Proc. Indian Acad. Sci. (Math. Sci.) Vol. 113, No. 2, May 2003, pp. 171–178.

Wavelet subspaces invariant under groups of translation operators

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MS received 6 November 2002

Abstract. We study the action of translation operators on wavelet subspaces. This action gives rise to an equivalence relation on the set of all wavelets. We show by explicit construction that each of the associated equivalence classes is non-empty.

Keywords. Wavelet; multiresolution analysis; multiresolution analysis wavelet; minimally supported frequency wavelet; wavelet set; translation invariance.

1. Introduction

A function $\psi \in L^2(\mathbb{R})$ is said to be a *wavelet* if the system of functions $\{\psi_{j,k} : j, k \in \mathbb{Z}\}$ forms an orthonormal basis for $L^2(\mathbb{R})$, where

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k), \quad j, k \in \mathbb{Z}.$$

Mallat [7] and Meyer [8] provided a framework to construct wavelets through the concept of multiresolution analysis (MRA). A sequence $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R})$ is called an MRA if the following conditions hold:

(i) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$,
(ii) $f \in V_j$ if and only if $f(2\cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$,
(iii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$,
(iv) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$, and
(v) there exists a function $\varphi \in V_0$, called the *scaling function*, such that $\{\varphi(\cdot - k) : k \in \mathbb{Z}\}$ forms an orthonormal basis for V_0.

One can always construct a wavelet from an MRA (see, for instance, Ch. 2 in [5]), but not every wavelet can be obtained in this manner. The first example of a wavelet which cannot be obtained from an MRA was given by Journé. In [1] we characterized a large class of wavelets, which also includes Journé's wavelet, and proved that none of them is associated with an MRA (see Theorem 2.1).

The following theorem, which characterizes all wavelets of $L^2(\mathbb{R})$, was proved independently by Gripenberg [3] and Wang [12] (see also [4,5]).

Theorem 1.1. *Let $\psi \in L^2(\mathbb{R})$ with $\|\psi\|_2 = 1$. Then ψ is a wavelet of $L^2(\mathbb{R})$ if and only if*

$$\sum_{j\in\mathbb{Z}} |\hat{\psi}(2^j\xi)|^2 = 1 \quad \textit{for a.e. } \xi \in \mathbb{R}. \tag{1}$$

$$\sum_{j\geq 0} \hat{\psi}(2^j\xi)\overline{\hat{\psi}(2^j(\xi + 2m\pi))} = 0 \quad \textit{for a.e. } \xi \in \mathbb{R}, \textit{ for all } m \in 2\mathbb{Z}+1. \tag{2}$$

We use the following definition of the Fourier transform:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\mathrm{e}^{-i\xi x}\mathrm{d}x, \quad \xi \in \mathbb{R}.$$

If ψ is a wavelet, then the support of $\hat{\psi}$ must have measure at least 2π. This minimal measure is achieved if and only if $|\hat{\psi}|$ is the characteristic function of some measurable subset K of $\mathbb{R}$. Such a wavelet is called a minimally supported frequency (MSF) wavelet and the associated set K is called a *wavelet set*. We refer to [5] for proofs of the above statements.

As we mentioned earlier, a wavelet need not be associated with an MRA, but it can be made to be associated with an MRA-like structure in the following manner. Given a wavelet ψ we define the closed subspaces V_j, $j \in \mathbb{Z}$, by

$$V_j = \overline{\mathrm{span}}\{\psi_{l,k} : l < j, k \in \mathbb{Z}\}, \quad j \in \mathbb{Z}. \tag{3}$$

It is easy to verify that these subspaces satisfy properties (i)–(iv) of an MRA. Moreover, instead of (v), the following weaker property holds:

(v′) V_0 is invariant under translation by integers.

A sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ which satisfies properties (i)–(iv) and (v′) is called a generalized MRA (GMRA).

Madych (§3 in [6]) characterized all MRAs for which each V_j is invariant under translation by all real numbers. Let T_α, $\alpha \in \mathbb{R}$, be the translation operator defined by $T_\alpha f(x) = f(x - \alpha)$. An MRA $\{V_j : j \in \mathbb{Z}\}$ is called a *translation invariant* MRA if $T_\alpha(V_j) \subset V_j$ for all $j \in \mathbb{Z}$ and for all $\alpha \in \mathbb{R}$. Madych proved that the only translation invariant MRAs are those for which the Fourier transform of the scaling function is the characteristic function of a set. In other words, the associated wavelet is an MSF wavelet.

Walter [10,11] modified the definition of translation invariance to include wavelets other than the MSF wavelets. An MRA is called *weakly translation invariant* if $T_\alpha(V_j) \subset V_{j+1}$ for $j \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. Clearly, every translation invariant MRA is weakly translation invariant, since $V_j \subset V_{j+1}$. Walter gave necessary conditions for an MRA to be weakly translation invariant. With an additional condition, these conditions also turned out to be sufficient for the weak translation invariance of MRAs associated with a class of Meyer-type wavelets (see §10.5 in [11]). Thus, we see that the notion of translation invariance and weak translation invariance are applicable only to a limited class of wavelets.

In this article we investigate the translation invariance from a slightly different point of view. We work in the more general set up of GMRAs in order to include *all* wavelets. We still demand the spaces V_j to be invariant under translations, but only by dyadic rationals at a fixed level instead of translation by all reals. More precisely, we ask the following question:

Let $n \in \mathbb{N}$. Does there exist a wavelet such that the space V_0 of the associated GMRA is invariant under translation by elements of the form $m/2^n$ for all $m \in \mathbb{Z}$?

The purpose of this paper is to give an affirmative answer to this question. We do this by explicitly constructing such wavelets for each n.

Let $\mathcal{G}_n$, $n \in \mathbb{N} \cup \{0\}$, and $\mathcal{G}_\infty$ be the following groups of (unitary) translation operators:

$$\mathcal{G}_n = \{T_{m/2^n} : m \in \mathbb{Z}\}, \quad \mathcal{G}_\infty = \{T_\alpha : \alpha \in \mathbb{R}\}.$$

Denote by $\mathcal{L}_n$ the collection of all wavelets ψ such that the space V_0 (of the GMRA $\{V_j\}$ associated with ψ) is invariant under the group $\mathcal{G}_n$. Clearly, $\mathcal{L}_0$ is the set of all wavelets, and

$$\mathcal{L}_0 \supset \mathcal{L}_1 \supset \mathcal{L}_2 \supset \cdots \supset \mathcal{L}_n \supset \mathcal{L}_{n+1} \supset \cdots \supset \mathcal{L}_\infty.$$

These inclusions naturally defines an equivalence relation on the set of all wavelets. The equivalence classes are given by $\mathcal{M}_n = \mathcal{L}_n \setminus \mathcal{L}_{n+1}$, $n \in \mathbb{N} \cup \{0\}$, and $\mathcal{M}_\infty = \mathcal{L}_\infty$. Therefore, $\mathcal{M}_n$, $n \in \mathbb{N} \cup \{0\}$, is the class of wavelets such that V_0 is invariant under the group $\mathcal{G}_n$ but not under $\mathcal{G}_{n+1}$. This equivalence relation was defined by Weber in [13] and the equivalence classes were characterized in terms of the support of the Fourier transform of the wavelets.

Given a wavelet ψ, let $E(\psi, k) = \text{supp}\, \hat{\psi} \cap (\text{supp}\, \hat{\psi} + 2k\pi)$, $k \in \mathbb{Z}$, and $\mathcal{E}(\psi) = \{k \in \mathbb{Z} : E(\psi, k) \neq \emptyset\}$. In this notation, the characterization of $\mathcal{M}_n$, $n \in \mathbb{N} \cup \{0, \infty\}$, as given in [13], is the following.

Theorem 1.2. (a) $\mathcal{M}_\infty$ *is precisely the collection of all MSF wavelets.* (b) *The equivalence class* $\mathcal{M}_n$, $n \in \mathbb{N}$, *consists of all wavelets* ψ *such that every element of* $\mathcal{E}(\psi)$ *is divisible by* 2^n *but there exists an element of* $\mathcal{E}(\psi)$ *not divisible by* 2^{n+1}. (c) *A wavelet* ψ *belongs to* $\mathcal{M}_0$ *if and only if* $\mathcal{E}(\psi)$ *contains an odd integer.*

In the same paper, Weber produced examples of wavelets belonging to the first few equivalence classes $\mathcal{M}_n$, namely for $n = 0, 1, 2$ and 3. This motivated us to construct wavelets for each $\mathcal{M}_n$. After we constructed these wavelets we came to know about the article [9], in which Schaffer and Weber also constructed wavelets in $\mathcal{M}_n$, $n \in \mathbb{N}$, using the method of 'operator interpolation' (see chapter 5 in [2]).

In this paper we will construct wavelets belonging to each of these classes by a different method. Our approach is simpler than that of [9] in the sense that for each integer $n \geq 3$, we construct a function ψ_n such that ψ_n has the required properties to be in $\mathcal{M}_{n-2}$ as characterized in Theorem 1.2. Then we show that ψ_n is a wavelet. This is the content of §2. In §3 we construct a family of wavelets belonging to the equivalence class $\mathcal{M}_0$.

2. Construction of wavelets in $\mathcal{M}_n$, $n \geq 1$

Let $n \geq 2$ be an integer. Put

$$a_n = \frac{2^{n-1}}{2^n - 1}\pi, \qquad b_n = 2a_n = \frac{2^n}{2^n - 1}\pi,$$
$$c_n = \frac{2^{n-1}(2^n - 2)}{2^n - 1}\pi, \quad d_n = 2^n a_n = \frac{2^{2n-1}}{2^n - 1}\pi,$$
$$e_n = \frac{2^n - 2}{2^n - 1}\pi.$$

Define $S_n = S_n^+ \cup S_n^-$, where $S_n^+ = [a_n, b_n] \cup [c_n, d_n]$ and $S_n^- = -(S_n^+)$.

Observe that

$$[-b_n, -e_n] = [e_n, b_n] - 2\pi, \quad [c_n, d_n] = 2^{n-1}[e_n, b_n] \tag{4}$$

and

$$[-d_n, -c_n] = 2^{n-1}([e_n, b_n] - 2\pi). \tag{5}$$

The following theorem, proved by the authors in [1], characterizes all wavelets ψ of $L^2(\mathbb{R})$ such that the support of $\hat{\psi}$ is contained in the set S_n.

Theorem 2.1. *Let* $n \geq 2$, $\psi \in L^2(\mathbb{R})$, $\operatorname{supp} \hat{\psi} \subseteq S_n$ *and* $b(\xi) = |\hat{\psi}(\xi)|$. *Then* ψ *is a wavelet for* $L^2(\mathbb{R})$ *if and only if*

(i) $b(\xi) = 1$ *for a.e.* $\xi \in [a_n, e_n] \cup [-e_n, -a_n]$,
(ii) $b^2(\xi) + b^2(2^{n-1}\xi) = 1$ *for a.e.* $\xi \in [e_n, b_n]$,
(iii) $b^2(\xi) + b^2(\xi - 2\pi) = 1$ *for a.e.* $\xi \in [e_n, b_n]$,
(iv) $b(\xi) = b(2^{n-1}(\xi - 2\pi))$ *for a.e.* $\xi \in [e_n, b_n]$,
(v) $\hat{\psi}(\xi) = \mathrm{e}^{i\theta(\xi)} b(\xi)$, *where* θ *satisfies*

$$\theta(\xi) + \theta(2^{n-1}(\xi - 2\pi)) - \theta(\xi - 2\pi) - \theta(2^{n-1}\xi) = (2m(\xi) + 1)\pi,$$

for some $m(\xi) \in \mathbb{Z}$, *for a.e.* $\xi \in [e_n, b_n] \cap (\operatorname{supp} b) \cap (\frac{1}{2^{n-1}} \operatorname{supp} b)$. *Moreover, if* $n \geq 3$, *then none of these wavelets is associated with an MRA.*

Remark 2.2. (a) It follows from Theorem 2.1 that $\hat{\psi}$ is completely determined by its values on $[e_n, b_n]$. On this set, let $b = |\hat{\psi}|$ be an arbitrary measurable function taking values between 0 and 1. Then, in view of (4) and (5), $|\hat{\psi}|$ can be extented to other sets of S_n using properties (i)–(iv) (see Figure 1). Any function θ satisfying (v) now completely defines $\hat{\psi}$ on $\mathbb{R}$. (b) The function $\theta(\xi) = 2^{-(n-1)}\xi$ is a solution of the functional equation in Theorem 2.1(v). (c) If $(\operatorname{supp} b) \cap ((1/2^{n-1})\operatorname{supp} b)$ has an empty interior in $[e_n, b_n]$, then θ can be chosen to be any measurable function. In particular, we can take $\theta(\xi) = 0$.

Let $n \geq 3$. By choosing $b = 0$ a.e. on the interval $[e_n, b_n]$ and extending it to the other sets of S_n we get a wavelet γ_n, where

$$\hat{\gamma}_n = \chi_{W_n}, \quad W_n = [-b_n, -a_n] \cup [a_n, e_n] \cup [c_n, d_n].$$

We construct ψ_n from this function in the following manner. We translate the interval $[a_n/2, e_n/2] + 2^{n-1}\pi$ (which is a subset of $[c_n, d_n]$) to the left by a factor of $2^{n-1}\pi$ and

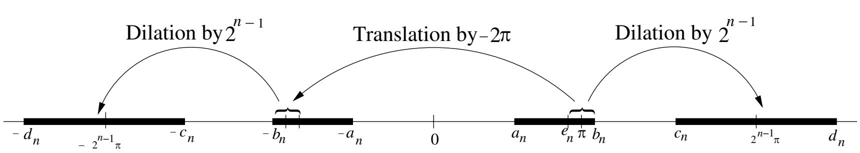

Figure 1. The set S_n.

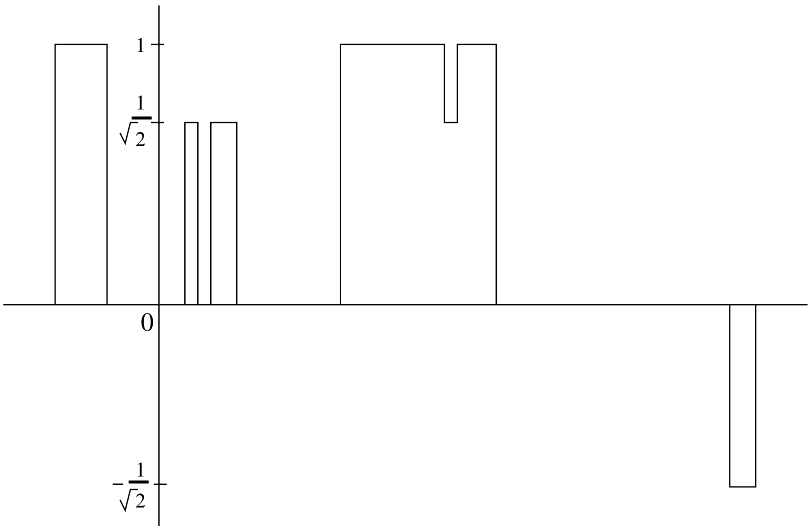

Figure 2. The function $\hat{\psi}_n$.

assign values $1/\sqrt{2}$ to $\hat{\psi}_n$ on both these sets. Then, we translate $[a_n, e_n]$ to the right by a factor of $2^n\pi$ and assign to $\hat{\psi}_n$ the value $1/\sqrt{2}$ on $[a_n, e_n]$ and $-1/\sqrt{2}$ on $[a_n, e_n] + 2^n\pi$. Assign the value 1 to $\hat{\psi}_n$ on the remaining sets of W_n and 0 elsewhere. More precisely, we have the following function (see figure 2):

$$\hat{\psi}_n(\xi) = \begin{cases} 1 & \text{if } \xi \in [-b_n, -a_n] \cup [c_n, \frac{a_n}{2} + 2^{n-1}\pi] \cup [\frac{e_n}{2} + 2^{n-1}\pi, d_n], \\ \frac{1}{\sqrt{2}} & \text{if } \xi \in [\frac{a_n}{2}, \frac{e_n}{2}] \cup [a_n, e_n] \cup [\frac{a_n}{2} + 2^{n-1}\pi, \frac{e_n}{2} + 2^{n-1}\pi], \\ -\frac{1}{\sqrt{2}} & \text{if } \xi \in [a_n + 2^n\pi, e_n + 2^n\pi], \\ 0 & \text{otherwise.} \end{cases}$$

In the following lemma we list the translates and dilates of subsets of supp $\hat{\psi}_n$ which are again in supp $\hat{\psi}_n$. The proof is a straightforward calculation and is ommited.

Lemma 2.3. *Let* $F_n = \text{supp}\, \hat{\psi}_n$.

(i) *If* $\xi \in [-b_n, -a_n]$, *then* $\xi + 2k\pi \in F_n$ *iff* $k = 0$, *and* $2^j\xi \in F_n$ *iff* $j = 0$.
(ii) *If* $\xi \in [a_n/2, e_n/2]$, *then* $\xi + 2k\pi \in F_n$ *iff* $k = 0, 2^{n-2}$, *and* $2^j\xi \in F_n$ *iff* $j = 0, 1$.
(iii) *If* $\xi \in [a_n, e_n]$, *then* $\xi + 2k\pi \in F_n$ *iff* $k = 0, 2^{n-1}$, *and* $2^j\xi \in F_n$ *iff* $j = 0, -1$.
(iv) *If* $\xi \in [c_n, (a_n/2) + 2^{n-1}\pi]$, *then* $\xi + 2k\pi \in F_n$ *iff* $k = 0$, *and* $2^j\xi \in F_n$ *iff* $j = 0$.
(v) *If* $\xi \in [(a_n/2) + 2^{n-1}\pi, (e_n/2) + 2^{n-1}\pi]$, *then* $\xi + 2k\pi \in F_n$ *iff* $k = 0, -2^{n-2}$, *and* $2^j\xi \in F_n$ *iff* $j = 0, 1$.
(vi) *If* $\xi \in [(e_n/2) + 2^{n-1}\pi, d_n]$, *then* $\xi + 2k\pi \in F_n$ *iff* $k = 0$, *and* $2^j\xi \in F_n$ *iff* $j = 0$.
(vii) *If* $\xi \in [a_n + 2^n\pi, e_n + 2^n\pi]$, *then* $\xi + 2k\pi \in F_n$ *iff* $k = 0, -2^{n-1}$, *and* $2^j\xi \in F_n$ *iff* $j = 0, -1$.

Theorem 2.4. *For each* $n \geq 3$, *the function* ψ_n *defined above is a wavelet and belongs to the equivalence class* $\mathcal{M}_{n-2}$.

Proof. To prove that ψ_n is a wavelet, it is sufficient to show that ψ satisfies the following three properties (see Theorem 1.1):

(a) $\|\psi_n\|_2 = 1$,
(b) $\rho(\xi) := \sum_{j\in\mathbb{Z}} |\hat{\psi}_n(2^j\xi)|^2 = 1$ for a.e. $\xi \in \mathbb{R}$,
(c) $t_q(\xi) := \sum_{j\geq 0} \hat{\psi}_n(2^j\xi)\overline{\hat{\psi}_n(2^j(\xi + 2q\pi))} = 0$ for a.e. $\xi \in \mathbb{R}$, for all $q \in 2\mathbb{Z}+1$.

Proof of (a). We have

$$\begin{aligned}\|\hat{\psi}_n\|_2^2 &= (b_n - a_n) + \left(\frac{a_n}{2} + 2^{n-1}\pi - c_n\right) + d_n - \left(\frac{e_n}{2} + 2^{n-1}\pi\right)\\ &\quad + \frac{1}{2}\left\{\left(\frac{e_n}{2} - \frac{a_n}{2}\right) + (e_n - a_n) + \left(\frac{e_n}{2} - \frac{a_n}{2}\right) + (e_n - a_n)\right\}\\ &= b_n - a_n + d_n - c_n + e_n - a_n = 2\pi.\end{aligned}$$

Therefore, $\|\psi_n\|_2 = (1/2\pi)\|\hat{\psi}_n\|_2^2 = 1$.

Proof of (b). Let $\xi > 0$. Since $\rho(\xi) = \rho(2\xi)$, it is enough to show that $\rho(\xi) = 1$ for a.e. $\xi \in [\alpha, 2\alpha]$ for some $\alpha > 0$. We will prove that $\rho(\xi) = 1$ for a.e. $\xi \in [a_n, 2a_n] = [a_n, e_n] \cup [e_n, b_n]$.

Suppose $\xi \in [a_n, e_n]$. Then $2^j\xi \in F_n$ if and only if $j = 0, -1$. So, $\rho(\xi) = |\hat{\psi}_n(\xi)|^2 + |\hat{\psi}_n(\xi/2)|^2 = (1/\sqrt{2})^2 + (1/\sqrt{2})^2 = 1$. Now, $\xi \in [e_n, b_n]$ if and only if $2^{n-1}\xi \in [c_n, d_n]$. We write $[c_n, d_n]$ as a union of three intervals:

$$\begin{aligned}[c_n, d_n] &= \left[c_n, \frac{a_n}{2} + 2^{n-1}\pi\right] \cup \left[\frac{a_n}{2} + 2^{n-1}\pi, \tfrac{e_n}{2} + 2^{n-1}\pi\right] \cup \left[\frac{e_n}{2} + 2^{n-1}\pi, d_n\right]\\ &= I_1 \cup I_2 \cup I_3, \text{ say.}\end{aligned}$$

If $2^{n-1}\xi \in (I_1 \cup I_3)$, then $2^j(2^{n-1}\xi) \in F_n$ if and only if $j = 0$ (see Lemma 2.3). So, $\rho(\xi) = |\hat{\psi}_n(2^{n-1}\xi)|^2 = 1$. Also if $2^{n-1}\xi \in I_2$, then $2^j(2^{n-1}\xi) \in F_n$ if and only if $j = 0$ or 1, hence, $\rho(\xi) = |\hat{\psi}_n(2^{n-1}\xi)|^2 + |\hat{\psi}_n(2^n\xi)|^2 = (1/\sqrt{2})^2 + (-1/\sqrt{2})^2 = 1$. Therefore, we get $\rho(\xi) = 1$ for a.e. $\xi > 0$.

For $\xi < 0$, it suffices to show that $\rho(\xi) = 1$ on $[-b_n, -a_n]$. On this set, $2^j\xi \in F_n$ if and only if $j = 0$. Hence, $\rho(\xi) = |\hat{\psi}_n(\xi)|^2 = 1$ for a.e. $\xi \in [-b_n, -a_n]$.

Proof of (c). Since $t_q(\xi) = \overline{t_{-q}(\xi + 2q\pi)}$, it is enough to show that $t_q = 0$ a.e., if q is a negative odd integer. Suppose $q \neq -1$, and is odd. We have $2^j q \neq 0, \pm 2^{n-1}, \pm 2^{n-2}$. Therefore, if $2^j\xi \in F_n$, then by Lemma 2.3 we observe that $2^j\xi + 2\cdot 2^j q\pi \notin F_n$, which shows that each term of the sum $t_q(\xi)$ is 0. Hence, $t_q = 0$ a.e.

It remains to prove that $t_{-1}(\xi) = 0$ for a.e. $\xi \in \mathbb{R}$. We have

$$t_{-1}(\xi) = \sum_{j\geq 0} \hat{\psi}_n(2^j\xi)\overline{\hat{\psi}_n(2^j\xi - 2\cdot 2^j\pi)}.$$

By Lemma 2.3 (see (vii) and (v)), we observe that both $2^j\xi$ and $2^j\xi - 2\cdot 2^j\pi$ belong to F_n only in the following two cases:

(i) $2^j\xi \in [a_n + 2^n\pi, e_n + 2^n\pi]$ and $j = n-1$.
(ii) $2^j\xi \in [\frac{a_n}{2} + 2^{n-1}\pi, \frac{e_n}{2} + 2^{n-1}\pi]$ and $j = n-2$.

But both are equivalent to saying that $2^{n-1}\xi \in [a_n + 2^n\pi, e_n + 2^n\pi]$. So we get $t_{-1}(\xi) = 0$, if $2^{n-1}\xi \notin [a_n + 2^n\pi, e_n + 2^n\pi]$.

Now, if $2^{n-1}\xi \in [a_n + 2^n\pi, e_n + 2^n\pi]$, then

$$\begin{aligned} t_{-1}(\xi) &= \hat{\psi}_n(2^{n-2}\xi)\overline{\hat{\psi}_n(2^{n-2}\xi - 2^{n-1}\pi)} + \hat{\psi}_n(2^{n-1}\xi)\overline{\hat{\psi}_n(2^{n-1}\xi - 2^n\pi)} \\ &= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + \left(-\frac{1}{\sqrt{2}}\right) \cdot \frac{1}{\sqrt{2}} \\ &= 0. \end{aligned}$$

This completes the proof of (c). Therefore, ψ_n is a wavelet.

Our claim now is that $\psi_n \in \mathcal{M}_{n-2}$. By referring to Lemma 2.3 again we observe that $\mathcal{E}(\psi_n) = \{0, \pm 2^{n-2}, \pm 2^{n-1}\}$. Hence, by Theorem 1.2(b), $\psi_n \in \mathcal{M}_{n-2}$.

Since $n \geq 3$, we have constructed examples of wavelets in each of the equivalence classes $\mathcal{M}_n$, $n \geq 1$. □

3. A family of wavelets belonging to the class $\mathcal{M}_0$

In this section we give examples of a family of wavelets in $\mathcal{M}_0$. In fact, we shall show that all non-MSF wavelets characterized in Theorem 2.1 belong to $\mathcal{M}_0$.

For $n \geq 3$, let a_n, b_n, c_n, d_n and e_n be as in §2. Define the function b on $[e_n, b_n]$ as follows:

$$b(\xi) = \begin{cases} 1/\sqrt{2} & \text{if } \xi \in [e_n, \pi] \\ 0 & \text{if } \xi \in [\pi, b_n]. \end{cases}$$

Then we extend b to the whole of S_n by using (i)–(iv) of Theorem 2.1. Observe that $[e_n, b_n] \cap \operatorname{supp} b \cap ((1/2^{n-1}) \operatorname{supp} b) = [e_n, \pi]$. We define θ on $\mathbb{R}$ as

$$\theta(\xi) = \begin{cases} \pi & \text{if } \xi \in [e_n, \pi], \\ 0 & \text{otherwise}. \end{cases}$$

It is easy to see that θ satisfies the functional equation in (v) of Theorem 2.1. This choice of b and θ will give us the wavelet w_n, where $\widehat{w}_n(\xi) = \mathrm{e}^{i\theta(\xi)}b(\xi)$. That is,

$$\widehat{w}_n(\xi) = \begin{cases} 1 & \text{if } \xi \in [-\pi, -a_n] \cup [a_n, e_n] \cup [2^{n-1}\pi, d_n], \\ 1/\sqrt{2} & \text{if } \xi \in [-d_n, -2^{n-1}\pi] \cup [-b_n, -\pi] \cup [c_n, 2^{n-1}\pi], \\ -1/\sqrt{2} & \text{if } \xi \in [e_n, \pi], \\ 0 & \text{otherwise}. \end{cases}$$

Since $[-b_n, -\pi] + 2\pi = [e_n, \pi]$, and $\widehat{w}_n$ does not vanish on the sets $[-b_n, -\pi]$ and $[e_n, \pi]$, it is clear that $1 \in \mathcal{E}(w_n)$. Hence $w_n \in \mathcal{M}_0$, by Theorem 1.2(c).

Remark 3.1. The above example is a particular case of the fact that all wavelets characterized in Theorem 2.1 belong to $\mathcal{M}_0 \cup \mathcal{M}_\infty$. To see this, let ψ be such a wavelet. If ψ is an MSF wavelet, then $\psi \in \mathcal{M}_\infty$ by the characterization of $\mathcal{M}_\infty$. On the other hand, if ψ is not MSF, then there is a non-trivial set $K \subset [e_n, b_n]$ such that $0 < |\hat{\psi}(\xi)| < 1$ for a.e. $\xi \in K$. By condition (iii) of Theorem 2.1, $0 < |\hat{\psi}(\xi)| < 1$ for a.e. $\xi \in K - 2\pi$ as well. As above, $1 \in \mathcal{E}(\psi)$ proving that $\psi \in \mathcal{M}_0$.

Acknowledgements

The research done by the first author was supported by the National Board for Higher Mathematics (NBHM), Govt. of India.

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