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Positive solutions of singular boundary value problem of negative exponent Emden–Fowler equation

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Abstract. This paper investigates the existence of positive solutions of a singular boundary value problem with negative exponent similar to standard Emden–Fowler equation. A necessary and sufficient condition for the existence of $C[0, 1]$ positive solutions as well as $C^1[0, 1]$ positive solutions is given by means of the method of lower and upper solutions with the Schauder fixed point theorem.

Keywords. Singular boundary value problem; lower and upper solutions; positive solution.

1. Introduction

Consider the singular boundary value problems for the Emden–Fowler equation

$$u'' + p(t)u^{-\lambda}(t) = 0, \quad 0 < t < 1, \tag{1}$$

$$\alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0, \tag{2}$$

where $\alpha, \beta, \gamma, \delta \geq 0, \lambda \in R$ and $\rho := \gamma\beta + \alpha\gamma + \alpha\delta > 0$; $p \in C((0, 1), [0, \infty))$ and may be singular at $t = 0, t = 1$. When $\lambda < 0$, see [3,4,7,8] for the result concerning the above problem. When $\lambda > 0$, [6] shows the existence and uniqueness to (1) and (2) in the case of $\beta = \delta = 0$ by means of the shooting method. For the following problem

$$u'' + p(t)u^{-\lambda}(t) + q(t)u^{-m}(t) = 0, \quad 0 < t < 1, \tag{3}$$

$$\alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0, \tag{4}$$

where $\alpha, \beta, \gamma, \delta \geq 0$, $\rho = \gamma\beta + \alpha\gamma + \alpha\delta > 0$ and $p, q \in ((0, 1), [0, \infty))$. Mao [5] gave a sufficient and necessary condition when $\lambda < 0, m < 0$. In this paper we shall consider the case of $\lambda > 0, m > 0$ for the problems (3) and (4).

A function $u(t) \in C^1[0, 1] \cap C^2(0, 1)$ is a positive solution of (3) and (4) if u satisfies (3) and (4) and $u(t) > 0, t \in (0, 1)$.

2. Main results

We state the following hypothesis, which is used throughout this paper.

(H) $p(t), q(t) \in C(0, 1)$, $p(t) \geq 0$, $p(t) \not\equiv 0$, $q(t) \geq 0$, $q(t) \not\equiv 0$, $t \in (0, 1)$, $\lambda, m > 0$.

We now state the main results of this paper as follows:

Theorem 2.1. *Suppose that* (H) *is satisfied. Then*

(I) *If* $\beta\delta \neq 0$, *the problems* (3) *and* (4) *have a positive solution if and only if*

$$0 < \int_0^1 [p(t) + q(t)]\,\mathrm{d}t < \infty. \tag{5}$$

(II) *If* $\beta = 0, \delta \neq 0$, *the problems* (3) *and* (4) *have a positive solution if and only if*

$$0 < \int_0^1 t[p(t) + q(t)]\,\mathrm{d}t < \infty. \tag{6}$$

(III) *If* $\beta \neq 0, \delta = 0$, *the problems* (3) *and* (4) *have a positive solution if and only if*

$$0 < \int_0^1 (1-t)[p(t) + q(t)]\,\mathrm{d}t < \infty. \tag{7}$$

(IV) *If* $\beta = \delta = 0$, *the problems* (3) *and* (4) *have a positive solution if and only if*

$$0 < \int_0^1 t(1-t)[p(t) + q(t)]\,\mathrm{d}t < \infty. \tag{8}$$

Theorem 2.2. *Suppose that* (H) *is satisfied. Then problems* (3) *and* (4) *have a* $C^1[0,1]$ *positive solution if and only if the following inequalities hold.*

(H1)

$$0 < \int_0^1 [t^{-\lambda} p(t) + t^{-m}(t) q(t)]\,\mathrm{d}t < \infty, \quad \beta = 0, \delta \neq 0. \tag{9}$$

(H2)

$$0 < \int_0^1 [(1-t)^{-\lambda} p(t) + (1-t)^{-m} q(t)]\,\mathrm{d}t < \infty, \quad \beta \neq 0, \delta = 0. \tag{10}$$

(H3)

$$0 < \int_0^1 [t^{-\lambda}(1-t)^{-\lambda} p(t) + t^{-m}(1-t)^{-m} q(t)]\,\mathrm{d}t < \infty, \ \beta=\delta=0. \tag{11}$$

3. Proofs of the main results

First we prove Theorem 2.1. We will prove all the necessary conditions first then all the sufficient conditions.

1. *Necessity*

Case I: $\beta\delta \neq 0$. Let $u(t) \in C^1[0,1] \cap C^2(0,1)$ is a positive solution of (3) and (4). From (4) and the nontrivial concave function $u(t)$, we know that $u(t)$ must satisfy the following case:

$$u(0) \geq 0, \quad u(1) \geq 0, \quad u'(0) \geq 0, \quad u'(1) \leq 0.$$

Then there exists $t_0 \in [0,1]$ with $u'(t_0)=0$, $u''(t)<0$ yield $u'(t)\le 0$, $t\in[t_0,1)$; $u'(t)\ge 0$, $t\in(0,t_0]$. Let C_0 be a constant which satisfies $C_0u(t)<1/4$, $t\in[0,1]$, and $1/C_0\ge 4$. Then

$$p(t)u^{-\lambda}(t) \ge p(t)(4C_0)^{\lambda}, \tag{12}$$

$$q(t)u^{-m}(t) \ge q(t)(4C_0)^{m}. \tag{13}$$

By means of (12) and (13), we have

$$\begin{aligned}
u'(t) &= \int_t^{t_0} [p(s)u^{-\lambda}(s)+q(s)u^{-m}(s)]\,\mathrm{d}s\\
&\ge \int_t^{t_0} [(4C_0)^{\lambda}p(s)+(4C_0)^{m}q(s)]\,\mathrm{d}s\\
&\ge (4C_0)^{\min\{\lambda,m\}}\int_t^{t_0}[p(s)+q(s)]\,\mathrm{d}s,\quad t\in[0,t_0)\\
-u'(t) &= \int_{t_0}^{t} [p(s)u^{-\lambda}(s)+q(s)u^{-m}(s)]\,\mathrm{d}s\\
&\ge (4C_0)^{\min\{\lambda,m\}}\int_{t_0}^{t}[p(s)+q(s)]\,\mathrm{d}s,\quad t\in[t_0,1].
\end{aligned}$$

So,

$$\begin{aligned}
0 &< \int_0^1 [p(s)+q(s)]\,\mathrm{d}s\\
&= \int_0^{t_0}[p(s)+q(s)]\,\mathrm{d}s + \int_{t_0}^1 [p(s)+q(s)]\,\mathrm{d}s\\
&\le (4C_0)^{-\min\{\lambda,m\}}[u'(0)-u'(1)]\\
&< \infty.
\end{aligned}$$

Therefore, (5) holds.

Case II: $\beta=0$, $\delta>0$. Let $u\in C[0,1]\cap C^1(0,1]\cap C^2(0,1)$ be a positive solution of (3) and (4). From (4) we obtain $u(0)=0$, $u(1)\ge 0$, $u'(1)=-\gamma\delta^{-1}u(1)\le 0$. Then by the concavity of u there exists $t_0\in(0,1]$ with $u'(t_0)=0$.

Let C_1 be a constant satisfying $C_1u(t)\le 1/4$, $1/C_1\ge 4$. By means of (12) and (13), we obtain,

$$\begin{aligned}
0 &\le \int_0^{t_0} s[p(s)+q(s)]\,\mathrm{d}s = \int_0^{t_0}\mathrm{d}\tau\int_\tau^{t_0}[p(s)+q(s)]\,\mathrm{d}s\\
&\le (4C_1)^{-\lambda}\int_0^{t_0}\mathrm{d}\tau\int_\tau^{t_0} p(s)u^{-\lambda}(s)\,\mathrm{d}s\\
&\quad + (4C_1)^{-m}\int_0^{t_0}\mathrm{d}\tau\int_\tau^{t_0} q(s)u^{-m}(s)\,\mathrm{d}s
\end{aligned}$$

$$\leq (4C_1)^{-\min\{\lambda,m\}} \int_0^{t_0} \mathrm{d}\tau \int_\tau^{t_0} [p(s)u^{-\lambda}(s) + q(s)u^{-m}(s)]\,\mathrm{d}s$$

$$= (4C_1)^{-\min\{\lambda,m\}} \int_0^{t_0} (-u'(t_0) + u'(\tau))\,\mathrm{d}\tau$$

$$= (4C_1)^{-\min\{\lambda,m\}} u(t_0)$$

$$< +\infty.$$

Similarly,

$$0 \leq \int_{t_0}^1 s[p(s) + q(s)]\,\mathrm{d}s < +\infty.$$

Hence we conclude

$$0 < \int_0^1 t[p(t) + q(t)]\,\mathrm{d}t < \infty.$$

Case III: $\beta > 0$, $\delta = 0$. The proof for Case III is almost the same as that for Case II.

Case IV: $\beta = \delta = 0$. Let $u \in C[0,1]$ be a positive solution of (3) and (4). Integrating (3) twice gives

$$u'\left(\frac{1}{2}\right) - u'(t) = \int_{1/2}^t [p(s)u^{-\lambda}(s) + q(s)u^{-m}(s)]\,\mathrm{d}s,$$

$$\begin{aligned} u'\left(\frac{1}{2}\right)\left(t - \frac{1}{2}\right) - u(t) + u\left(\frac{1}{2}\right) &= \int_{1/2}^t \mathrm{d}\eta \int_{1/2}^\eta [p(s)u^{-\lambda}(s) \\ &\quad + q(s)u^{-m}(s)]\,\mathrm{d}s \\ &= \int_{1/2}^t (t-s)[p(s)u^{-\lambda}(s) \\ &\quad + q(s)u^{-m}(s)]\,\mathrm{d}s. \end{aligned} \tag{14}$$

Since the limit of (23) as $t \to 1$ exists and is finite, by the monotone convergence theorem,

$$0 < \int_{1/2}^1 (1-s)[p(s)u^{-\lambda}(s) + q(s)u^{-m}(s)]\,\mathrm{d}s < \infty.$$

So,

$$0 < \int_{1/2}^1 (1-s)[p(s) + q(s)]\,\mathrm{d}s < \infty.$$

Similarly

$$0 < \int_0^{1/2} (1-s)[p(s) + q(s)]\,\mathrm{d}s < \infty.$$

Hence

$$0 < \int_0^1 s(1-s)[p(s) + q(s)]\,\mathrm{d}s < \infty.$$

2. *Sufficiency*

Case I: $\beta\delta \neq 0$. Suppose that (5) is satisfied. Let

$$q_1(t) = \frac{\gamma(1-t)+\delta}{\rho} \int_0^t (\alpha s + \beta)[p(s) + q(s)]\,\mathrm{d}s$$
$$+ \frac{\alpha t + \beta}{\rho} \int_t^1 (\gamma(1-s)+\delta)[p(s)+q(s)]\,\mathrm{d}s,\ t \in [0,1].$$

Then $q_1 \in C^1[0,1] \cap C^2(0,1)$ satisfies (4) and solve the equation $q_1''(t) = -[p(t) + q(t)]$, $t \in (0,1)$. Let $L_1 = (\beta\delta/\rho)\int_0^1 [p(s)+q(s)]\,\mathrm{d}s$, $L_2 = 1/\rho \int_0^1 (\alpha s + \beta)(\gamma(1-s) + \delta)[p(s)+q(s)]\,\mathrm{d}s$. Then it is easy to check that $0 < L_1 \le q_1(t) \le L_2$, $\forall t \in [0,1]$. Let $\alpha(t) = k_1 q_1(t)$, $\beta(t) = k_2 q_1(t)$, $t \in [0,1]$, where $k_1 = \min\{1, L_2^{-\lambda/(1+\lambda)}, L_2^{-m/(1+m)}\}$, $k_2 = \max\{1, L_1^{-\lambda/(1+\lambda)}, L_1^{-m/(1+m)}\}$.Then $\alpha(t), \beta(t) \in C^1[0,1] \cap C^2(0,1)$, $0 < \alpha(t) \le \beta(t)$, $t \in [0,1]$, and $\alpha(t)$, $\beta(t)$ satisfy the boundary condition (4). Furthermore,

$$\begin{aligned}
\alpha''(t) + p(t)\alpha^{-\lambda}(t) + q(t)\alpha^{-m}(t) &= -k_1[p(t)+q(t)] + p(t)[k_1 q_1(t)]^{-\lambda}\\
&\quad + q(t)[k_1 q_1(t)]^{-m}\\
&\ge p(t)[(k_1 L_2)^{-\lambda} - k_1]\\
&\quad + q(t)[(k_1 L_2)^{-m} - k_1]\\
&\ge 0,\quad t \in (0,1),\\
\beta''(t) + p(t)\beta^{-\lambda}(t) + q(t)\beta^{-m}(t) &= -k_2[p(t)+q(t)] + p(t)[k_2 q_1(t)]^{-\lambda}\\
&\quad + q(t)[k_2 q_1(t)]^{-m}\\
&\le p(t)[(k_2 L_1)^{-\lambda} - k_2]\\
&\quad + q(t)[(k_2 L_1)^{-m} - k_2]\\
&\le 0,\quad t \in (0,1).
\end{aligned}$$

Thus, $\alpha(t)$ and $\beta(t)$ are respectively lower and upper solutions of problems (3) and (4). We will now prove that problems (3) and (4) admit a $C^1[0,1]$ positive solution u^* satisfying $0 < \alpha(t) \le u^*(t) \le \beta(t)$, $t \in [0,1]$.

First, define an auxiliary function

$$f(t,u) = \begin{cases} p(t)\alpha^{-\lambda}(t) + q(t)\alpha^{-m}(t), & u < \alpha(t),\\ p(t)u^{-\lambda}(t) + q(t)u^{-m}(t), & \alpha \le u \le \beta(t),\\ p(t)\beta^{-\lambda}(t) + q(t)\beta^{-m}(t), & u > \beta(t). \end{cases}$$

From (H), $f : (0,1) \times R \to [0,\infty)$ is continuous.

Consider the boundary value problem

$$u''(t) + f(t, u(t)) = 0, \quad t \in (0, 1), \tag{15}$$

$$\alpha u(0) - \beta u'(0) = 0, \quad \gamma u(1) + \delta u'(1) = 0. \tag{16}$$

It is clear that the above problem is equivalent to the integral equation

$$u(t) = Au(t) = \int_0^1 G(t, s) f(s, u(s))\, \mathrm{d}s, \tag{17}$$

where

$$G(t, s) = \frac{1}{\rho} \begin{cases} (\alpha s + \beta)(\gamma(1 - t) + \delta), \ s < t, \\ (\alpha t + \beta)(\gamma(1 - s) + \delta), \ t \geq s. \end{cases}$$

Let $X = C[0, 1]$. For $u \in X$, if for some $t \in [0, 1]$ $\alpha(t) \leq u(t) \leq \beta(t)$, we obtain that

$$\begin{aligned} 0 &\leq p(t)u^{-\lambda}(t) + q(t)u^{-m}(t) \leq p(t)\alpha^{-\lambda}(t) + q(t)\alpha^{-m}(t) \\ &\leq p(t)(k_1 L_1)^{-\lambda} + q(t)(k_1 L_1)^{-m} \leq (k_1 L_1)^{-\min\{\lambda, m\}}[p(t) + q(t)]. \end{aligned} \tag{18}$$

Therefore, from (5), (14) and (18), we know that $A : X \to X$ is continuous and $A(X)$ is a bounded set. In addition, $u \in X \cap C^1[0, 1]$ is a solution of problems (15) and (16) if and only if $Au = u$.

Since $AX \subseteq C^2[0, 1]$, by the standard application of Arzela–Ascoli theorem, we obtain that A is compact. By means of Schauder fixed point theorem, we obtain that A has at least one fixed point $u^* \in X \cap C^1[0, 1]$. We will show

$$\alpha(t) \leq u^*(t) \leq \beta(t), \quad t \in [0, 1], \tag{19}$$

which will imply that $u^*(t) \in C^1[0, 1]$ is a positive solution of (3) and (4). Suppose (19) is not satisfied. Then there exists $t^* \in [0, 1]$ such that either $u^*(t^*) < \alpha(t^*)$ or $u^*(t^*) > \beta(t^*)$. Let us consider the second case. Let $I \subseteq [0, 1]$ denote the maximal interval containing t^* such that $u > \beta$ on I. Then, it is clear that either $u = \beta$ on ∂I or both u and β satisfy the same boundary conditions given by (4) on ∂I. Let $Z(t) = \beta(t) - u^*(t)$, $t \in I$. Then $Z''(t) \leq 0$ on I and either $Z(t) = 0$ for $t \in \partial I$ or $Z(t)$ satisfies the boundary condition (4) for $t \in \partial I$. From the maximum principle $Z(t) \geq 0$ for $t \in I$, i.e. $\beta(t) \geq u^*(t)$ for $t \in I$. This is a contradiction. In the same way, $\alpha(t) \leq u^*(t)$ for $t \in I$. So, $u^*(t)$ is a $C^1[0, 1]$ positive solution of (3) and (4).

Case II: $\beta = 0, \delta > 0$. Suppose (6) is satisfied. Choose $n \geq 4$ so that $n \min\{\lambda, m\} > 1$. Let

$$\begin{aligned} R(t) = \Bigg(&\frac{\gamma(1 - t) + \delta}{\gamma + \delta} \int_0^t s[p(s) + q(s)]\, \mathrm{d}s \\ &+ t \int_t^1 \frac{\gamma(1 - s) + \delta}{\gamma + \delta} [p(s) + q(s)]\, \mathrm{d}s \Bigg)^{1/(n \min\{\lambda, m\})}. \end{aligned}$$

Then $R(t) \in C[0,1] \cap C^2(0,1)$, satisfies $R(t) > 0$, $R''(t) \le 0, t \in (0,1)$, and the boundary conditions $R(0) = 0$, $R(1) > 0$, $\gamma R(1) + \delta R'(1) \ge 0$. We now estimate

$$\begin{aligned}
0 &\le \int_0^1 t[p(t)+q(t)]R^{-\min\{\lambda,m\}}(t)\,\mathrm{d}t \\
&= \int_0^1 t[p(t)+q(t)]\Bigg(\frac{\gamma(1-t)+\delta}{\gamma+\delta}\int_0^t s[p(s)+q(s)]\,\mathrm{d}s \\
&\quad + t\int_t^1 \frac{\gamma(1-s)+\delta}{\gamma+\delta}[p(s)+q(s)]\,\mathrm{d}s\Bigg)^{-(1/n)}\mathrm{d}t \\
&\le \int_0^1 t[p(t)+q(t)]\left(\int_0^1 s[p(s)+q(s)]\,\mathrm{d}s\right)^{-(1/n)}\mathrm{d}t.
\end{aligned}$$

Let

$$\begin{aligned}
\Gamma_1(t) &= \frac{\gamma(1-t)+\delta}{\gamma+\delta}\int_0^t s[p(s)+q(s)]\,\mathrm{d}s \\
&\quad + t\int_t^1 \frac{\gamma(1-s)+\delta}{\gamma+\delta}[p(s)+q(s)]\,\mathrm{d}s, \\
\Gamma_2(t) &= \frac{\gamma(1-t)+\delta}{\gamma+\delta}\int_0^t s[p(s)+q(s)]R^{-\min\{\lambda,m\}}\,\mathrm{d}s \\
&\quad + t\int_t^1 \frac{\gamma(1-s)+\delta}{\gamma+\delta}[p(s)+q(s)]R^{-\min\{\lambda,m\}}(s)\,\mathrm{d}s \\
&\quad + R(t), \quad t \in [0,1].
\end{aligned}$$

It is clear that $\gamma\Gamma_1(1)+\delta\Gamma_1'(1) = 0$, $\gamma\Gamma_2(1)+\delta\Gamma_2'(1) \ge 0$ and $L_3 t(\gamma(1-t)+\delta/(\gamma+\delta)) \le \Gamma_1(t) \le L_3$, $R(t) \le \Gamma_2(t) \le L_4$, $t \in [0,1]$, where

$$\begin{aligned}
L_3 &= \int_0^1 s[p(s)+q(s)]\left(\frac{\gamma(1-s)+\delta}{\gamma+\delta}\right)\mathrm{d}s, \\
L_4 &= \int_0^1 s\left(\frac{\gamma(1-s)+\delta}{\gamma+\delta}\right)[p(s)+q(s)]R^{-\min\{\lambda,m\}}(s)\,\mathrm{d}s \\
&\quad + R_0, \;\; R_0 = \max_{t\in[0,1]} R(t).
\end{aligned}$$

We also check by direct computation that $\Gamma_1''(t) = -[p(t)+q(t)]$, $\Gamma_2''(t) \le -(p(t)+q(t))R^{-\min\{\lambda,m\}}(t)$, $t \in (0,1)$. Let $\alpha(t) = k_1\Gamma_1(t)$, $\beta(t) = k_2\Gamma_2(t)$, $t \in [0,1]$, where $k_1 = \min\{1, L_3^{-\lambda/1+\lambda}, L_3^{-m/1+m}\}$, $k_2 = \max\{1, L_4^{\min\{\lambda,m\}}\}$. Then we have

$$\begin{aligned}
\alpha''(t) + p(t)\alpha^{-\lambda}(t) + q(t)\alpha^{-m}(t) &\ge -k_1[p(t)+q(t)] + p(t)(k_1L_3)^{-\lambda} \\
&\quad + q(t)(k_1L_3)^{-m} \ge 0,
\end{aligned}$$

$$\begin{aligned}\beta''(t) + p(t)\beta^{-\lambda}(t) + q(t)\beta^{-m}(t) \leq -k_2[p(t) + q(t)]R^{-\min\{\lambda,m\}}(t)\\ + p(t)(k_2R(t))^{-\lambda}(t)\\ + q(t)(k_2R(t))^{-m} \leq 0.\end{aligned}$$

In addition, $\alpha(t), \beta(t) \in C[0, 1] \cap C^1[0, 1] \cap C^2(0, 1), \gamma\alpha(1) + \delta\alpha'(1) = 0, \gamma\beta(1) + \delta\beta'(1) \geq 0$. Let $Z(t) = \beta(t) - \alpha(t)$. Then $Z(0) = 0, \gamma Z(1) + \delta Z'(1) \geq 0$. Also, if we assume $\alpha \leq \beta$ on an interval $I \subseteq [0, 1]$, we find $Z''(t) \leq -k_2[p(t) + q(t)]R^{-\min\{\lambda,m\}} + k_1[p(t) + q(t)] \leq -k_2[p(t) + q(t)]L_4^{-\min\{\lambda,m\}} + [p(t) + q(t)] \leq 0$ in I. As before, for $t \in \partial I$, we have either $Z(t) = 0$ or if $t = 1, \gamma Z(1) + \delta Z'(1) \geq 0$. From the maximum principle, $Z(t) \geq 0, t \in [0, 1]$, i.e. $\alpha(t) \leq \beta(t), t \in [0, 1]$, which is a contradiction. Hence $\alpha(t), \beta(t)$ are respectively the lower and upper solutions of (3) and (4).

In the following we will prove that problems (3) and (4) have a $C^1[0, 1]$ positive solution $u(t)$ satisfying $0 < \alpha(t) \leq u(t) \leq \beta(t), t \in (0, 1)$. Let a_n be a sequence satisfying $0 < \cdots < a_{n+1} < a_n < \cdots < a_2 < a_1 < 1/2$ with $a_n \to 0$ as $n \to \infty$. Let r_n be a sequence satisfying $0 < \alpha(a_n) \leq r_n \leq \beta(a_n), n = 1, 2, \ldots$. For each n, consider the following singular boundary problem:

$$u''(t) + p(t)u^{-\lambda}(t) + q(t)u^{-m}(t) = 0, \quad t \in (a_n, 1), \tag{20}$$

$$u(a_n) = r_n, \quad \gamma u(1) + \delta u'(1) = 0. \tag{21}$$

From (6), we see that

$$\int_0^1 s[p(s) + q(s)]\,\mathrm{d}s \geq \int_{a_n}^1 s[p(s) + q(s)]\,\mathrm{d}s \geq a_n \int_{a_n}^1 [p(s) + q(s)]\,\mathrm{d}s.$$

Therefore,

$$0 \leq \int_{a_n}^1 [p(s) + q(s)]\,\mathrm{d}s \leq \frac{1}{a_n}\int_0^1 s[p(s) + q(s)]\,\mathrm{d}s < \infty. \tag{22}$$

Following the proof of Case I, with (22), we can say that for each n, the singular boundary value problems (20) and (21) have at least one positive solution $u_n(t) \in C^1[a_n, 1]$ satisfying $\alpha(t) \leq u_n(t) \leq \beta(t), t \in [a_n, 1]$. So

$$|u_n(1)| \leq M = \alpha(1) + \beta(1),$$

$$|u_n'(1)| = |(\gamma/\delta)u_n(1)| \leq \frac{\gamma}{\delta}M, \quad n = 1, 2, \ldots.$$

Without loss of generality, we can assume

$$u_n(1) \to u_0 \in [\alpha(1), \beta(1)], \quad n \to \infty,$$

$$u_n'(1) \to -(\gamma/\delta)u_0, \quad n \to \infty.$$

Similar to Theorem 3.2 in [3], we can prove (3) has a positive solution $u(t)$ with $u(1) = u_0, u'(1) = -(\gamma/\delta)u_0$. Its maximal interval of existence is (ω^-, ω^+), and $u_n(t)$ converges to $u(t)$ uniformly in any compact subset of (ω^-, ω^+) ($u_n'(t)$ converges to $u'(t)$ uniformly),

$n \to \infty$. Since $\alpha(t) \le u_n(t) \le \beta(t), t \in [a_n, 1]$ and $\bigcup_{n=1}^{\infty}[a_n, 1] = [0, 1]$, we have $\alpha(t) \le u(t) \le \beta(t), t \in (\omega^-, \omega^+) \cap [0, 1]$. From continuation theorem we obtain that $[0, 1] \subset (\omega^-, \omega^+)$. Since $\alpha(0) = \beta(0) = 0$, we also obtain that $u(0) = 0$. In addition $\delta u(1) + \gamma u'(1) = 0$. Thus $u(t) \in C[0, 1] \cap C^1(0, 1) \cap C^2(0, 1)$ is a positive solution of problems (3) and (4). The proof for Case II is complete.

Case III: $\beta > 0, \delta = 0$. The proof for Case III is almost the same as that for Case II.

Case IV: $\beta = \delta = 0$. Let $Q(t) = ((1-t)\int_0^t s[p(s)+q(s)]\,\mathrm{d}s + t\int_t^1 (1-s)[p(s) + q(s)]\,\mathrm{d}s)^{1/n\min\{\lambda,m\}}$, $t \in [0, 1]$. Then $Q(t) \in C[0, 1] \cap C^2(0, 1)$ with $Q(t) > 0$, $Q''(t) \le 0, t \in (0, 1)$ and $Q(0) = Q(1) = 0$. Let

$$\Gamma_1(t) = (1-t)\int_0^t s[p(s)+q(s)]\,\mathrm{d}s + t\int_t^1 (1-s)[p(s)+q(s)]\,\mathrm{d}s,$$

$$\Gamma_2(t) = (1-t)\int_0^t s[p(s)+q(s)]Q^{-\min\{\lambda,m\}}(s)\,\mathrm{d}s$$

$$+ t\int_t^1 (1-s)[p(s)+q(s)]Q^{-\min\{\lambda,m\}}(s)\,\mathrm{d}s + Q(t)\ t \in [0, 1].$$

Then $t(1-t)L_5 \le \Gamma_1(t) \le L_5, Q(t) \le \Gamma_2(t) \le L_6, L_5 = \int_0^1 s(1-s)[p(s) + q(s)]\,\mathrm{d}s,\ L_6 = \int_0^1 s(1-s)[p(s)+q(s)]Q^{-\min\{\lambda,m\}}(s)\,\mathrm{d}s + Q_0,\ \ Q_0 = \max Q(t)$. Let $\alpha(t) = k_1\Gamma_1(t), \beta(t) = k_2\Gamma_2(t), t \in [0, 1]$. Here

$$k_1 = \min\{1, L_5^{-\lambda/1+\lambda}, L_5^{-m/1+m}\}, \quad k_2 = \{1, L_6^{\min\{\lambda,m\}}\}.$$

Then $\alpha(t), \beta(t)$ are respectively the lower and upper solutions of (3) and (4).The remaining proof is analogous to that of Case II. Thus we complete the proof of Theorem 2.1.

Proof of Theorem 2.2

Proof for Case (H1): $\beta = 0, \delta > 0$.

1. *Necessity*

Suppose that u is a $C^1[0, 1]$ positive solution of (3) and (4). Then both $u'(0)$ and $u'(1)$ exist, and $p(t), q(t) \not\equiv 0, t \in (0, 1)$. From (4) and the fact that u is a positive concave function, we know that $u(0) = 0, u(1) > 0, u'(0) > 0, u'(1) \le 0$. Then there exists $t_0 \in (0, 1]$ such that $u'(t_0) = 0$. Since $u''(t) \le 0$, and $u(t) > 0,\ t \in [0, 1]$, one easily sees that $0 < u(1) \le u(t) \le u(t_0),\ t \in [t_0, 1]$ and there exist constants I_1 and I_2 which satisfy

$$I_1 t \le u(t) \le I_2 t, \quad t \in [0, 1]. \tag{23}$$

Hence

$$0 < \int_0^1 [p(s)s^{-\lambda} + q(s)s^{-m}]\,\mathrm{d}s \le I_2^{\max\{\lambda,m\}}\int_0^1 [p(s)u^{-\lambda}(s) + q(s)u^{-m}(s)]\,\mathrm{d}s$$

$$= -I_2^{\max\{\lambda,m\}}\int_0^1 u''(s)\,\mathrm{d}s = I_2^{\max\{\lambda,m\}}(u'(0) - u'(1)) < \infty.$$

The above inequality shows that (9) holds.

2. *Sufficiency*

Suppose that (9) is satisfied. Let

$$\Gamma(t) = \frac{\gamma(1-t)+\delta}{\gamma+\delta}\int_0^t s[p(s)s^{-\lambda}+q(s)s^{-m}]\,\mathrm{d}s$$

$$+\,t\int_t^1 \frac{\gamma(1-s)+\delta}{\gamma+\delta}[p(s)s^{-\lambda}+q(s)s^{-m}]\,\mathrm{d}s,\; t\in[0,1].$$

Then $\Gamma(t)\in C^1[0,1]\cap C^2(0,1)$. Replace $u(t)$ with $\Gamma(t)$ in (24) and let

$$I_1 = \frac{\delta}{\gamma+\delta}\int_0^1 s\left(\frac{\gamma(1-s)+\delta}{\gamma+\delta}\right)[p(s)s^{-\lambda}+q(s)s^{-m}]\,\mathrm{d}s,$$

$$I_2 = \int_0^1 [p(s)s^{-\lambda}+q(s)s^{-m}]\,\mathrm{d}s.$$

Then $\Gamma(t)$ satisfies (24). Let

$$k_1 = \min\{1, (I_2)^{-\lambda/1+\lambda}, (I_2)^{-m/1+m}\},$$
$$k_2 = \max\{1, (I_1)^{-\lambda/1+\lambda}, (I_1)^{-m/1+m}\}.$$

Then $\alpha(t) = k_1\Gamma(t), \beta(t) = k_2\Gamma(t),\; t\in[0,1]$. Then $\alpha(t)$ and $\beta(t)$ are respectively lower and upper solutions of problems (3) and (4). It is clear that $0<\alpha(t)\le\beta(t), t\in(0,1], \alpha(0)=\beta(0)=0, \gamma\alpha(1)+\delta\alpha'(1)=0, \gamma\beta(1)+\delta\beta'(1)=0$. On the other hand, when $t\in(0,1), \alpha(t)\le u\le\beta(t)$, we have

$$0\le p(t)u^{-\lambda}(t)+q(t)u^{-m}(t)\le p(t)\alpha^{-\lambda}(t)+q(t)\alpha^{-m}(t)$$

$$= p(t)(k_1I_1t)^{-\lambda}+q(t)(k_1I_1t)^{-m} = F(t).$$

From (9), we have $\int_0^1 F(t)\,\mathrm{d}t<\infty$. The same argument that we have given in the sufficiency of Theorem 2.1 assures us that problems (3) and (4) admit a positive solution $u\in C^1[0,1]\cap C^2(0,1)$ such that $\alpha(t)\le u(t)\le\beta(t),\; t\in[0,1]$. The proof for Case (H1) is complete. Similarly we can prove cases (H2) and (H3). Thus we complete the proof of Theorem 2.2.

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