

On the absolute N_{q_α} -summability of r th derived conjugate series

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Abstract. The object of the present paper is to study the absolute N_{q_α} -summability of r th derived conjugate series generalizing a known result.

Keywords. Fourier series; conjugate series; derived conjugate series; Nevanlinna summability; kernel.

1. Introduction

1.1.

In the year 1921, Nevanlinna [7] suggested and discussed an interesting method of summation called N_q -method. Moursund [5] applied this method for summation of Fourier series and its conjugate series. Later, Moursund [6] developed N_{q_p} -method (where p is a positive integer) and applied it to p th derived Fourier series. Samal [9] discussed N_{q_α} -method ($0 \leq \alpha < 1$) and studied absolute N_{q_α} -summability of some series associated with Fourier series. In his Ph.D. thesis [10] he extended N_{q_p} -method of summation to N_{q_α} -method for any $\alpha \geq 0$ and studied absolute N_{q_α} -summability of Fourier series. Earlier we [8] have studied absolute N_{q_α} -summability of a series conjugate to a Fourier series. In the present paper we shall study the absolute N_{q_α} -summability of r th ($r < \alpha$) derived series of a conjugate series.

1.2.

DEFINITION 1 [6,10]

Let $F(w)$ be a function of a continuous parameter w defined for all $w > 0$. The N_{q_α} -method consists in forming the N_{q_α} -transform or mean

$$N_{q_\alpha} F(w) = \int_0^1 q_\alpha(t) F(wt) dt$$

and then considering the limit

$$\lim_{w \rightarrow \infty} N_{q_\alpha} F(w),$$

where the class of functions $q_\alpha(t)$ is such that

- (1) $q_\alpha(t) \geq 0$ for $0 \leq t \leq 1$,
- (2) $\int_0^1 q_\alpha(t) dt = 1$,
- (3) $\frac{d^\beta}{dt^\beta} q_\alpha(t)$ exists and is absolutely continuous for $0 \leq t \leq 1, \beta = 1, 2, \dots, k - 1$,
where $[\alpha] = k$,
- (4) $\frac{d^\beta}{dt^\beta} q_\alpha(t) = 0$ for $t = 1, \beta = 0, 1, 2, \dots, k - 1$,
- (5) $\frac{d^k}{dt^k} q_\alpha(t)$ exists for $0 < t < 1$,
- (6) $(-1)^k \frac{d^k}{dt^k} q_\alpha(t) \geq 0$ and monotonic increasing for $0 < t < 1$,
- (7) $\int_0^t \frac{Q_k(u)}{u^{1+\alpha-k}} du = O\left(\frac{Q_k(t)}{t^{\alpha-k}}\right)$,

where

$$Q_k(t) = \int_{1-t}^1 (-1)^k \frac{d^k}{du^k} q_\alpha(u) du.$$

Also we set

$$Q(t) = \int_{1-t}^1 q_\alpha(u) du.$$

If $\lim_{w \rightarrow \infty} N_{q_\alpha} F(w)$ exists, we say that N_{q_α} -limit of $F(w)$ exists.

DEFINITION 2 [9,10]

Let $\sum_{n=0}^\infty u_n$ be an infinite series with $S(w) = \sum_{n \leq w} u_n$. If $\lim_{w \rightarrow \infty} \{ \sum_{n \leq w} u_n Q(1 - (n/w)) \} = 1$, we say that $\sum u_n$ is summable by N_{q_α} -method to the sum 1. In short we write that $\sum u_n = 1(N_{q_\alpha})$. Further the series $\sum u_n$ is said to be $|N_{q_\alpha}|$ -summable (absolute N_{q_α} -summable) if

$$\int_A^\infty \frac{dw}{w^2} \left| \sum_{n \leq w} n u_n q_\alpha\left(\frac{n}{w}\right) \right| < \infty$$

for some positive constant A .

For $\alpha = 0$, the method reduces to the original N_q -method [7] and if α is any positive integer p , then the method reduces to N_{q_p} -method of Moursund [6].

1.3.

Let $f(t)$ be a periodic function with period 2π and Lebesgue integrable over $(-\pi, \pi)$.

Let

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) \equiv \sum_{n=0}^{\infty} A_n(t). \tag{1.3.1}$$

The series conjugate to (1.3.1) at $t = x$ is given by

$$\sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) \equiv \sum_{n=1}^{\infty} B_n(x), \tag{1.3.2}$$

$$P(u) = \sum_{i=0}^{r-1} \frac{\theta_i}{i!} u^i \quad \text{for } -\pi \leq u \leq \pi,$$

where θ_i s for $i = 0, 1, 2, \dots, r - 1$ are arbitrary constants.

$$h(u) = \frac{\{f(x + u) - P(u)\} - (-1)^r \{f(x - u) - P(-u)\}}{2u^r},$$

$$H_0(t) = h(t),$$

$$H_\beta(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - u)^{\beta-1} h(u) du, \quad (\beta > 0),$$

$$h_\beta(t) = \Gamma(1 + \beta) t^{-\beta} H_\beta(t), \quad (\beta \geq 0).$$

2. Purpose of the present paper

In the present paper we shall prove the following theorems:

Theorem 1. *Let $\beta = \alpha - r$. If $H_\beta(+0) = 0$ and $\int_0^\pi t^{-\beta} |dH_\beta(t)| < \infty$, where $\beta > 0$, then the r th derived series of the conjugate series of $f(t)$ at $t = x$ is $|N_{q_\alpha}|$ -summable.*

Theorem 2. *Let $\rho = \alpha - r - 1$. If $\rho \geq 0$ and $\int_0^\pi t^{-1} |h_\rho(t)| dt < \infty$, then the r th derived series of the conjugate series of $f(t)$ at $t = x$ is $|N_{q_\alpha}|$ -summable.*

By taking $\beta = \rho + 1, \rho \geq 0$ in Theorem 1, we can obtain Theorem 2 at once as it is known [4] that

$$\begin{aligned} H_{\rho+1}(+0) = 0 \quad \text{and} \quad \int_0^\pi t^{-\rho-1} |dH_{\rho+1}(t)| < \infty \\ \iff h_{\rho+1}(t) \in BV(0, \pi) \quad \text{and} \quad \frac{h_{\rho+1}(t)}{t} \in L(0, \pi) \\ \iff \frac{h_\rho(t)}{t} \in L(0, \pi). \end{aligned}$$

By taking $q_\alpha(t) = (\alpha + \delta)(1 - t)^{\alpha+\delta-1}$, where $\delta > 0$ and $\alpha + \delta < k + 1$ ($[\alpha] = k$) in Theorems 1 and 2, we obtain the following corollaries respectively.

COROLLARY 1 [3]

If $H_\beta(+0) = 0$ and $\int_0^\pi t^{-\beta} |dH_\beta(t)| < \infty$, then the r th derived series of the conjugate series of $f(t)$ at $t = x$ is summable $|C, \beta + r + \delta|$, where $\beta > 0$ and $\delta > 0$.

COROLLARY 2 [3]

If $\rho \geq 0$ and $\int_0^\pi t^{-1} |h_\rho(t)| dt < \infty$, then the r th derived series of the conjugate series of $f(t)$ at $t = x$ is summable $|C, \rho + r + 1 + \delta|$.

3. Notations and lemmas

3.1. *Notations*

For our purpose we use the following notations throughout this paper.

$$\begin{aligned}
 [\alpha] &= k, \\
 m &= \min(k - r, r), \\
 q^k(u) &= (-1)^k \frac{d^k}{du^k} q_\alpha(u), \\
 (\cos nu)_j &= \left(\frac{d}{du}\right)^j \cos nu, \\
 S^{i,j}(x, u) &= \sum_{n \leq x} (x - n)^i (\cos nu)_j, \\
 G_i(w, u) &= \sum_{n \leq w} q_\alpha\left(\frac{n}{w}\right) \left(\frac{d}{du}\right)^{k+1-i} \cos nu, \quad \text{for } i = 0, 1, 2, \dots, m, \\
 g_i(x, w, u) &= \frac{1}{k!} (-1)^k \left(\frac{d}{dx}\right)^k q_\alpha\left(\frac{x}{w}\right) \frac{d}{dx} S^{k,k+1-i}(x, u) \\
 &\quad \text{for } i = 0, 1, 2, \dots, m.
 \end{aligned}$$

3.2. *Lemmas*

We need the following lemmas for the proof of our theorem.

Lemma 1 [1]. *If $\beta > \alpha > 0$, $H_\alpha(t)$ is of $BV(0, \pi)$ and $H_\alpha(+0) = 0$, then $H_\beta(t)$ is an integral in $(0, \pi)$ and for almost all values of t ,*

$$H'_\beta(t) = \frac{1}{\Gamma(\beta - \alpha)} \int_0^t (t - u)^{\beta - \alpha - 1} dH_\alpha(u).$$

Lemma 2 [6]. *If $\alpha \geq 1$, the kernel $q_\alpha(t)$ is monotonic decreasing, its derivatives of odd orders less than k are negative and monotonic increasing, its derivatives of even orders less than k are positive and monotonic decreasing and there exists a constant A_k such that*

$$\left| \frac{d^\beta}{dt^\beta} q_\alpha(t) \right| < A_k \quad (\beta = 0, 1, 2, \dots, k - 1)$$

and

$$\int_0^1 \left| \frac{d^k}{dt^k} q_\alpha(t) \right| dt < A_k.$$

Lemma 3 [10]. *$Q_k(t)$ is continuous and monotonic increasing function of t , $Q_k(t) \geq 0$, $Q(0) = 0$ and $Q(1) = 1$.*

This follows directly from the definition of $Q(t)$ and $Q_k(t)$.

Lemma 4 [10,8]. *$\int_0^1 q^k(t) / ((1 - t)^{\alpha - k}) dt$ exists.*

Lemma 5 [8]. Let $x > 0$.

(i) If $1/x < u \leq \pi$, then

$$S^{i,j}(x, u) = \begin{cases} O(x^i u^{-j-1}) & \text{for } 0 \leq j \leq i, \\ O(x^j u^{-i-1}) & \text{for } j > i \geq 0. \end{cases}$$

(ii) If $1/x \geq u > 0$, then

$$S^{i,j}(x, u) = O(x^{i+j+1}).$$

Lemma 6 [2]. Let $\lambda = \{\lambda_n\}$ be a positive monotonic increasing sequence with $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$. Then

$$A_\lambda(x) = A_\lambda^0(x) = \sum_{\lambda_n \leq x} a_n$$

and

$$A_\lambda^r(x) = \sum_{\lambda_n \leq x} (x - \lambda_n)^r a_n (r > 0).$$

If k is a positive integer,

$$A_\lambda(x) = \frac{1}{k!} \left(\frac{d}{dx} \right)^k A_\lambda^k(x).$$

Lemma 7 [8,10]. For $\alpha \geq 1$,

$$\sum_{n \leq w} (-1)^n n^k q_\alpha \left(\frac{n}{w} \right) = O \left\{ q^k \left(1 - \frac{1}{w} \right) \right\} + O \left\{ w Q_k \left(\frac{1}{w} \right) \right\}.$$

Lemma 8 [8,10]. For $\alpha \geq 1$ and $r = 0, 1, 2, \dots, k - 1$,

$$\sum_{n \leq w} (-1)^n n^r q_\alpha \left(\frac{n}{w} \right) = O(1).$$

Lemma 9. For $i = 0, 1, 2, \dots, k - 1$,

$$\sum_{n \leq w} (-1)^n n^i \in |Nq_\alpha|.$$

Proof. For $i = 0, 1, 2, \dots, k - 2$,

$$\begin{aligned} \int_1^\infty \frac{dw}{w^2} \left| \sum_{n \leq w} n (-1)^n n^i q_\alpha \left(\frac{n}{w} \right) \right| &= \int_1^\infty O(1) \frac{dw}{w^2} \quad \text{by Lemma 8} \\ &= O(1) \end{aligned}$$

and

$$\begin{aligned} & \int_1^\infty \frac{dw}{w^2} \left| \sum_{n \leq w} (-1)^n n^k q_\alpha \left(\frac{n}{w} \right) \right| \\ &= \int_1^\infty O \left\{ q^k \left(1 - \frac{1}{w} \right) \right\} \frac{dw}{w^2} \\ & \quad + \int_1^\infty O \left\{ w Q_k \left(\frac{1}{w} \right) \right\} \frac{dw}{w^2} \text{ by Lemma 7} \\ &= O \left(\int_0^1 q^k(u) du \right) + O \left(\int_0^1 \frac{Q_k(u)}{u} du \right) \\ &= O(1) \end{aligned}$$

by Lemma 2 and the definitions of $q^k(u)$ and $Q_k(u)$. This completes the proof of Lemma 9.

Lemma 10. For $i = 0, 1, 2, \dots, m$,

$$G_i(w, u) = \int_1^w g_i(x, w, u) dx.$$

Proof.

$$\begin{aligned} G_i(w, u) &= \sum_{n \leq w} q_\alpha \left(\frac{n}{w} \right) \left(\frac{d}{du} \right)^{k+1-i} \cos nu \\ &= q_\alpha(1) \sum_{n \leq w} \left(\frac{d}{du} \right)^{k+1-i} \cos nu \\ & \quad - \int_1^w \frac{d}{dx} q_\alpha \left(\frac{x}{w} \right) \left\{ \sum_{n \leq x} \left(\frac{d}{du} \right)^{k+1-i} \cos nu \right\} dx \\ &= - \int_1^w \frac{d}{dx} q_\alpha \left(\frac{x}{w} \right) \frac{1}{k!} \left(\frac{d}{dx} \right)^k \left\{ \sum_{n \leq x} (x-n)^k \left(\frac{d}{du} \right)^{k+1-i} \cos nu \right\} dx \\ & \quad \text{by Lemma 6} \\ &= \left[\frac{1}{k!} \sum_{\rho=1}^{k-1} (-1)^\rho \left(\frac{d}{dx} \right)^\rho q_\alpha \left(\frac{x}{w} \right) \left(\frac{d}{dx} \right)^{k-\rho} S^{k, k+1-i}(x, u) \right]_{x=1}^w \\ & \quad + \int_1^w \frac{(-1)^k}{k!} \left(\frac{d}{dx} \right)^k q_\alpha \left(\frac{x}{w} \right) \frac{d}{dx} S^{k, k+1-i}(x, u) dx \\ & \quad \text{(integrating by parts for } (k-1) \text{ times)} \\ &= \int_1^w g_i(x, w, u) dx \end{aligned}$$

as the integrated part vanishes for $x = w$ and $x = 1$.

Lemma 11. For $wt \leq \pi$ and $i = 0, 1, 2, \dots, m$,

$$\int_t^{t+(1/w)} u^{r-i} (u-t)^{k-\alpha} G_i(w, u) du = O(w^{\alpha-r+1}).$$

Proof. For $i = 0, 1, 2, \dots, m$,

$$\begin{aligned} & \int_t^{t+(1/w)} u^{r-i} (u-t)^{k-\alpha} G_i(w, u) du \\ &= \int_t^{t+(1/w)} u^{r-i} (u-t)^{k-\alpha} \left(\sum_{n \leq w} q_\alpha \left(\frac{n}{w} \right) \left(\frac{d}{du} \right)^{k+1-i} \cos nu \right) du \\ &= \int_t^{t+(1/w)} u^{r-i} (u-t)^{k-\alpha} O(w^{k+2-i}) du \\ &= O \left\{ \left(t + \frac{1}{w} \right)^{r-i} w^{k+2-i} \int_t^{t+(1/w)} (u-t)^{k-\alpha} du \right\} \\ &= O \left\{ \left(\frac{wt+1}{w} \right)^{r-i} w^{k+2-i} \cdot \frac{1}{w^{k-\alpha+1}} \right\} \\ &= O(w^{\alpha-r+1}) \text{ as } wt \leq \pi. \end{aligned}$$

Lemma 12. For $i = 0, 1, 2, \dots, m$ and $wt \leq \pi$,

$$\int_{t+(1/w)}^\pi u^{r-i} (u-t)^{k-\alpha} G_i(w, u) du = O(w^{\alpha-r+1}).$$

Proof. By the use of Lemma 10,

$$\begin{aligned} & \int_{t+(1/w)}^\pi u^{r-i} (u-t)^{k-\alpha} G_i(w, u) du \\ &= \int_{t+(1/w)}^\pi u^{r-i} (u-t)^{k-\alpha} du \int_1^w g_i(x, w, u) dx \\ &= \int_{t+(1/w)}^\pi u^{r-i} (u-t)^{k-\alpha} du \frac{1}{k!} \\ & \quad \times \int_1^w (-1)^k \left(\frac{d}{dx} \right)^k q_\alpha \left(\frac{x}{w} \right) \frac{d}{dx} S^{k, k+1-i}(x, u) dx \\ &= \frac{1}{(k-1)!} \int_1^w (-1)^k \left(\frac{d}{dx} \right)^k q_\alpha \left(\frac{x}{w} \right) dx \\ & \quad \times \int_{t+(1/w)}^\pi u^{r-i} (u-t)^{k-\alpha} S^{k-1, k+1-i}(x, u) du \\ &= \frac{1}{(k-1)!} \int_1^w (-1)^k \left(\frac{d}{dx} \right)^k q_\alpha \left(\frac{x}{w} \right) dx w^{\alpha-k} \\ & \quad \times \int_{t+(1/w)}^\xi u^{r-i} S^{k-1, k+1-i}(x, u) du, \tag{3.2.1} \end{aligned}$$

for some $t + (1/w) < \xi < \pi$, by an application of the mean value theorem. For $i \geq 2$, using Lemma 5(i) in (3.2.1), we get

$$\begin{aligned}
 & \int_{t+(1/w)}^{\pi} u^{r-i} (u-t)^{k-\alpha} G_i(w, u) du \\
 &= \frac{1}{(k-1)!} \int_1^w (-1)^k \left(\frac{d}{dx}\right)^k q_\alpha\left(\frac{x}{w}\right) w^{\alpha-k} dx \\
 & \quad \times \int_{t+(1/w)}^{\xi} u^{r-i} O(x^{k-1} u^{-k-2+i}) du \\
 &= \frac{1}{(k-1)!} \int_1^w (-1)^k \left(\frac{d}{dx}\right)^k q_\alpha\left(\frac{x}{w}\right) w^{\alpha-k} O\left\{\frac{x^{k-1}}{\left(t+\frac{1}{w}\right)^{k-r+1}}\right\} dx \\
 &= O\left(w^{\alpha-r+1} \int_1^w (-1)^k \left(\frac{d}{dx}\right)^k q_\alpha\left(\frac{x}{w}\right) x^{k-1} dx\right) \\
 &= O\left(w^{\alpha-r+1} \int_0^1 q^k(\theta) d\theta\right) \\
 &= O\left(w^{\alpha-r+1}\right) \text{ by Lemma 2.}
 \end{aligned}$$

For $i = 1$,

$$\begin{aligned}
 & \int_{t+(1/w)}^{\xi} u^{r-i} S^{k-1, k+1-i}(x, u) du \\
 &= \int_{t+(1/w)}^{\xi} u^{r-1} S^{k-1, k}(x, u) du \\
 &= \left[u^{r-1} S^{k-1, k-1}(x, u) \right]_{t+(1/w)}^{\xi} \\
 & \quad - (r-1) \int_{t+(1/w)}^{\xi} u^{r-2} S^{k-1, k-1}(x, u) du \\
 &= O\left\{\frac{x^{k-1}}{\left(t+\frac{1}{w}\right)^{k-r+1}}\right\} + \int_{t+(1/w)}^{\xi} u^{r-2} O(x^{k-1} u^{-k}) du \\
 & \hspace{15em} \text{by Lemma 5(i)} \\
 &= O(w^{2k-r}).
 \end{aligned}$$

Similarly, for $i = 0$, integrating by parts twice and using Lemma 5(i), it follows that

$$\int_{t+(1/w)}^{\xi} u^{r-i} S^{k-1, k+1-i}(x, u) du = O(w^{2k-r}).$$

Hence, for $i \leq 1$, using the above estimation in (3.2.1)

$$\begin{aligned} & \int_{t+(1/w)}^\pi u^{r-i}(u-t)^{k-\alpha} G_i(w, u) du \\ &= O\left(\int_1^w (-1)^k \left(\frac{d}{dx}\right)^k q_\alpha\left(\frac{x}{w}\right) w^{\alpha+k-r} dx\right) \\ &= O\left(w^{\alpha-r+1} \int_0^1 q^k(\theta) d\theta\right) \\ &= O(w^{\alpha-r+1}) \text{ by Lemma 2.} \end{aligned}$$

This completes the proof of Lemma 12.

Lemma 13.

$$\begin{aligned} & \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} du \int_1^{w-(\pi/t)} g_i(x, w, u) dx \\ &= O\left(\frac{w^{\alpha-k}}{t^{k+1-r}} q^k\left(1 - \frac{\pi}{wt}\right)\right). \end{aligned}$$

Proof. For some $1 < \xi < w - (\pi/t)$, by an application of the mean value theorem,

$$\begin{aligned} & \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} du \int_1^{w-(\pi/t)} g_i(x, w, u) dx \\ &= \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} du \\ & \quad \times \int_1^{w-(\pi/t)} \frac{(-1)^k}{k!} \left(\frac{d}{dx}\right)^k q_\alpha\left(\frac{x}{w}\right) \frac{d}{dx} S^{k,k+1-i}(x, u) dx \\ &= \int_t^{t+(1/w)} \frac{1}{k!} u^{r-i}(u-t)^{k-\alpha} \left[(-1)^k \left(\frac{d}{dx}\right)^k q_\alpha\left(\frac{x}{w}\right)\right]_{x=w-(\pi/t)} du \\ & \quad \times \int_\xi^{w-(\pi/t)} \frac{d}{dx} S^{k,k+1-i}(x, u) dx \\ &= \frac{1}{k!} \int_t^{t+(1/w)} \frac{u^{r-i}(u-t)^{k-\alpha}}{w^k} q^k\left(1 - \frac{\pi}{wt}\right) \left[S^{k,k+1-i}(x, u)\right]_{x=\xi}^{w-(\pi/t)} du. \end{aligned} \tag{3.2.2}$$

For $i = 0$, using Lemma 5(i) in (3.2.2), we get

$$\begin{aligned} & \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} du \int_1^{w-(\pi/t)} g_i(x, w, u) dx \\ &= \frac{1}{k!} \int_t^{t+(1/w)} \frac{u^r(u-t)^{k-\alpha}}{w^k} q^k\left(1 - \frac{\pi}{wt}\right) O\left\{\frac{\left(w - \frac{\pi}{t}\right)^{k+1}}{u^{k+1}}\right\} du \end{aligned}$$

$$\begin{aligned}
&= O\left(\frac{wq^k\left(1-\frac{\pi}{wt}\right)}{t^{k+1-r}}\int_t^{t+(1/w)}(u-t)^{k-\alpha}du\right) \\
&= O\left(\frac{w^{\alpha-k}q^k\left(1-\frac{\pi}{kt}\right)}{t^{k+1-r}}\right).
\end{aligned}$$

For $i \geq 1$, using Lemma 5(i) in (3.2.2), we obtain

$$\begin{aligned}
&\int_t^{t+(1/w)}u^{r-i}(u-t)^{k-\alpha}du\int_1^{w-(\pi/t)}g_i(x,w,u)dx \\
&= \frac{1}{k!}\int_t^{t+(1/w)}\frac{u^{r-i}(u-t)^{k-\alpha}}{w^k}q^k\left(1-\frac{\pi}{wt}\right)O\left(\frac{w^k}{u^{k+2-i}}\right)du \\
&= O\left(\frac{q^k\left(1-\frac{\pi}{wt}\right)}{t^{k+2-r}}\int_t^{t+(1/w)}(u-t)^{k-\alpha}du\right) \\
&= O\left(\frac{w^{\alpha-k+1}}{t^{k+2-r}}q^k\left(1-\frac{\pi}{wt}\right)\right) \\
&= O\left(\frac{w^{\alpha-k}}{t^{k+1-r}}q^k\left(1-\frac{\pi}{wt}\right)\right) \text{ as } wt > \pi.
\end{aligned}$$

This completes the proof of Lemma 13.

Lemma 14. For $i = 0, 1, 2, \dots, m$,

$$\int_{t+(1/w)}^{\pi}u^{r-i}(u-t)^{k-\alpha}S^{k,k+1-i}\left(w-\frac{\pi}{t},u\right)du=O\left(\frac{w^{\alpha}}{t^{k+1-r}}\right).$$

Proof. By an application of the mean value theorem for some $t+(1/w) < \xi < \pi$,

$$\begin{aligned}
&\int_{t+(1/w)}^{\pi}u^{r-i}(u-t)^{k-\alpha}S^{k,k+1-i}\left(w-\frac{\pi}{t},u\right)du \\
&= w^{\alpha-k}\int_{t+(1/w)}^{\xi}u^{r-i}S^{k,k+1-i}\left(w-\frac{\pi}{t},u\right)du \\
&= w^{\alpha-k}\left[u^{r-i}S^{k,k-i}\left(w-\frac{\pi}{t},u\right)\right]_{u=t+(1/w)}^{\xi} \\
&\quad - (r-i)w^{\alpha-k}\int_{t+(1/w)}^{\xi}u^{r-i-1}S^{k,k-i}\left(w-\frac{\pi}{t},u\right)du \\
&= w^{\alpha-k}O\left\{\frac{w^k}{\left(t+\frac{1}{w}\right)^{k+1-r}}\right\}+w^{\alpha-k}\int_{t+(1/w)}^{\xi}u^{r-i-1}O\left(\frac{w^k}{u^{k+1-i}}\right)du \\
&\hspace{20em}\text{by Lemma 5(i)} \\
&= O\left(\frac{w^{\alpha}}{t^{k+1-r}}\right)+O\left(w^{\alpha}\int_{t+(1/w)}^{\xi}\frac{1}{u^{k+2-r}}du\right) \\
&= O\left(\frac{w^{\alpha}}{t^{k+1-r}}\right).
\end{aligned}$$

Lemma 15. For $i = 0, 1, 2, \dots, m$,

$$\begin{aligned} & \int_{t+(1/w)}^\pi u^{r-i}(u-t)^{k-\alpha} du \int_1^{w-(\pi/t)} g_i(x, w, u) dx \\ &= O \left\{ \frac{w^{\alpha-k} q^k \left(1 - \frac{\pi}{wt}\right)}{t^{k+1-r}} \right\}. \end{aligned}$$

Proof. For some $1 < \eta < w - (\pi/t)$, by an application of the mean value theorem

$$\begin{aligned} & \int_{t+(1/w)}^\pi u^{r-i}(u-t)^{k-\alpha} du \int_t^{w-(\pi/t)} g_i(x, w, u) dx \\ &= \frac{1}{k!} \int_{t-(1/w)}^\pi u^{r-i}(u-t)^{k-\alpha} du \left[(-1)^k \left(\frac{d}{dx}\right)^k q_\alpha \left(\frac{x}{w}\right) \right]_{x=w-(\pi/t)} \\ & \quad \times \int_\eta^{w-(\pi/t)} \frac{d}{dx} S^{k,k+1-i}(x, u) dx \\ &= \frac{q^k \left(1 - \frac{\pi}{wt}\right)}{k! w^k} \left\{ \int_{t+(1/w)}^\pi u^{r-i}(u-t)^{k-\alpha} S^{k,k+1-i} \left(w - \frac{\pi}{t}, u\right) du \right. \\ & \quad \left. - \int_{t+(1/w)}^\pi u^{r-i}(u-t)^{k-\alpha} S^{k,k+1-i}(\eta, u) du \right\} \\ &= O \left(\frac{q^k \left(1 - \frac{\pi}{wt}\right) w^\alpha}{w^k t^{k+1-r}} \right) \\ &= O \left(\frac{q^k \left(1 - \frac{\pi}{wt}\right) w^{\alpha-k}}{t^{k+1-r}} \right), \end{aligned}$$

since by Lemma 14, the first integral is $O(w^\alpha/(t^{k+1-r}))$ and the second integral is dominated by the first integral.

Lemma 16. For $i = 0, 1, 2, \dots, m$ and $wu > \pi$,

$$\int_{w-(\pi/t)}^w g_i(x, w, u) dx = O \left(w^2 u^{-k+i} Q_k \left(\frac{\pi}{wt}\right) \right).$$

Proof. For $0 \leq i$, by use of Lemma 5(i),

$$\begin{aligned} & \int_{w-(\pi/t)}^w g_i(x, w, u) dx \\ &= \int_{w-(\pi/t)}^w \frac{(-1)^k}{(k-1)!} \left(\frac{d}{dx}\right)^k q_\alpha \left(\frac{x}{w}\right) S^{k-1,k+1-i}(x, u) dx \\ &= \frac{1}{(k-1)!} \int_{w-(\pi/t)}^w (-1)^k \left(\frac{d}{dx}\right)^k q_\alpha \left(\frac{x}{w}\right) O \left(\frac{x^{k+1-i}}{u^k}\right) dx \end{aligned}$$

$$\begin{aligned}
 &= O\left(\frac{w^{2-i}}{u^k} \int_{1-(\pi/wt)}^1 q^k(\theta) d\theta\right) \\
 &= O\left(w^2 u^{-k+i} Q_k\left(\frac{\pi}{wt}\right)\right) \text{ as } wu > \pi.
 \end{aligned}$$

For $i \geq 2$, by use of Lemma 5(i)

$$\begin{aligned}
 &\int_{w-(\pi/t)}^w g_i(x, w, u) dx \\
 &= \int_{w-(\pi/t)}^w \frac{(-1)^k}{(k-1)!} \left(\frac{d}{dx}\right)^k q_\alpha\left(\frac{x}{w}\right) S^{k-1, k+1-i}(x, u) dx \\
 &= \frac{1}{(k-1)!} \int_{w-(\pi/t)}^w (-1)^k \left(\frac{d}{dx}\right)^k q_\alpha\left(\frac{x}{w}\right) O\left(\frac{x^{k-1}}{u^{k+2-i}}\right) dx \\
 &= O\left(u^{-k-2+i} \int_{1-(\pi/wt)}^1 q^k(\theta) d\theta\right) \\
 &= O\left(w^2 u^{-k+i} Q_k\left(\frac{\pi}{wt}\right)\right) \text{ as } wu > \pi.
 \end{aligned}$$

Hence

$$\int_{w-(\pi/t)}^w g_i(x, w, u) dx = O\left(w^2 u^{-k+i} Q_k\left(\frac{\pi}{wt}\right)\right).$$

Lemma 17. For $i = 0, 1, 2, \dots, m$ and $wt > \pi$,

$$\int_{t+(1/w)}^\pi u^{r-i} (u-t)^{k-\alpha} \frac{d}{dx} S^{k, k+1-i}(x, u) du = O\left(\frac{w^\alpha}{t^{k-r}}\right).$$

Proof. Let $i = 0$. By mean value theorem for some $t + (1/w) < \xi < \pi$,

$$\begin{aligned}
 &\int_{t+(1/w)}^\pi u^{r-i} (u-t)^{k-\alpha} \frac{d}{dx} S^{k, k+1-i}(x, u) du \\
 &= k \int_{t+(1/w)}^\pi u^r (u-t)^{k-\alpha} S^{k-1, k+1}(x, u) du \\
 &= kw^{\alpha-k} \int_{t+(1/w)}^\xi u^r S^{k-1, k+1}(x, u) du \\
 &= kw^{\alpha-k} [u^r S^{k-1, k}(x, u)]_{u=t+(1/w)}^\xi \\
 &\quad - krw^{\alpha-k} \int_{t+(1/w)}^\xi u^{r-1} S^{k-1, k}(x, u) du
 \end{aligned}$$

$$\begin{aligned}
 &= kw^{\alpha-k} O \left\{ \frac{x^k}{(t + (1/w))^{k-r}} \right\} \\
 &\quad - krw^{\alpha-k} [u^{r-1} S^{k-1,k-1}(x, u)]_{u=t+(1/w)}^\xi \\
 &\quad + kr(r-1)w^{\alpha-k} \int_{t+(1/w)}^\xi u^{r-2} S^{k-1,k-1}(x, u) du \quad \text{by Lemma 5(i)} \\
 &= O \left(\frac{w^\alpha}{t^{k-r}} \right) + krw^{\alpha-k} O \left\{ \frac{x^{k-1}}{(t + \frac{1}{w})^{k-r+1}} \right\} \\
 &\quad + kr(r-1)w^{\alpha-k} \int_{t+(1/w)}^\xi u^{r-2} O \left(\frac{x^{k-1}}{u^k} \right) du \\
 &= O \left(\frac{w^\alpha}{t^{k-r}} \right) + O \left(\frac{w^{\alpha-1}}{t^{k-r+1}} \right) \\
 &= O \left(\frac{w^\alpha}{t^{k-r}} \right) \quad \text{as } wt > \pi.
 \end{aligned}$$

For $i > 1$, using the technique used in the proof of Lemma 14, it can be proved that

$$\begin{aligned}
 &\int_{t+(1/w)}^\pi u^{r-i}(u-t)^{k-\alpha} \frac{d}{dx} S^{k,k+1-i}(x, u) du \\
 &= O \left(\frac{w^{\alpha-1}}{t^{k-r+1}} \right) \\
 &= O \left(\frac{w^\alpha}{t^{k-r}} \right) \quad \text{as } wt > \pi.
 \end{aligned}$$

This completes the proof of Lemma 17.

Lemma 18. For $i = 0, 1, 2, \dots, m$,

$$\int_1^{\pi/t} \frac{dw}{w^2} \left| \int_t^\pi u^{r-i}(u-t)^{k-\alpha} G_i(w, u) du \right| = O \left(\frac{1}{t^{\alpha-r}} \right).$$

Proof.

$$\begin{aligned}
 &\int_1^{\pi/t} \frac{dw}{w^2} \left| \int_t^\pi u^{r-i}(u-t)^{k-\alpha} G_i(w, u) du \right| \\
 &\leq \int_1^{\pi/t} \frac{dw}{w^2} \left| \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} G_i(w, u) du \right| \\
 &\quad + \int_1^{\pi/t} \frac{dw}{w^2} \left| \int_{t+(1/w)}^\pi u^{r-i}(u-t)^{k-\alpha} G_i(w, u) du \right| \\
 &= \int_1^{\pi/t} \frac{dw}{w^2} O(w^{\alpha-r+1}), \quad \text{by Lemmas 11 and 12} \\
 &= O \left(\frac{1}{t^{\alpha-r}} \right).
 \end{aligned}$$

Lemma 19. For $i = 0, 1, 2, \dots, m$ and $wt > \pi$,

$$\begin{aligned} & \int_t^\pi u^{r-i}(u-t)^{k-\alpha} G_i(w, u) \, du \\ &= O\left(\frac{w^{\alpha-k} q^k \left(1 - \frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + O\left(\frac{w^{\alpha-k+1} Q_k\left(\frac{\pi}{wt}\right)}{t^{k-r}}\right). \end{aligned}$$

Proof. Using Lemma 10,

$$\begin{aligned} & \int_t^\pi u^{r-i}(u-t)^{k-\alpha} G_i(w, u) \, du \\ &= \int_t^\pi u^{r-i}(u-t)^{k-\alpha} \, du \int_1^w g_i(x, w, u) \, dx \\ &= \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} \, du \int_1^w g_i(x, w, u) \, dx \\ &\quad + \int_{t+(1/w)}^\pi u^{r-i}(u-t)^{k-\alpha} \, du \int_1^w g_i(x, w, u) \, dx \\ &= J_1 + J_2, \quad \text{say.} \end{aligned}$$

Using Lemmas 13 and 16,

$$\begin{aligned} J_1 &= \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} \, du \int_1^{w-(\pi/t)} g_i(x, w, u) \, dx \\ &\quad + \int_t^{t+(1/w)} u^{r-i}(u-t)^{k-\alpha} \, du \int_{w-(\pi/t)}^w g_i(x, w, u) \, dx \\ &= O\left(\frac{w^{\alpha-k} q^k \left(1 - \frac{\pi}{wt}\right)}{t^{k-r+1}}\right) \\ &\quad + O\left(\int_t^{t+(1/w)} u^{r-k}(u-t)^{k-\alpha} w^2 Q_k\left(\frac{\pi}{wt}\right) \, du\right) \\ &= O\left(\frac{w^{\alpha-k} q^k \left(1 - \frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + O\left(\frac{w^{\alpha-k+1} Q_k\left(\frac{\pi}{wt}\right)}{t^{k-r}}\right) \quad \text{as } k \geq r \end{aligned}$$

and

$$\begin{aligned} J_2 &= \int_{t+(1/w)}^\pi u^{r-i}(u-t)^{k-\alpha} \, du \int_1^{w-(\pi/t)} g_i(x, w, u) \, dx \\ &\quad + \int_{t+(1/w)}^\pi u^{r-i}(u-t)^{k-\alpha} \, du \int_{w-(\pi/t)}^w g_i(x, w, u) \, dx \\ &= O\left(\frac{w^{\alpha-k} q^k \left(1 - \frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + \int_{w-(\pi/t)}^w dx \\ &\quad \times \int_{t+(1/w)}^\pi u^{r-i}(u-t)^{k-\alpha} g_i(x, w, u) \, dx \quad \text{by Lemma 15,} \end{aligned}$$

$$\begin{aligned}
 &= O\left(\frac{w^{\alpha-k} q^k \left(1 - \frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + \frac{1}{k!} \int_{w-(\pi/t)}^w (-1)^k \left(\frac{d}{dx}\right)^k q_\alpha\left(\frac{x}{w}\right) dx \\
 &\quad \times \int_{t+(1/w)}^\pi u^{r-i} (u-t)^{k-\alpha} \frac{d}{dx} S^{k,k+1-i}(x, u) dx \\
 &= O\left(\frac{w^{\alpha-k} q^k \left(1 - \frac{\pi}{wt}\right)}{t^{k-r+1}}\right) \\
 &\quad + \frac{1}{k!} \int_{w-(\pi/t)}^w (-1)^k \left(\frac{d}{dx}\right)^k q_\alpha\left(\frac{x}{w}\right) O\left(\frac{w^\alpha}{t^{k-r}}\right) dx \quad \text{by Lemma 17} \\
 &= O\left(\frac{w^{\alpha-k} q^k \left(1 - \frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + O\left(\frac{w^{\alpha-k+1}}{t^{k-r}} \int_{1-(\pi/wt)}^1 q^k(\theta) d\theta\right) \\
 &= O\left(\frac{w^{\alpha-k} q^k \left(1 - \frac{\pi}{wt}\right)}{t^{k-r+1}}\right) + O\left(\frac{w^{\alpha-k+1} Q_k\left(\frac{\pi}{wt}\right)}{t^{k-r}}\right).
 \end{aligned}$$

This completes the proof of Lemma 19.

Lemma 20. For $i = 0, 1, 2, \dots, m$,

$$\int_{\pi/t}^\infty \frac{dw}{w^2} \left| \int_t^\pi u^{r-i} (u-t)^{k-\alpha} G_i(w, u) du \right| = O\left(\frac{1}{t^{\alpha-r}}\right).$$

Proof. By the use of Lemma 19, we get

$$\begin{aligned}
 &\int_{\pi/t}^\infty \frac{dw}{w^2} \left| \int_t^\pi u^{r-i} (u-t)^{k-\alpha} G_i(w, u) du \right| \\
 &= O\left(\int_{\pi/t}^\infty \frac{w^{\alpha-k-2} q^k \left(1 - \frac{\pi}{wt}\right)}{t^{k-r+1}} dw\right) + O\left(\int_{\pi/t}^\infty \frac{w^{\alpha-k-1} Q_k\left(\frac{\pi}{wt}\right)}{t^{k-r}} dw\right) \\
 &= O\left(\frac{1}{t^{\alpha-r}} \int_0^1 \frac{q^k(\theta)}{(1-\theta)^{\alpha-k}} d\theta\right) + O\left(\frac{1}{t^{\alpha-r}} \int_0^1 \frac{Q_k(u)}{u^{\alpha-k+1}} du\right) \\
 &= O\left(\frac{1}{t^{\alpha-r}}\right) \quad \text{by Lemma 4.}
 \end{aligned}$$

4. Proof of the theorem

Proof of Theorem 1. We have for $r \geq 1$,

$$\begin{aligned}
 \left(\frac{d}{dx}\right)^r B_n(x) &= \frac{2}{\pi} \int_0^\pi \frac{(-1)^r}{2} \{f(x+u) - (-1)^r f(x-w)\} \\
 &\quad \times \left(\frac{d}{du}\right)^r \sin nu du
 \end{aligned}$$

$$\begin{aligned}
 &= (-1)^r \frac{2}{\pi} \int_0^\pi h(u) u^r \left(\frac{d}{du}\right)^r \sin nu \, du \\
 &\quad + (-1)^r \frac{2}{\pi} \int_0^\pi \frac{1}{2} \{P(u) - (-1)^r P(-u)\} \left(\frac{d}{du}\right)^r \sin nu \, du \\
 &= \alpha_n + \beta_n, \quad \text{say.}
 \end{aligned}$$

For the proof of our theorem it is enough to show that

$$\sum \alpha_n \in |N_{q_\alpha}|$$

and

$$\sum \beta_n \in |N_{q_\alpha}|.$$

Now

$$\begin{aligned}
 n\alpha_n &= (-1)^r \frac{2}{\pi} \int_0^\pi nh(u) u^r \left(\frac{d}{du}\right)^r \sin nu \, du \\
 &= (-1)^{r+1} \frac{2}{\pi} \int_0^\pi h(u) u^r \left(\frac{d}{du}\right)^{r+1} \cos nu \, du \\
 &= (-1)^{r+1} \frac{2}{\pi} \left[\sum_{j=1}^{k-r} (-1)^{j-1} H_j(u) \left(\frac{d}{du}\right)^{j-1} \right. \\
 &\quad \left. \times \left\{ u^r \left(\frac{d}{du}\right)^{r+1} \cos nu \right\} \right]_{u=0}^\pi \\
 &\quad + (-1)^{k+1} \frac{2}{\pi} \int_0^\pi H_{k-r}(u) \left(\frac{d}{du}\right)^{k-r} \left\{ u^r \left(\frac{d}{du}\right)^{r+1} \cos nu \right\} du \\
 &= J_1(n) + J_2(n), \quad \text{say.} \tag{4.1}
 \end{aligned}$$

Since for $j = 1, 2, \dots, k - r, H_j(+0) = O$ it is clear that $J_1(n)$ is the sum of the terms containing $(-1)^n n^p$, where p is even and $r + 1 \leq p \leq k$.

By the use of Lemma 9, for $p = 1, 2, \dots, k$,

$$\int_1^\infty \frac{dw}{w^2} \left| \sum_{n \leq w} n^p (-1)^n q_\alpha \left(\frac{n}{w}\right) \right| < \infty.$$

Hence

$$\int_1^\infty \frac{dw}{w^2} \left| \sum_{n \leq w} J_1(n) q_\alpha \left(\frac{n}{w}\right) \right| < \infty. \tag{4.2}$$

Now

$$\begin{aligned}
 J_2(n) &= (-1)^{k+1} \frac{2}{\pi} \int_0^\pi H_{k-r}(u) \left(\frac{d}{du}\right)^{k-r} \left\{ u^r \left(\frac{d}{du}\right)^{r+1} \cos nu \right\} du \\
 &= \frac{2(-1)^{k+1}}{\pi \Gamma(k-\alpha+1)} \int_0^\pi \left(\frac{d}{du}\right)^{k-r} \left\{ u^r \left(\frac{d}{du}\right)^{r+1} \cos nu \right\} du \\
 &\quad \times \int_0^u (u-t)^{k-\alpha} dH_\beta(t) \text{ by Lemma 1 as } \beta = \alpha - r \text{ and } [\alpha] = k \\
 &= \frac{2(-1)^{k+1}}{\pi \Gamma(k-\alpha+1)} \int_0^\pi dH_\beta(t) \int_t^\pi (u-t)^{k-\alpha} \left(\frac{d}{du}\right)^{k-r} \\
 &\quad \times \left\{ u^r \left(\frac{d}{du}\right)^{r+1} \cos nu \right\} du \\
 &= \frac{2(-1)^{k+1}}{\pi \Gamma(k-\alpha+1)} \int_0^\pi dH_\beta(t) \int_t^\pi (u-t)^{k-\alpha} \\
 &\quad \times \left\{ \sum_{i=0}^m \binom{k-r}{i} \left(\frac{d}{du}\right)^i u^r \left(\frac{d}{du}\right)^{k+1-i} \cos nu \right\} du \\
 &\hspace{15em} \text{where } m = \min(k-r, r) \\
 &= \frac{2(-1)^{k+1}}{\pi \Gamma(k-\alpha+1)} \sum_{i=0}^m \binom{k-r}{i} \frac{r!}{(r-i)!} \int_0^\pi dH_\beta(t) \\
 &\quad \times \int_t^\pi (u-t)^{k-\alpha} u^{r-i} \left(\frac{d}{du}\right)^{k+1-i} \cos nu du.
 \end{aligned}$$

By the use of Lemmas 20 and 18,

$$\begin{aligned}
 &\int_1^\infty \frac{dw}{w^2} \left| \sum_{n \leq w} J_2(n) q_\alpha \left(\frac{n}{w}\right) \right| \\
 &\leq \frac{2}{\pi \Gamma(k-\alpha+1)} \sum_{i=0}^m \binom{k-r}{i} \frac{r!}{(r-i)!} \int_0^\pi |dH_\beta(t)| \int_1^\infty \frac{dw}{w^2} \\
 &\quad \times \left| \int_t^\pi u^{r-i} (u-t)^{k-\alpha} G_i(w, u) du \right| \\
 &= \frac{2}{\pi \Gamma(k-\alpha+1)} \sum_{i=0}^m \binom{k-r}{i} \frac{r!}{(r-i)!} \int_0^\pi |dH_\beta(t)| \\
 &\quad \times \left\{ \int_1^{\pi/t} \frac{dw}{w^2} \left| \int_t^\pi u^{r-1} (u-t)^{k-\alpha} G_i(w, u) du \right| \right. \\
 &\quad \left. + \int_{\pi/t}^\infty \frac{dw}{w^2} \left| \int_t^\pi u^{r-i} (u-t)^{k-\alpha} G_i(w, u) du \right| \right\}
 \end{aligned}$$

$$\begin{aligned}
&= \frac{2}{\pi \Gamma(k - \alpha + 1)} \sum_{i=0}^m \binom{k-i}{i} \frac{r!}{(r-i)!} \int_0^\pi |dH_\beta(t)| O\left(\frac{1}{t^{\alpha-r}}\right) \\
&\quad \text{by Lemmas 18 and 20} \\
&= O\left(\sum_{i=0}^m \binom{k-r}{i} \frac{r!}{(r-i)!} \int_0^\pi \frac{|dH_\beta(t)|}{t^\beta}\right) \text{ as } \alpha - r = \beta \\
&= O(1). \tag{4.3}
\end{aligned}$$

From (4.1), (4.2) and (4.3) it is clear that

$$\sum \alpha_n \in |N_{q_\alpha}|.$$

Let r be an odd number, i.e. $r = 2p + 1$, where $p = 0, 1, 2, \dots$. Then

$$\begin{aligned}
\beta_n &= -\frac{2}{\pi} \int_0^\pi \frac{1}{2} \{P(u) + P(-u)\} \left(\frac{d}{du}\right)^{2p+1} \sin nu \, du \\
&= (-1)^{p+1} \frac{2}{\pi} n^{2p+1} \int_0^\pi \left(\sum_{j=0}^p \frac{\theta_{2j} u^{2j}}{(2j)!}\right) \cos nu \, du \\
&= (-1)^{p+1} \frac{2}{\pi} n^{2p+1} \sum_{j=0}^p \frac{\theta_{2j}}{(2j)!} \int_0^u u^{2j} \cos nu \, du \\
&= (-1)^{p+1} \frac{2}{\pi} n^{2p+1} \sum_{j=1}^p \frac{\theta_{2j}}{(2j)!} (-1)^n \\
&\quad \times \left(\sum_{\mu=1}^j (-1)^{\mu+1} n^{-2\mu} \pi^{2j-2\mu+1} \frac{(2j)!}{(2j-2\mu)!}\right) \\
&= 2(-1)^n \sum_{\mu=1}^p (-1)^{p+\mu} n^{2p-2\mu+1} \sum_{j=\mu}^p \frac{\theta_{2j}}{(2j-2\mu)!} \pi^{2j-2\mu}.
\end{aligned}$$

Let r be an even number, i.e. $r = 2p$, where $p = 1, 2, \dots$. Then

$$\begin{aligned}
\beta_n &= \frac{2}{\pi} \int_0^\pi \frac{1}{2} \{P(u) - P(-u)\} \left(\frac{d}{du}\right)^{2p} \sin nu \, du \\
&= (-1)^p \frac{2}{\pi} n^{2p} \sum_{j=1}^p \frac{\theta_{2j-1}}{(2j-1)!} \int_0^\pi u^{2j-1} \sin nu \, du \\
&= (-1)^p \frac{2}{\pi} n^{2p} \sum_{j=1}^p \frac{\theta_{2j-1}}{(2j-1)!} (-1)^n \\
&\quad \times \left(\sum_{\mu=1}^j (-1)^{\mu-1} n^{-2\mu+1} \pi^{2j-2\mu+1} \frac{(2j-1)!}{(2j-2\mu)!}\right) \\
&= 2(-1)^n \sum_{\mu=1}^p (-1)^{p+\mu-1} n^{2p-2\mu+1} \sum_{j=\mu}^p \frac{\theta_{2j-1}}{(2j-2\mu)!} \pi^{2j-2\mu}.
\end{aligned}$$

So by the use of Lemma 9,

$$\int_1^\infty \frac{dw}{w^2} \left| \sum_{n \leq w} n \beta_n q_\alpha \left(\frac{n}{w} \right) \right| < \infty,$$

i.e. $\sum \beta_n \in |N_{q_\alpha}|$. This terminates the proof of Theorem 1.

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