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A note on absolute summability factors

H S ÖZARSLAN

Department of Mathematics, Erciyes University, 38039 Kayseri, Turkey
E-mail: seyhan@erciyes.edu.tr

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Abstract. In this paper, by using an almost increasing and δ-quasi-monotone sequence, a general theorem on $\varphi - |C, \alpha|_k$ summability factors, which generalizes a result of Bor [3] on $\varphi - |C, 1|_k$ summability factors, has been proved under weaker and more general conditions.

Keywords. Absolute summability; almost increasing sequences.

1. Introduction

A sequence (b_n) of positive numbers is said to be δ-quasi-monotone, if $b_n \to 0$, $b_n > 0$ ultimately and $\Delta b_n \geq -\delta_n$, where (δ_n) is a sequence of positive numbers (see [3]). Let (φ_n) be a sequence of complex numbers and let $\sum a_n$ be a given infinite series with partial sums (s_n). We denote by σ_n^α and t_n^α the nth Cesàro means of order α, with $\alpha > -1$, of the sequences (s_n) and (na_n), respectively, i.e.,

$$\sigma_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=0}^{n} A_{n-v}^{\alpha-1} s_v, \tag{1}$$

$$t_n^\alpha = \frac{1}{A_n^\alpha} \sum_{v=1}^{n} A_{n-v}^{\alpha-1} v a_v, \tag{2}$$

where

$$A_n^\alpha = O(n^\alpha), \quad \alpha > -1, \quad A_0^\alpha = 1 \quad \text{and} \quad A_{-n}^\alpha = 0 \quad \text{for} \quad n > 0. \tag{3}$$

The series $\sum a_n$ is said to be summable $|C, \alpha|_k$, $k \geq 1$ and $\alpha > -1$, if (see [6])

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} \frac{1}{n} |t_n^\alpha|^k < \infty, \tag{4}$$

and it is said to be summable $|C, \alpha; \beta|_k$, $k \geq 1$, $\alpha > -1$ and $\beta \geq 0$, if (see [7])

$$\sum_{n=1}^{\infty} n^{\beta k+k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k = \sum_{n=1}^{\infty} n^{\beta k-1} |t_n^\alpha|^k < \infty. \tag{5}$$

The series $\sum a_n$ is said to be summable $\varphi - |C, \alpha|_k$, $k \geq 1$ and $\alpha > -1$, if (see [2])

$$\sum_{n=1}^{\infty} n^{-k} |\varphi_n t_n^\alpha|^k < \infty. \tag{6}$$

In the special case when $\varphi_n = n^{1-\frac{1}{k}}$ (resp. $\varphi_n = n^{\beta+1-\frac{1}{k}}$) $\varphi - |C,\alpha|_k$ summability is the same as $|C,\alpha|_k$ (resp. $|C,\alpha;\beta|_k$) summability. Bor [4] has proved the following theorem for $\varphi - |C,1|_k$ summability factors of infinite series.

Theorem A. *Let (X_n) be a positive non-decreasing sequence and let (λ_n) be a sequence such that*

$$|\lambda_n| X_n = O(1) \quad as \quad n \to \infty, \tag{7}$$

$$\sum_{v=1}^{n} vX_v |\Delta^2 \lambda_v| = O(1) \quad as \quad n \to \infty. \tag{8}$$

If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} |\varphi_n|^k)$ is non-increasing and

$$\sum_{v=1}^{n} v^{-k} |\varphi_v t_v|^k = O(X_n) \quad \text{as} \quad n \to \infty, \tag{9}$$

then the series $\sum a_n \lambda_n$ is summable $\varphi - |C,1|_k$, $k \geq 1$.

2. The main result

The aim of this paper is to extend Theorem A, by using an almost increasing and δ-quasi monotone sequence, under weaker and more general conditions for $\varphi - |C,\alpha|_k$ summability. For this we need the concept of almost increasing sequence. A positive sequence (b_n) is said to be *almost increasing* if there exists a positive increasing sequence c_n and two positive constants A and B such that $A(c_n) \leq b_n \leq B(c_n)$ (see [1]). Obviously every increasing sequence is an almost increasing sequence, but the converse need not be true, as can be seen from the example $b_n = n\mathrm{e}^{(-1)^n}$. So we are weakening the hypotheses of the theorem in replacing the increasing sequence by an almost increasing sequence.

Now, we shall prove the following:

Theorem. *Let (X_n) be an almost increasing sequence and the sequence (λ_n) such that condition* (7) *of Theorem A is satisfied. Suppose that there exists a sequence of numbers (A_n) such that it is δ-quasi monotone with $\sum nA_nX_n$ convergent and $|\Delta\lambda_n| \leq |A_n|$ for all n. If there exists an $\epsilon > 0$ such that the sequence $(n^{\epsilon-k} |\varphi_n|^k)$ is non-increasing and if the sequence (w_n^α), defined by (see* [8]*)*

$$w_n^\alpha = \begin{cases} |t_n^\alpha|, & \alpha = 1 \\ \max_{1\leq v\leq n} |t_v^\alpha|, & 0 < \alpha < 1 \end{cases} \tag{10}$$

satisfies the condition

$$\sum_{n=1}^{m} n^{-k} (w_n^\alpha |\varphi_n|)^k = O(X_m) \quad as \quad m \to \infty, \tag{11}$$

then the series $\sum a_n \lambda_n$ is summable $\varphi - |C,\alpha|_k$, $k \geq 1$, $0 < \alpha \leq 1$ and $k\alpha + \epsilon > 1$. We need the following lemma for the proof of our theorem.

Lemma 1. [5]. *If $0 < \alpha \leq 1$ and $1 \leq v \leq n$, then*

$$\left|\sum_{p=0}^{v} A_{n-p}^{\alpha-1} a_p\right| \leq \max_{1\leq m\leq v} \left|\sum_{p=0}^{m} A_{m-p}^{\alpha-1} a_p\right|. \tag{12}$$

3. Proof of the theorem

Let (T_n^α), with $0 < \alpha \leq 1$, be the nth (C, α) mean of the sequence $(na_n\lambda_n)$. Then, by (2), we have

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} va_v\lambda_v. \tag{13}$$

Using Abel's transformation, we get

$$T_n^\alpha = \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} \Delta\lambda_v \sum_{p=1}^{v} A_{n-p}^{\alpha-1} pa_p + \frac{\lambda_n}{A_n^\alpha}\sum_{v=1}^{n} A_{n-v}^{\alpha-1} va_v,$$

so that making use of Lemma 1, we have

$$\begin{aligned}
|T_n^\alpha| &\leq \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} |\Delta\lambda_v| \left|\sum_{p=1}^{v} A_{n-p}^{\alpha-1} pa_p\right| + \frac{|\lambda_n|}{A_n^\alpha}\left|\sum_{v=1}^{n} A_{n-v}^{\alpha-1} va_v\right| \\
&\leq \frac{1}{A_n^\alpha}\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha\, |\Delta\lambda_v| + |\lambda_n|\, w_n^\alpha \\
&= T_{n,1}^\alpha + T_{n,2}^\alpha, \quad \text{say.}
\end{aligned}$$

Since

$$|T_{n,1}^\alpha + T_{n,2}^\alpha|^k \leq 2^k(|T_{n,1}^\alpha|^k + |T_{n,2}^\alpha|^k),$$

to complete the proof of the theorem, it is sufficient to show that

$$\sum_{n=1}^{\infty} n^{-k}\, |\varphi_n T_{n,r}^\alpha|^k < \infty \quad \text{for} \quad r = 1, 2, \quad \text{by} \quad (6).$$

Now, when $k > 1$, applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we get

$$\begin{aligned}
\sum_{n=2}^{m+1} n^{-k}\, |\varphi_n T_{n,1}^\alpha|^k &\leq \sum_{n=2}^{m+1} n^{-k}(A_n^\alpha)^{-k}\, |\varphi_n|^k \left\{\sum_{v=1}^{n-1} A_v^\alpha w_v^\alpha\, |\Delta\lambda_v|\right\}^k \\
&= O(1)\sum_{n=2}^{m+1} n^{-k} n^{-\alpha k}\, |\varphi_n|^k \left\{\sum_{v=1}^{n-1} v^{\alpha k}(w_v^\alpha)^k\, |A_v|\right\} \\
&\quad \times \left\{\sum_{v=1}^{n-1} |A_v|\right\}^{k-1} \\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^\alpha)^k\, |A_v| \sum_{n=v+1}^{m+1} \frac{n^{-k}\, |\varphi_n|^k}{n^{\alpha k}} \\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^\alpha)^k\, |A_v| \sum_{n=v+1}^{m+1} \frac{n^{\epsilon-k}\, |\varphi_n|^k}{n^{\alpha k+\epsilon}} \\
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^\alpha)^k\, |A_v|\; v^{\epsilon-k}\, |\varphi_v|^k \sum_{n=v+1}^{m+1} \frac{1}{n^{\alpha k+\epsilon}}
\end{aligned}$$

$$
\begin{aligned}
&= O(1)\sum_{v=1}^{m} v^{\alpha k}(w_v^{\alpha})^k \mid A_v \mid v^{\epsilon-k} \mid \varphi_v \mid^k \int_v^{\infty} \frac{\mathrm{d}x}{x^{\alpha k+\epsilon}} \\
&= O(1)\sum_{v=1}^{m} v \mid A_v \mid v^{-k}(w_v^{\alpha} \mid \varphi_v \mid)^k \\
&= O(1)\sum_{v=1}^{m-1} \Delta(v \mid A_v \mid)\sum_{r=1}^{v} r^{-k}(w_r^{\alpha} \mid \varphi_r \mid)^k \\
&\quad + O(1)m \mid A_m \mid \sum_{v=1}^{m} v^{-k}(w_v^{\alpha} \mid \varphi_v \mid)^k \\
&= O(1)\sum_{v=1}^{m-1} \mid \Delta(v \mid A_v \mid) \mid X_v + O(1)m \mid A_m \mid X_m \\
&= O(1)\sum_{v=1}^{m-1} v \mid A_v \mid X_v + O(1)\sum_{v=1}^{m-1}(v+1) \mid A_{v+1} \mid X_v \\
&\quad + O(1)m \mid A_m \mid X_m = O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of the Theorem.

Again, since $|\lambda_n| = O(1/X_n) = O(1)$, by (7), we have

$$
\begin{aligned}
\sum_{n=1}^{m} n^{-k} \mid \varphi_n T_{n,2}^{\alpha} \mid^k &= \sum_{n=1}^{m} \mid \lambda_n \mid^{k-1} \mid \lambda_n \mid n^{-k}(w_n^{\alpha} \mid \varphi_n \mid)^k \\
&= O(1)\sum_{n=1}^{m} \mid \lambda_n \mid n^{-k}(w_n^{\alpha} \mid \varphi_n \mid)^k \\
&= O(1)\sum_{n=1}^{m-1} \mid \Delta \mid \lambda_n \| \sum_{v=1}^{n} v^{-k}(w_v^{\alpha} \mid \varphi_v \mid)^k \\
&\quad + O(1) \mid \lambda_m \mid \sum_{n=1}^{m} n^{-k}(w_n^{\alpha} \mid \varphi_n \mid)^k \\
&= O(1)\sum_{n=1}^{m-1} \mid \Delta\lambda_n \mid X_n + O(1) \mid \lambda_m \mid X_m \\
&= O(1)\sum_{n=1}^{m-1} \mid A_n \mid X_n + O(1) \mid \lambda_m \mid X_m \\
&= O(1) \quad \text{as} \quad m \to \infty,
\end{aligned}
$$

by virtue of the hypotheses of the Theorem.

Therefore, we get

$$
\sum_{n=1}^{m} n^{-k} \mid \varphi_n T_{n,r}^{\alpha} \mid^k = O(1) \quad \text{as} \quad m \to \infty, \quad \text{for} \quad r = 1, 2.
$$

This completes the proof of the Theorem.

If we take $\epsilon = 1$ and $\varphi_n = n^{1-\frac{1}{k}}$ (resp. $\epsilon = 1$ and $\varphi_n = n^{\beta+1-\frac{1}{k}}$), then we get a new result related to $|C, \alpha|_k$ (resp. $|C, \alpha; \beta|_k$) summability factors.

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