooth manifold $N = \psi^{-1}(s(M))$ is contained in $0_{E\times\mathbb{R}}$.

We now note that since every stratum $s(M_i)$ of $s(M)$ meets 0_E transversely, $s(M_i) \subset \overline{s(M_j)} = s(\overline{M}_j)$ for $i \neq j$ will imply that the codimension of $s(M_i) \cap 0_E$ in $\overline{s(M_j)} \cap 0_E$ is the same as the codimension of $s(M_i)$ in $\overline{s(M_j)}$, which is non-zero. Thus if $\overline{s(M_j)} \cap 0_E \neq \phi$, the union

$$\bigcup \left\{ s(M_i) \cap 0_E : \ s(M_i) \subset \overline{s(M_j)}, \ \text{ and } i \neq j \right\}$$

has non-zero codimension in $\overline{s(M_j)} \cap 0_E$. Thus

$$s(M_j) \cap 0_E = \left(\overline{s(M_j)} \cap 0_E\right) \Big\backslash \left(\bigcup \left\{ s(M_i) \cap 0_E : \ s(M_i) \subset \overline{s(M_j)} \ \text{ and } i \neq j \right\}\right) \neq \phi.$$

That is, $\overline{s(M_j)} \cap 0_E \neq \phi$ if and only if $s(M_j) \cap 0_E \neq \phi$, if and only if $j = 0, 1, ..., r$.

By noting that $N_j = \psi^{-1}(s(M_j))$, and $\psi(0_{E\times\mathbb{R}}) = 0_E$, the same fact obtains for the N_j, viz. $\overline{N}_j \ \cap \ 0_{E\times\mathbb{R}} \neq \phi$ if and only if $N_j \cap 0_{E\times\mathbb{R}} \neq \phi$ which, in turn, happens if and only if $j = 0, 1, ..., r$.

Consider the closed subset C of $E \times \mathbb{R}$ defined by

$$C := \cup_{i=r+1}^{m} \overline{N}_i$$

which is disjoint from $0_{E\times\mathbb{R}}$, by the preceding paragraph. Since C is closed and $0_E \times [-2, 2]$ is compact, there will exist an $\epsilon > 0$ such that the restricted 2ϵ-disc bundle $E(2\epsilon) \times [-2, 2]$ does not intersect C. Thus $E(2\epsilon) \times [-2, 2]$ is disjoint from N_i for $i = r + 1, ..., m$. We saw above in Claim $(2)_i$ that $N_i \cap 0_{E\times\mathbb{R}} = s(Z_i) \times \mathbb{R}$, and similarly $N_i \cap (0_E \times [-2, 2]) = s(Z_i) \times [-2, 2]$. Thus N_i intersects $E(2\epsilon) \times [-2, 2]$ if and only if N_i intersects $0_E \times [-2, 2]$, and this happens if and only if $Z_i \neq \phi$, i.e. if and only if $i = 0, 1, ..., r$. We record this fact in claim (7)

Claim (7). $(E(2\epsilon) \times [-2, 2]) \cap N_i = \phi$ iff $i = r + 1, r + 2, ..., m$.

By reducing ϵ if necessary, and from Claims (5) and (6), using the compactness of $0_E \ \times \ [-2, 2]$, we can further assert the following Claim (8).

Claim (8). For $i \ = \ 0, 1, ..., r$, the derivative $Dp_i(e, s)$ is surjective for all $(e, s) \in (E(2\epsilon) \times [-2, 2]) \cap N_i$.

Claim (9). For $i = 0, 1, ..., r$, the intersection of $S(\delta) \times [-2, 2]$ with N_i is transverse for all $\delta < 2\epsilon$.

We need an analogue of Claim (8) above for $S(\epsilon)$. That is

Claim (10). For ϵ small enough, and $0 \leq i \leq r$ such that the intersection $(S(\epsilon) \times [-2, 2]) \cap N_i \neq \phi$, the derivative $Dp_i(e, s)$ is surjective for all $(e, s) \in (S(\delta) \times [-2, 2]) \cap N_i$ and all $\delta < 2\epsilon$.

We note that the tangent subspace $T_{(e,s)}(S(\delta) \times \mathbb{R})$ has the line complement $\mathbb{R}e \subset E_{\pi(e)}$ in the tangent space $T_{(e,s)}(E \times \mathbb{R}) = T_{(e,s)}(E(2\epsilon) \times \mathbb{R})$. Since Dp annihilates all vectors in $T_e(E(2\epsilon)) \oplus \{0\} \subset T_{(e,s)}(E)$, it annihilates the subspace $E_{\pi(e)}$, and hence $\mathbb{R}e$. Thus the Dp_i image of $T_{(e,s)}((S(\delta) \times [-2, 2]) \cap N_i)$ is the same as the Dp_i image of $T_{(e,s)}((E(2\epsilon) \times [-2, 2]) \cap N_i)$, which is all of $\mathbb{R} = T_s([-2, 2])$ by Claim (5) above. Thus Claim (10) is proved.

Consider $\overline{E(\epsilon)} \times (-2, 2)$ as a SW-stratified subspace of the analytic manifold $E \times (-2, 2)$ with just the two strata $E(\epsilon) \times (-2, 2)$ and $S(\epsilon) \times (-2, 2)$, as in Remark 5.7. By the fact that $E(\epsilon) \times (-2, 2)$ is open (and subanalytic in $E \times (-2, 2)$, since it is the inverse image of $(-\epsilon^2, \epsilon^2)$ under the analytic map $\| \; \|^2$), and the fact that N is SW-stratified by the connected components of N_i, it follows that $A := N \cap (E(\epsilon) \times (-2, 2))$ is SW-stratified by the connected components of the analytic submanifolds $N_i \cap (E(\epsilon) \times (-2, 2))$. By Claim (7) above, we have

$$A = \cup_{i=1}^{r} (N_i \cap (E(\epsilon) \times (-2, 2)).$$

Similarly by Claim (9) above, and Remark 5.7, the subset $\partial A = N \cap (S(\epsilon) \times (-2, 2))$ is SW-stratified by the connected components of the analytic submanifolds $N_i \cap (S(\epsilon) \times (-2, 2))$. (Note that $S(\epsilon)$ is a smooth real analytic subspace of E since $\| \; \|^2$ is a real analytic function, and $\overline{E}(\epsilon)$ is a real analytic manifold with boundary $S(\epsilon)$). In fact, by Remark 5.7,

$$\overline{A} := N \cap (\overline{E}(\epsilon) \times (-2, 2))$$

is SW-stratified by the connected components of the real analytic submanifolds $N_i \cap (S(\epsilon) \times (-2, 2))$ and $N_i \cap (E(\epsilon) \times (-2, 2))$ for $i = 1, ..., r$. Let us call these connected components A_α, where $\alpha \in F$ for some finite set F. Thus the subset $\overline{A}$ is a SW-stratified space in $E \times (-2, 2)$ with stratification $\mathcal{S}_A$ by A_α.

By Claims (8) and (10) above, for $i = 0, 1, ..., r$ the maps $p_i : N_i \cap (S(\epsilon) \times (-2, 2)) \to (-2, 2)$ and $p_i : N_i \cap (E(\epsilon) \times (-2, 2)) \to (-2, 2)$ are submersions. Thus they are submersions when restricted to each connected component. In particular, $p_i : A_\alpha \to (-2, 2)$ is a submersion for each $\alpha \in F$. That is $p : \overline{A} \to (-2, 2)$ is a stratified submersion in the sense of Definition 5.8.

For any compact subset $K \subset (-2, 2)$, $p_{|\overline{A}}^{-1}(K) = p^{-1}(K) \cap \overline{A}$ is a closed set, and contained in the compact set $\overline{E}(\epsilon) \times K$. ($\overline{E}(\epsilon)$ is compact since M is compact!). Thus $p_{|\overline{A}}^{-1}(K)$ is compact, implying that $p_{|\overline{A}}$ is proper.

By the First Isotopy Lemma 5.9, applied to the analytic map $p : E \times (-2, 2) \to (-2, 2)$ and the closed SW-stratified subset $\overline{A} \subset E \times (-2, 2)$, it follows that for each point $t \in (-2, 2)$, there is a neighbourhood $U_t := (t - \delta, t + \delta)$ of t, and a stratum preserving rugeux homeomorphism

$$h : \overline{A} \cap p^{-1}(U_t) \to \overline{A}_t \times (t - \delta, t + \delta),$$

where $\overline{A}_t := \overline{A} \cap (E \times \{t\})$, such that $pr_2 \circ h = p$ (here pr_2 is the second projection on the right hand side). That is, $\overline{A} \to (-2, 2)$ is a topologically locally-trivial stratified fibre bundle.

By the compactness and connectedness of $[0, 1] \subset (-2, 2)$, there is therefore a stratification preserving rugeux homeomorphism

$$h : \overline{A}_1 \to \overline{A}_0$$

between the fibres $\overline{A}_1$ and $\overline{A}_0$. But $\overline{A}_1 = N \cap (\overline{E}(\epsilon) \times \{1\}) = (\psi_1^{-1}(s(M)) \cap (\overline{E}(\epsilon)) \times \{1\})$ by definition. Since $\psi_1 : E \to E$ is the identity map, this last set is stratified homeomorphic to $(s(M) \cap \overline{E}(\epsilon))$. On the other hand, the map $\psi_0 : E \to E$ is the bundle projection map $e \to 0_{\pi(e)}$. Thus $\psi_0^{-1}(s(M)) = E_{|Z}$ where $Z = s^{-1}(0_E)$ is the zero-locus of s. Thus $\overline{A}_0$ is stratified homeomorphic to $\overline{E}(\epsilon)_{|Z}$.

Note that $s(M) \cap E(\epsilon)$ is an open neighbourhood of $s(M) \cap 0_E = s(Z) \cap 0_E$. By the last para, this neighbourhood is stratified homeomorphic to $E(\epsilon)_{|Z}$. The fact that this homeomorphism preserves strata shows that it maps $s(M_i) \cap E(\epsilon)$ homeomorphically to $E(\epsilon)_{|Z_i}$, where $Z_i = Z \cap M_i$ for $i = 1, 2, ..., r$ (recall for $i > r$, the above sets are empty). Since X is a union of strata from among the M_i, and this last homeomorphism is stratum preserving, it follows that this homeomorphism when restricted to the open neighbourhood $s(X) \cap E(\epsilon)_{|X}$ of $s(X) \cap 0_E = s_{|X}(Z \cap X)$, is again a stratum preserving homeomorphism $s(X) \cap E(\epsilon)_{|X} \to E(\epsilon)_{|Z \cap X}$.

Since the analytic embedding $s : M \to s(M)$ defines a homeomorphism between a neighbourhood of $s^{-1}(0_E) \cap X$ and $s(X) \cap E(\epsilon)_{|X}$, Theorem 6.5 follows. □

Theorem 6.6 (**Stratified Tubular Neighbourhood Theorem 2**). *Let M be a compact real analytic manifold, and $X \subset M$ be a subanalytic set. Let E be a subanalytic bundle of rank k over X generated by an n-dimensional vector space of global sections P in the sense of Definition* 4.1. *Then, for a Baire subset $U \subset P$, and $s \in U$, there exists a neighbourhood of $Z := s^{-1}(0_E)$ in X which is homeomorphic to an ϵ-disc bundle $E(\epsilon)_{|Z}$ for some $\epsilon > 0$ (i.e. $s : X \to E_{|X}$ has a tubular neighbourhood in the sense of Definition* 6.1).

Proof. By Lemma 4.4, there is a subanalytic pseudoequivalence $j : (X, M) \to (X_1, M_1)$, with X_1 a subanalytic set in M_1, such that our given bundle E is the pullback j^*C of a real analytic rank-k vector bundle C on M_1, generated by a space of global analytic sections P. Further, since M is compact, and $M_1 = M \times G(n-k, P)$, M_1 is also compact.

By (ii) of Remark 6.2, the Tubular Neighbourhood Theorem for E on X follows from the Stratified Tubular Neighbourhood Theorem 1 (i.e. Theorem 6.5) applied to C, X_1 and M_1 above. □

Remark 6.7. Theorem 6.6 covers the case of X being any real projective or affine variety. In fact, any real projective or affine algebraic constructible set can be regarded as a subanalytic subset in projective space. In the analytic situation too, any real analytically constructible subset of a compact real analytic manifold automatically becomes a subanalytic set. The main Theorem 6.6 applies in all of the above situations, provided the bundle E is a subanalytic bundle generated by global sections in the sense of Definition 4.1.

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