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Fixed point of multivalued mapping in uniform spaces

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Abstract. In this paper we prove some new fixed point theorems for multivalued mappings on orbitally complete uniform spaces.

Keywords. Fixed point; multivalued mappings; orbitally complete; uniform space.

1. Introduction

Let $(X, \mathcal{U})$ be a uniform space. A family $\{d_i : i \in I\}$ of pseudometrics on X with indexing set I, is called an associated family for the uniformity $\mathcal{U}$ if the family

$$\beta = \{V(i, \varepsilon) : i \in I; \varepsilon > 0\},$$

where

$$V(i, \varepsilon) = \{(x, y) : x, y \in X, \ \ d_i(x, y) < \varepsilon\}$$

is a sub-base for the uniformity $\mathcal{U}$. We may assume that β itself is a base by adjoining finite intersection of members of β, if necessary. The corresponding family of pseudometrics is called an associated family for $\mathcal{U}$. An associated family for $\mathcal{U}$ will be denoted by p^*. For details the reader is referred to [1,3–8].

Let A be a nonempty subset of a uniform space X. Define

$$\Delta^*(A) = \sup\{d_i(x, y) : x, y \in A, \ i \in I\},$$

where

$$\{d_i : i \in I\} = p^*.$$

Then Δ^* is called an augmented diameter of A. Further, A is said to be p^*-*bounded* if $\Delta^*(A) < \infty$. Let

$$2^X = \left\{A : A \text{ is a nonempty, closed and } p^* \text{ - bounded subset of } X\right\}.$$

For any nonempty subsets A and B of X, define

$$d_i(x, A) = \inf\{d_i(x, a) : a \in A\}, i \in I$$

$$H_i(A,\ B) = \max\left\{\sup_{a\in A} d_i(a,\ B),\ \sup_{b\in B} d_i(A,\ b)\right\}$$
$$= \sup_{x\in X}\{|d_i(x,\ A) - d_i(x,\ B)|\}.$$

It is well-known that on 2^X, H_i is a pseudometric, called the Hausdorff pseudometric induced by d_i, $i \in I$.

Let $(X, \mathcal{U})$ be a uniform space with an augmented associated family p^*. p^* also induces a uniformity $\mathcal{U}^*$ on 2^X defined by the base

$$\beta^* = \left\{V^*(i,\ \varepsilon) : i \in I,\ \varepsilon > 0\right\},$$

where

$$V^*(i,\ \varepsilon) = \left\{(A,\ B) : A,\ B \in 2^X,\ H_i(A,\ B) < \varepsilon\right\}.$$

The space $(2^X, \mathcal{U}^*)$ is a uniform space called the hyperspace of $(X, \mathcal{U})$.

DEFINITION 1

The collection of all filters on a given set X is denoted by $\Phi(X)$. An order relation is defined on $\Phi(X)$ by the rule $\mathcal{F}_1 < \mathcal{F}_2$ iff $\mathcal{F}_1 \supset \mathcal{F}_2$. If $\mathcal{F}^* < \mathcal{F}$, then $\mathcal{F}^*$ is called a subfilter of $\mathcal{F}$.

DEFINITION 2

Let $(X, \mathcal{U})$ be a uniform space defined by $\{d_i : i \in I\} = p^*$. If $F : X \to 2^X$ is a multivalued mapping, then

(i) $x \in X$ is called a fixed point of F if $x \in Fx$;

(ii) An orbit of F at a point $x_0 \in X$ is a sequence $\{x_n\}$ given by

$$O(F, x_0) = \{x_n : x_n \in Fx_{n-1}, n = 1, 2, ...\};$$

(iii) A uniform space X is called *F-orbitally complete* if every Cauchy filter which is a subfilter of an orbit of F at each $x \in X$ converges to a point of X.

DEFINITION 3

Let $(X, \mathcal{U})$ be a uniform space and let $F : X \to X$ be a mapping. A single-valued mapping F is *orbitally continuous* if $\lim\,(T^{n_i}x) = u$ implies $\lim\ T(T^{n_i}x) = Tu$ for each $x \in X$.

2. Main results

Theorem 1. *Let $(X, \mathcal{U})$ be an F-orbitally complete Hausdorff uniform space defined by $\{d_i : i \in I\} = p^*$ and $(2^X, \mathcal{U}^*)$ a hyperspace and let $F : X \to 2^X$ be a continuous mapping with Fx compact for each x in X. Assume that*

$$\min\left\{H_i(Fx,\ Fy)^r,\ d_i(x, Fx)d_i(y,\ Fy)^{r-1},\ d_i(y, Fy)^r\right\}$$
$$+ a_i \min\{d_i(x, Fy), d_i(y, Fx)\} \le [b_i d_i(x, Fx)$$
$$+ c_i d_i(x, y)]d_i(y, Fy)^{r-1} \qquad (1)$$

for all $i \in I$ and $x,\ y \in X$, where $r \ge 1$ is an integer, $a_i, b_i,\ c_i$ are real numbers such that $0 < b_i + c_i < 1$, then F has a fixed point.

Proof. Let x_0 be an arbitrary point in X and consider the sequence $\{x_n\}$ defined by

$$x_1 \in Fx_0, x_2 \in Fx_1, ..., x_n \in Fx_{n-1}, ...$$

Let us suppose that $d_i(x_n, Fx_n) > 0$ for each $i \in I$ and $n = 0, 1, 2, ...$ (Otherwise for some positive integer n, $x_n \in Fx_n$ as desired.)

Let $U \in \mathcal{U}$ be an arbitrary entourage. Since β is a base for $\mathcal{U}$, there exists $V(i, \varepsilon) \in \beta$ such that $V(i, \varepsilon) \subseteq U$. Now $y \to d_i(x_0, y)$ is continuous on the compact set Fx_0 and this implies that there exists $x_1 \in Fx_0$ such that $d_i(x_0, x_1) = d_i(x_0, Fx_0)$. Similarly, Fx_1 is compact so there exists $x_2 \in Fx_1$ such that $d_i(x_1, x_2) = d_i(x_1, Fx_1)$. Continuing, we obtain a sequence $\{x_n\}$ such that $x_{n+1} \in Fx_n$ and $d_i(x_n, x_{n+1}) = d_i(x_n, Fx_n)$.

For $x = x_{n-1}$, and $y = x_n$ by condition (1) we have

$$\begin{aligned} \min & \left\{H_i(Fx_{n-1}, Fx_n)^r, d_i(x_{n-1}, Fx_{n-1})d_i(x_n, Fx_n)^{r-1}, d_i(x_n, Fx_n)^r\right\} \\ & + a_i \min\{d_i(x_{n-1}, Fx_n), d_i(x_n, Fx_{n-1})\} \le \big[b_i\, d_i(x_{n-1}, Fx_{n-1}) \\ & + c_i\, d_i(x_{n-1}, x_n)\big]\, d_i(x_n, Fx_n)^{r-1} \end{aligned}$$

or since $d_i(x_n, Fx_{n-1}) = 0$, $x_n \in Fx_{n-1}$. Hence we have

$$\begin{aligned} \min & \left\{d_i(x_n, x_{n+1})^r, d_i(x_{n-1}, x_n)d_i(x_n, x_{n+1})^{r-1}\right\} \\ & \le \big[b_i\, d_i(x_{n-1}, x_n) + c_i\, d_i(x_{n-1}, x_n)\big]\, d_i(x_n, x_{n+1})^{r-1} \end{aligned}$$

and it follows that

$$\begin{aligned} \min & \left\{d_i(x_n, x_{n+1})^r, d_i(x_{n-1}, x_n)d_i(x_n, x_{n+1})^{r-1}\right\} \\ & \le (b_i + c_i)\, d_i(x_{n-1}, x_n)\, d_i(x_n, x_{n+1})^{r-1}. \end{aligned}$$

Since

$$d_i(x_{n-1}, x_n)\, d_i(x_n, x_{n+1})^{r-1} \le (b_i + c_i)\, d_i(x_{n-1}, x_n)\, d_i(x_n, x_{n+1})^{r-1}$$

is not possible (as $0 < b_i + c_i < 1$), we have

$$d_i(x_n, x_{n+1})^r \le (b_i + c_i)\, d_i(x_{n-1}, x_n)\, d_i(x_n, x_{n+1})^{r-1}$$

or

$$d_i(x_n, x_{n+1})^r \le k_i\, d_i(x_{n-1}, x_n)\, d_i(x_n, x_{n+1})^{r-1},$$

where $k_i = b_i + c_i$, $0 < k_i < 1$.

Proceeding in this manner we get

$$\begin{aligned} d_i(x_n, x_{n+1}) & \le k_i d_i(x_{n-1}, x_n) \\ & \le k_i^2 d_i(x_{n-2}, x_{n-1}) \\ & \quad \vdots \\ & \le k_i^n d_i(x_0, x_1). \end{aligned}$$

Hence we obtain

$$\begin{aligned} d_i(x_n,\ x_m) &\le d_i(x_n,\ x_{n+1}) + d_i(x_{n+1,}\, x_{n+2}) + \cdots + d_i(x_{m-1},\ x_m) \\ &\le (k_i^n + k_i^{n+1} + \cdots + k_i^{m-1})\ d_i(x_0,\ x_1) \\ &\le k_i^n(1 + k_i + \cdots + k_i^{m-n-1})\ d_i(x_0,\ x_1) \\ &\le \frac{k_i^n}{1-k}\ d_i(x_0,\ x_1). \end{aligned}$$

Since $\lim_{n\to\infty} k_i^n = 0$, it follows that there exists $N(i, \varepsilon)$ such that $d_i(x_n,\ x_m) < \varepsilon$ and hence $(x_n,\ x_m) \in U$ for all $n, m \ge N(i, \varepsilon)$. Therefore the sequence $\{x_n\}$ is a Cauchy sequence in the d_i-uniformity on X.

Let $S_p = \{x_n : n \ge p\}$ for all positive integers p and let β be the filter basis $\{S_p : p = 1,\ 2,\ ...\}$. Then since $\{x_n\}$ is a d_i-Cauchy sequence for each $i \in I$, it is easy to see that the filter basis β is a Cauchy filter in the uniform space $(X,\ \mathcal{U})$. To see this we first note that the family $\{V(i, \varepsilon) : i \in I\}$ is a base for $\mathcal{U}$ as $p^* = \{d_i : i \in I\}$. Now since $\{x_n\}$ is a d_i-Cauchy sequence in X, there exists a positive integer p such that $d_i(x_n,\ x_m) < \varepsilon$ for $m \ge p,\ n \ge p$. This implies that $S_p \times S_p \subseteq V(i, \varepsilon)$. Thus given any $U \in \mathcal{U}$, we can find an $S_p \in \beta$ such that $S_p \times S_p \subset U$. Hence β is a Cauchy filter in $(X,\ \mathcal{U})$. Since $(X,\ \mathcal{U})$ is F-orbitally complete and Hausdorff space, $S_p \to z$ for some $z \in X$. Consequently $F(S_p) \to Fz$ (follows from the continuity of F). Also

$$S_{p+1} \subseteq F(S_p) = \cup\{Fx_n : n \ge p\}$$

for $p = 1,\ 2, \ldots$. It follows that $z \in Fz$. Hence z is a fixed point of F. This completes the proof.

If we take $r = 1$ in Theorem 1, then we obtain the following theorem.

Theorem 2. *Let $(X,\ \mathcal{U})$ be an F-orbitally complete Hausdorff uniform space defined by $\{d_i : i \in I\} = p^*$ and $(2^X,\ \mathcal{U}^*)$ a hyperspace, let $F:\ X \to 2^X$ be a continuous mapping and Fx compact for each x in X. Assume that*

$$\begin{aligned} &\min\{H_i(Fx,\ Fy),\ d_i(x,\ Fx),\ d_i(y,\ Fy)\} \\ &\quad + a_i \min\{d_i(x,\ Fy),\ d_i(y,\ Fx)\} \le b_i\, d_i(x,\ Fx) + c_i\, d_i(x,\ y) \end{aligned} \tag{2}$$

for all $i \in I$ and $x,\ y \in X$, where $a_i,\ b_i, c_i$ are real numbers such that $0 < b_i + c_i < 1$, then F has a fixed point.

We denote that if F is a single valued mapping on X, then we can write $d_i(Fx,\ Fy) = H_i(Fx,\ Fy), x,\ y \in X,\ i \in I$.

Thus we obtain the following theorem as a consequence of the Theorem 2.

Theorem 3. *Let $(X,\ \mathcal{U})$ be a T-orbitally complete Hausdorff uniform space and let $T : X \to X$ be a T-orbitally continuous mapping satisfying*

$$\begin{aligned} &\min\{d_i(Tx,\ Ty),\ d_i(x,\ Tx),\ d_i(y,\ Ty)\} \\ &\quad + a_i \min\{d_i(x,\ Ty),\ d_i(y,\ Tx)\} \le b_i\, d_i(x,\ Tx) + c_i\, d_i(x,\ y) \end{aligned} \tag{3}$$

for all $x,\ y \in X,\ i \in I$ and $a_i,\ b_i, c_i$ are real numbers such that $0 < b_i + c_i < 1$. Then T has a fixed point and which is unique whenever $a_i > c_i > 0$.

Proof. Define a mapping F of X into 2^X by putting $Fx = \{Tx\}$ for all x in X. It follows that F satisfies the conditions of Theorem 2. Hence T has a fixed point.

Now if $a_i > c_i > 0$, we show that T has a unique fixed point. Assume that T has two fixed points z and w which are distinct. Since $d_i(z, Tz) = 0$ and $d_i(w, Tw) = 0$, then by the condition (2),

$$a_i \min\{d_i(z, Tw), d_i(w, Tz)\} \le c_i\, d_i(z, w)$$

or

$$a_i\, d_i(z, w) \le c_i\, d_i(z, w),$$

$$d_i(z, w) \le \frac{c_i}{a_i} d_i(z, w)$$

which is impossible. Thus if $a_i > c_i > 0$, then T has a unique fixed point in X. This completes the proof.

We note that if $a_i = -1$ in condition (3), then one gets the following result as a corollary.

COROLLARY 4

Let T be an orbitally cotinuous self-map of a T-orbitally complete uniform space $(X, \mathcal{U})$ satisfying the condition

$$\min\{d_i(Tx, Ty), d_i(x, Tx), d_i(y, Ty)\}$$
$$- \min\{d_i(x, Ty), d_i(y, Tx)\} \le b_i\, d_i(x, Tx) + c_i\, d_i(x, y),$$

$x, y \in X$, $i \in I$ and $0 < b_i + c_i < 1$. Then for each $x \in X$, the sequence $\{T^n x\}$ converges to a fixed point of T.

Remark 1. If we replace the uniform space $(X, \mathcal{U})$ in Theorem 3 and Corollary 4 by a metric space (i.e. a metrizable uniform space), then Theorem 1 and Corollary 1 of Dhage [2] will follow as special cases of our results.

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