

## Reduction theory for a rational function field

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**Abstract.** Let  $G$  be a split reductive group over a finite field  $\mathbf{F}_q$ . Let  $F = \mathbf{F}_q(t)$  and let  $\mathbf{A}$  denote the adèles of  $F$ . We show that every double coset in  $G(F)\backslash G(\mathbf{A})/K$  has a representative in a maximal split torus of  $G$ . Here  $K$  is the set of integral adèlic points of  $G$ . When  $G$  ranges over general linear groups this is equivalent to the assertion that any algebraic vector bundle over the projective line is isomorphic to a direct sum of line bundles.

**Keywords.** Automorphic form; function field.

### 1. Introduction

Let  $F$  be a global field,  $\mathbf{A}$  its ring of adèles and  $G$  a reductive group defined over  $F$ . The theory of automorphic forms involves the study of spaces of functions on  $G(F)\backslash G(\mathbf{A})$  as representations of  $G(\mathbf{A})$ . The functions involved are often required to be right invariant under certain large compact subgroups  $K$  of  $G(\mathbf{A})$  because (among other reasons) the double coset space  $G(F)\backslash G(\mathbf{A})/K$  admits nice interpretations. For example, the classical study of the upper half plane modulo the action of arithmetic subgroups of the real special linear group is a special case of the above when  $F$  is the field of rational numbers (see e.g., ([13], §1). Another special case, which corresponds to taking  $F$  to be a field of rational functions in one variable and  $G$  to be  $GL(2)$  is discussed by Weil in [15]. When  $F$  is a function field, Harder describes a fundamental domain for the action of  $G(F)$  on  $G(\mathbf{A})$  in ([10], §1) using results from [8] and [9]. This is an analogue of the Siegel domain described by Godement in [6] for  $F = \mathbf{Q}$ . Proposition 14 in this article is analogous to these results and the proof proceeds along the lines of [6]. Harder's description of the fundamental domain is a very basic result in the theory of automorphic forms over function fields (see e.g., [12], §9 and Appendix E).

From now on let  $G$  be a split reductive group defined over a finite field  $\mathbf{F}_q$  with  $q$  elements. Fix a Borel subgroup  $B$  defined over  $\mathbf{F}_q$  with unipotent radical  $N$ , and a maximal  $\mathbf{F}_q$ -split torus  $T$  contained in  $B$ . Set  $F = \mathbf{F}_q(t)$ . For a valuation  $v$  of  $F$ , we denote the corresponding local field by  $F_v$  and its ring of integers by  $\mathbf{O}_v$ . For each  $v$ , fix a uniformizing element  $\pi_v \in F \cap \mathbf{O}_v$ . In particular, fix  $\pi_\infty = t^{-1}$  as a uniformizing element at the place  $\infty$  whose local field is  $\mathbf{F}_q((t^{-1}))$ . Let  $K$  be the maximal compact subgroup  $\prod_v G(\mathbf{O}_v)$  of  $G(\mathbf{A})$ . This article concerns the double coset space

$$G(F)\backslash G(\mathbf{A})/K$$

which may be interpreted as the set of isomorphism classes of principal  $G$ -bundles on the projective line. In [7], Grothendieck proves that when  $G$  is a complex reductive group any

holomorphic  $G$ -bundle over the complex projective line admits a reduction of structure group to a maximal torus. (In fact this result has been attributed to Dedekind and Weber for  $G = GL(n)$  by Geyer ([5], §6) who deduces it from a statement in ([3], §22).) In our adèlic setting, this should correspond to the assertion that every double coset has a representative in  $T(\mathbf{A})$ .

Let  $X_*(T)$  denote the lattice  $\text{Hom}(\mathbf{G}_m, T)$  of algebraic co-characters of  $T$ . Given  $\eta \in X_*(T)$ , and a valuation  $v$  denote by  $\pi_v^\eta$  the element  $\eta(\pi_v) \in T(F_v) \subset T(\mathbf{A})$ . Recall that  $\eta \in X_*(T)$  is called *antidominant* if  $|\alpha_i \circ \eta(\pi_v)|_v \geq 1$  for each simple root  $\alpha_i$  (see §3). Precisely stated, the main result of this article is the following:

**Theorem 1.** *Every double coset in*

$$G(F) \backslash G(\mathbf{A}) / K$$

*has a unique representative of the form  $(t^{-1})^\eta$ , where  $\eta \in X_*(T)$  is antidominant.*

In §6, we will deduce Theorem 1 from the following local result which is proved in §5. Let  $F_\bullet$  be the local field  $\mathbf{F}_q((\pi))$  of Laurent series in  $\pi$  with coefficients in  $\mathbf{F}_q$ . It contains, as its ring of integers, the discrete valuation ring  $\mathbf{O} = \mathbf{F}_q[[\pi]]$ , and as a discrete subring, the polynomial ring  $R = \mathbf{F}_q[\pi^{-1}]$ . Let  $\Gamma = G(R)$ .

**Theorem 2.** *Every double coset in*

$$\Gamma \backslash G(F_\bullet) / G(\mathbf{O})$$

*has a unique representative of the form  $\pi^\eta$ , where  $\eta \in X_*(T)$  is antidominant.*

The main results proved in this article should be known to the experts, but we have not found them in the literature beyond the case of  $GL(2)$ , for which Theorem 2 is proved in ([15], §3). The results proved in this paper have played an important role in the author’s work [14], as well as in the work of other authors on  $\mathbf{F}_q(t)$  [4,1,11].

## 2. Normed local vector spaces

Let  $V$  be a vector space defined over  $\mathbf{F}_q$ . Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be a basis of the free  $\mathbf{O}$ -module  $V(\mathbf{O})$  (so that  $V(\mathbf{O})$  is isomorphic to the free  $\mathbf{O}$ -module generated by the  $\mathbf{e}_i$ s). Given a vector  $\mathbf{x} \in V(F_\bullet)$ , we may write  $\mathbf{x} = x_1\mathbf{e}_1 + \dots + x_n\mathbf{e}_n$ , uniquely, with  $x_i \in F_\bullet$ . Define

$$\|\mathbf{x}\| = \sup\{|x_1|, \dots, |x_n|\}. \tag{1}$$

*Lemma 4.* *If  $g \in GL(V(\mathbf{O}))$ , then  $\|\mathbf{x}g\| = \|\mathbf{x}\|$ .*

*Proof.* Let  $(g_{ij})$  be the matrix of  $G$  with respect to the basis chosen above. Let  $\mathbf{y} = \mathbf{x}g$ . If  $\mathbf{y} = y_1\mathbf{e}_1 + \dots + y_n\mathbf{e}_n$ , then

$$y_j = \sum_{i=1}^n x_i g_{ij}$$

and

$$\begin{aligned}
 \|\mathbf{y}\| &= \sup_{1 \leq j \leq n} \left| \sum_{i=1}^n x_i g_{ij} \right| \\
 &\leq \sup_{1 \leq j \leq n} \sup_{1 \leq i \leq n} |x_i g_{ij}| \quad (\text{ultrametric inequality}) \\
 &\leq \sup_{1 \leq j \leq n} \sup_{1 \leq i \leq n} |x_i| \quad (\text{since } g_{ij} \in \mathbf{O}) \\
 &= \|\mathbf{x}\|.
 \end{aligned}$$

Hence

$$\|\mathbf{y}\| \leq \|\mathbf{x}\|.$$

We may apply the same reasoning to  $g^{-1}$  to show that

$$\|\mathbf{x}\| \leq \|\mathbf{y}\|.$$

Therefore,

$$\|\mathbf{y}\| = \|\mathbf{x}\|.$$

□

#### COROLLARY 5

The norm  $\|\cdot\|$  is independent of our choice of basis of  $V(\mathbf{O})$ .

*Proof.* The coordinates of a vector with respect to two different bases differ by a matrix with entries in  $\mathbf{O}$ . The argument in the proof of Lemma 4 shows that the norms with respect to two different bases are equal. □

*Lemma 6.* The norm  $\|\cdot\|$  satisfies the ultrametric triangle inequality, i.e., for vectors  $\mathbf{x}, \mathbf{y}$  in  $V(F_\bullet)$ ,

$$\|\mathbf{x} + \mathbf{y}\| \leq \sup\{\|\mathbf{x}\|, \|\mathbf{y}\|\}.$$

*Proof.* Write  $\mathbf{x} = x_1 \mathbf{e}_1 + \dots + x_n \mathbf{e}_n$  and  $\mathbf{y} = y_1 \mathbf{e}_1 + \dots + y_n \mathbf{e}_n$ . Then

$$\begin{aligned}
 \|\mathbf{x} + \mathbf{y}\| &= \sup\{|x_1 + y_1|, \dots, |x_n + y_n|\} \\
 &\leq \sup\{\sup\{|x_1|, |y_1|\}, \dots, \sup\{|x_n|, |y_n|\}\} \\
 &= \sup\{|x_1|, |y_1|, \dots, |x_n|, |y_n|\} \\
 &= \sup\{\|\mathbf{x}\|, \|\mathbf{y}\|\}.
 \end{aligned}$$

□

*Lemma 7.* For a scalar  $\lambda \in F_\bullet$  and a vector  $\mathbf{x} \in V(F_\bullet)$ ,

$$\|\lambda \mathbf{x}\| = |\lambda| \|\mathbf{x}\|.$$

*Lemma 8.* If  $g \in GL(V(F_\bullet))$ , then there is a constant  $C_g > 0$ , such that for any vector  $\mathbf{x} \in V(F_\bullet)$ ,

$$\|\mathbf{x}g\| \leq C_g \|\mathbf{x}\|.$$

*Proof.* Suppose that  $g$  has matrix  $(g_{ij})$ , and  $\mathbf{x}$  has coordinates  $(x_1, \dots, x_n)$  with respect to the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$ . Then

$$\begin{aligned} \|\mathbf{x}g\| &= \sup \left\{ \left| \sum_{i=1}^n x_i g_{i1} \right|, \dots, \left| \sum_{i=1}^n x_i g_{in} \right| \right\} \\ &\leq \sup_{1 \leq j \leq n} \sup_{1 \leq i \leq n} |g_{ij}| \|\mathbf{x}\|. \end{aligned}$$

Therefore, let

$$C_g = \sup_{1 \leq j \leq n} \sup_{1 \leq i \leq n} |g_{ij}|.$$

□

*Lemma 9.* If  $\mathbf{x} \in V(R)$  is a non-zero vector then  $\|\mathbf{x}\| \geq 1$ .

*Proof.* By Corollary 5, we may assume that the elements  $\mathbf{e}_i$  of a basis used to define  $\|\cdot\|$  lie in  $V(\mathbf{F}_q)$ . Then at least one coordinate of  $\mathbf{x}$  is non-zero in  $R$ . But any non-zero element in  $R$  has norm at least one. Therefore,  $\|\mathbf{x}\| \geq 1$ . □

#### PROPOSITION 10

For any non-zero vector  $\mathbf{x} \in V(\mathbf{F}_q)$  and any  $g \in GL(V(\mathbf{F}_\bullet))$ , there is a positive constant  $E$  such that for all  $\gamma \in GL(V(R))$ ,

$$\|\mathbf{x}\gamma g\| \geq E.$$

Consequently, for any subset  $S$  of  $GL(V(R))$ , the set  $\{\|\mathbf{x}sg\| : s \in S\}$  has a positive minimal element.

*Proof.* Applying Lemma 8 to  $g^{-1}$ , and Lemma 9 to  $\mathbf{x}\gamma$  (which lies in  $V(R)$ ), we have

$$\|\mathbf{x}\gamma g\| \geq C_{g^{-1}} \|\mathbf{x}\gamma\| \geq C_{g^{-1}} > 0.$$

The second part of the assertion follows by noting that the values taken by the norm  $\|\cdot\|$  are of the form  $q^j$ , where  $j$  is an integer. □

### 3. Fundamental representations

Let  $\alpha_1, \dots, \alpha_r$  be the simple roots with respect to  $B$  in the root system  $\Phi(G, T)$  of  $G$  with respect to  $T$ . Let  $W = N_G(T)/T$  be the Weyl group of  $G$  with respect to  $T$ . To each simple root  $\alpha_i$ , we associate an element  $s_i$  of order two in  $W$  in the usual way.

Given a subset  $D$  of  $\{1, \dots, r\}$ , let  $W_D$  denote the subgroup of  $W$  generated by  $\{s_j | j \in D\}$ , and let  $P_D$  denote the parabolic subgroup  $BW_D B$  of  $G$  containing  $B$ . This group has a Levi decomposition

$$P_D = L_D U_D,$$

where  $L_D$  is a reductive group of rank  $|D|$  and  $U_D$  is the unipotent radical of  $P_D$ .  $L_D \cap B$  is a Borel subgroup for  $L_D$  containing the split torus  $T$ . The set of simple roots of  $L_D$  with respect to  $L_D \cap B$  is  $\{\alpha_j | j \in D\}$ . Denote by  $P_i$  (resp.,  $L_i, U_i$ ) the parabolic subgroup (resp., Levi subgroup, unipotent subgroup) corresponding to the set  $\{1, \dots, i-1, i+1, \dots, r\}$ . These are the maximal proper parabolic subgroups of  $G$  containing  $B$ .

**Theorem 11 [2].** *There exist irreducible finite dimensional representations  $(\rho_i, V_i)$  of  $G$ , vectors  $\mathbf{v}_i \in V_i(\mathbf{F}_q)$  that are unique up to scaling, and characters  $\Delta_i : P_i \rightarrow \mathbf{G}_m$ , for  $i = 1, \dots, r$  all defined over  $\mathbf{F}_q$ , such that*

1.  $P_i$  is the stabilizer of the line generated by  $\mathbf{v}_i$  and  $\mathbf{v}_i \rho_i(p) = \Delta_i(p) \mathbf{v}_i$  for each  $p \in P_i$  for  $i = 1, \dots, r$ .
2. The restrictions  $\mu_i$  to  $T$  of  $\Delta_i$ s are antidominant weights of  $T$  with respect to  $B$ , which generate  $X^*(T) \otimes \mathbf{Q}$  as a vector space over the rational numbers.

#### 4. Ordering by roots

*Lemma 12. Let  $L$  be a Levi subgroup of  $G$  associated to a parabolic subgroup  $P$  containing  $B$ . Then there is a canonical surjection*

$$G(\mathbf{F}_\bullet)/G(\mathbf{O}) \xrightarrow{\Phi_L^G} L(\mathbf{F}_\bullet)/L(\mathbf{O}).$$

*If  $Q = MN$  is a parabolic subgroup of  $G$  containing  $B$  and contained in  $P$ , then  $M$  is a Levi subgroup for  $L$  corresponding to the parabolic subgroup  $L \cap Q$  of  $L$ , and  $\Phi_M^L \circ \Phi_L^G = \Phi_M^G$ .*

*Proof.* Given  $g \in G(\mathbf{F}_\bullet)$ , we may use the Iwasawa decomposition to write  $g = luk$ , where  $l \in L(\mathbf{F}_\bullet)$ ,  $u \in U(\mathbf{F}_\bullet)$  and  $k \in G(\mathbf{O})$ . Moreover, if  $g = l'u'k'$  is another such decomposition, then, setting  $l_0 = l'^{-1}l$  and  $k_0 = k'k^{-1}$ ,

$$u'^{-1}l_0u = k_0 \in G(\mathbf{O}).$$

On the other hand,

$$k_0 = u'^{-1}l_0u = l_0l_0^{-1}u'^{-1}l_0u.$$

Since  $L$  normalizes  $U$ ,  $l_0^{-1}u'^{-1}l_0 \in U(\mathbf{F}_\bullet)$ , and hence, setting  $u_0 = l_0^{-1}u'^{-1}l_0u \in U(\mathbf{F}_\bullet)$ ,

$$l_0 = k_0u_0 \in G(\mathbf{O})U(\mathbf{F}_\bullet) \cap L(\mathbf{F}_\bullet).$$

Therefore  $l_0u_0^{-1} = k_0 \in G(\mathbf{O}) \cap P(\mathbf{F}_\bullet) = P(\mathbf{O})$ , so that  $l_0 \in L(\mathbf{O})$ . This shows that  $luk \mapsto l$  induces a well defined map  $\Phi_L^G : G(\mathbf{F}_\bullet)/G(\mathbf{O}) \rightarrow L(\mathbf{F}_\bullet)/L(\mathbf{O})$ . It is clear that this map is surjective. To see that  $\Phi_M^L \circ \Phi_L^G = \Phi_M^G$ , note that we may write  $g = muk$  with  $m \in M(\mathbf{F}_\bullet)$ ,  $u \in N(\mathbf{F}_\bullet)$  and  $k \in G(\mathbf{O})$ . But  $N(\mathbf{F}_\bullet) = (N(\mathbf{F}_\bullet) \cap L(\mathbf{F}_\bullet))U(\mathbf{F}_\bullet)$ , so we may write  $u = u_1u_2$ , where  $u_1 \in N(\mathbf{F}_\bullet) \cap L(\mathbf{F}_\bullet)$  and  $u_2 \in U(\mathbf{F}_\bullet)$ . Therefore, we see that  $mM(\mathbf{O}) = \Phi_M^L(mu_1) = \Phi_M^G(g)$ .  $\square$

In the sequel we denote  $\Phi_T^G$  simply by  $\Phi$ . Define

$$\Omega_G := \{g \in G(\mathbf{F}_\bullet) : |\alpha_i \circ \Phi(g)| \geq 1 \text{ for } i = 1, \dots, r\}. \quad (2)$$

PROPOSITION 14

$$G(\mathbf{F}_\bullet) = \Gamma\Omega_G.$$

*Proof.*

The rank one case (following [15]): Here  $G$  has one simple root  $\alpha_1$ , and one fundamental representation  $(\rho_1, V_1)$  and a vector  $\mathbf{v}_1 \in V_1(\mathbf{F}_q)$  such that for any element  $p$  in the parabolic subgroup  $B = TN$ , where  $N$  is the unipotent radical of  $B$ ,

$$\mathbf{v}_1 \rho_1(b) = \Delta_1(b) \mathbf{v}_1, \quad (3)$$

where the character  $\Delta_1 : B \mapsto \mathbf{G}_m$  (defined over  $\mathbf{F}_q$ ) restricts to an anti-dominant weight  $\mu_1$  on the maximal split torus  $T$ . Let  $g \in G(\mathbf{F}_\bullet)$ . We wish to show that  $g \in \Gamma \Omega_G$ . To this end, by Proposition 10, and by replacing  $g$ , if necessary by an appropriate element of  $\Gamma g$ , we may assume that  $g$  has the property that

$$\|\mathbf{v}_1 \rho_1(\gamma g)\| \geq \|\mathbf{v}_1 \rho_1(g)\| \quad \text{for all } \gamma \in \Gamma. \quad (4)$$

Write  $g = tnk$ , where  $t \in T(\mathbf{F}_\bullet)$ ,  $n \in N(\mathbf{F}_\bullet)$  and  $k \in G(\mathbf{O})$ . By Theorem 11 and Lemma 4,

$$\|\mathbf{v}_1 \rho(g)\| = |\Delta_1(t)| \|\mathbf{v}_1\| = |\mu_1(t)|. \quad (5)$$

Fix an isomorphism  $u_{\alpha_1} : \mathbf{G}_a \rightarrow N$  defined over  $\mathbf{F}_q$ , and let  $x \in \mathbf{F}_\bullet$  be such that  $n = u_{\alpha_1}(x)$ . Choose  $\sigma$  in the nontrivial  $T(\mathbf{F}_q)$ -coset of  $N_G T(\mathbf{F}_q)$ . Note that if  $S \in R$ , then  $\sigma u_{\alpha_1}(S) \in \Gamma$ , therefore, using Proposition 10,

$$\begin{aligned} |\mu_1(t)| &= \|\mathbf{v}_1 \rho_1(g)\| \\ &\leq \|\mathbf{v}_1 \rho_1(\sigma u_{\alpha_1}(S) t u_{\alpha_1}(x))\| \\ &= \|\mathbf{v}_1 \rho_1({}^\sigma t \sigma u_{\alpha_1}(\alpha_1(t)^{-1}(S + \alpha_1(t)x)))\| \\ &= |\mu_1(t)|^{-1} \|\mathbf{v}_1 \rho_1(u_{-\alpha_1}(\alpha(t)^{-1}S + x))\|. \end{aligned}$$

Here  $u_{-\alpha_1} = \sigma u_{\alpha_1} \sigma^{-1}$ , and its image is the root subgroup for  $-\alpha_1$ . The element  $u_{-\alpha_1}(\alpha(t)^{-1}S + x)$  lies in the derived group of  $G$  which is isomorphic to either  $SL_2$  or  $PGL_2$  in the rank one case. When the derived group of  $G$  is isomorphic to  $SL_2$ , we may take  $V_1$  to be the right action of  $SL_2$  on the space of  $1 \times 2$ -matrices by right multiplication. One may take the torus  $T$  to consist of diagonal matrices in  $SL_2$ ,  $B$  the upper triangular matrices in  $SL_2$  and  $\mathbf{v}_1$  to be the vector  $(0, 1)$ . Calculating with matrices, one may verify that

$$\|\mathbf{v}_1 \rho_1(u_{-\alpha_1}(\alpha(t)^{-1}S + x))\| \leq \sup\{1, |\alpha(t)^{-1}S + x|\}.$$

Therefore,

$$\sup\{1, |\alpha_1(t)^{-1}S + x|\} \geq |\mu_1(t)|^2. \quad (6)$$

Choose  $S$  in  $R$  such that  $|S + \alpha(t)x| < 1$ . Then  $|\alpha_1(t)^{-1}S + x| < |\alpha_1(t)|^{-1}$ . Suppose that  $|\alpha_1(t)^{-1}S + x| \geq |\mu_1(t)|^2$ . Then  $|\alpha_1(t)|^{-1} > |\mu_1(t)|^2$ . This is impossible, since  $\alpha_1(t)^{-1} = \mu_1(t)^2$ . It follows that  $|\alpha_1(t)^{-1}S + x| < |\mu_1(t)|^2$ . Therefore, (6) can hold only if  $1 \geq |\mu_1(t)|^2$ , which is the same as  $|\alpha_1(t)| \geq 1$ . This completes the proof of Proposition 14 when the derived group of  $G$  is isomorphic to  $SL_2$ .

When the derived group of  $G$  is isomorphic to  $PGL_2$ , then  $G$  is the product of its centre with  $PGL_2$ . Therefore, the assertion of Proposition 14 for  $G$  follows from that for  $PGL_2$ . However, the assertion for  $PGL_2$  follows easily from that for  $GL_2$ . The derived group

of  $GL_2$  is  $SL_2$ , hence the proposition holds for  $GL_2$  by the argument in the previous paragraph, completing the proof of Proposition 14 in the rank one case.

*The general case:* Let  $G$  be a group of rank  $r$ , and  $g \in G(F_\bullet)$ . By modifying  $g$  on the left by an element of  $\Gamma$ , we may, for the purposes of this proof, assume, using the second assertion of Proposition 10, that

$$\|\mathbf{v}_1 \rho_1(g)\| \leq \|\mathbf{v}_1 \rho_1(\gamma g)\| \quad \text{for all } \gamma \in \Gamma. \quad (7)$$

Note that if  $\gamma \in P_1(F_\bullet) \cap \Gamma$ , then  $\mathbf{v}_1 \rho_1(\gamma g) = \Delta_1(\gamma) \mathbf{v}_1 \rho_1(g)$ . Since  $\Delta_1(\gamma) \in \mathbf{F}_q[\pi^{-1}]^\times$ ,  $|\Delta_1(\gamma)| = 1$ . Therefore,  $\|\mathbf{v}_1 \rho_1(\gamma g)\| = \|\Delta_1(\gamma) \mathbf{v}_1 \rho_1(g)\|$ . We may use the second assertion of Proposition 10 again, to assume, for the purposes of this proof, that

$$\|\mathbf{v}_2 \rho_2(g)\| \leq \|\mathbf{v}_2 \rho_2(\gamma g)\| \quad \text{for all } \gamma \in \Gamma \cap P_1(F_\bullet) \quad (8)$$

while preserving (7). Continuing in this manner, we may assume that

$$\|\mathbf{v}_j \rho_j(g)\| \leq \|\mathbf{v}_j \rho_j(\gamma g)\| \quad \text{for all } \gamma \in \Gamma \cap P_1(F) \cap \dots \cap P_{j-1}(F), \quad (9)$$

for  $j = 1, \dots, r$ . Therefore, it suffices to prove the following:

*Lemma 22.* *If an element  $g \in G(F_\bullet)$  satisfies the inequalities (9) for each integer  $1 \leq j \leq r$ , then  $g \in \Omega_G$ .*

The proof of Proposition 14 in the rank one case shows that Lemma 22 is true when  $G$  is of semisimple rank one. We prove it in general assuming the validity of Theorem 2 in the rank one case.

Suppose that  $g$  satisfies the inequalities (9) for each  $1 \leq j \leq r$ . Write  $g = bk$ , with  $b \in B(F_\bullet)$  and  $k \in G(\mathbf{O})$ . Then  $b$  can be written as  $lu$ , where  $l \in L_{\{i\}}(F_\bullet) \cap B(F_\bullet)$  and  $u \in U_{\{i\}}(F_\bullet)$ . Since  $U_{\{i\}}$  fixes  $\mathbf{v}_i$ , the inequalities (9) imply that

$$\|\mathbf{v}_i \rho_i(l)\| \leq \|\mathbf{v}_i \rho_i(\gamma l)\| \quad \text{for all } \gamma \in L_{\{i\}}(R). \quad (10)$$

From the rank one case,  $l = \gamma \pi^\eta k$  for some  $\gamma \in L_{\{i\}}(R)$ ,  $k \in L_{\{i\}}(\mathbf{O})$  and  $\eta \in X_*(T)$  such that  $|\alpha_i(\pi^\eta)| \geq 1$ .  $\rho_i(\gamma)$  maps  $\mathbf{v}_i$  into  $V(R)$ . From Lemma 24 it follows that

$$\|\mathbf{v}_i \rho_i(l)\| \geq \|\mathbf{v}_i \rho_i(\pi^\eta)\|.$$

Equation (10) implies that the above must be an equality. This forces  $\gamma \in L_{\{i\}}(R) \cap P_i(R)$ , and hence also  $k \in L_{\{i\}}(\mathbf{O}) \cap P_i(\mathbf{O})$ . Write  $b = tn$  with  $t \in T(F_\bullet)$  and  $n \in N(F_\bullet)$ . Then viewing  $\alpha_i$  as a rational character of  $B(F_\bullet)$  that is trivial on  $N(F_\bullet)$ , we have

$$|\alpha_i(t)| = |\alpha_i(l)| = |\alpha_i(\pi^\eta)| \geq 1.$$

Repeating this argument for each  $i$  completes the proof of Lemma 22. □

### 5. Local reduction theory

In order to prove the existence part of Theorem 2, it suffices to show that every element  $g$  in  $\Omega_G$  may be written as  $g = \gamma \pi^\eta k$ , where  $\gamma \in \Gamma$ ,  $\eta \in X_*(T)$  is antidominant and  $k \in G(\mathbf{O})$ . To this end, we may assume (using the Iwasawa decomposition) that we are

given  $g \in \Omega_G$ , with  $g = tn$ , with  $t \in T(F_\bullet)$  and  $n \in N(F_\bullet)$ . Since  $g$ , and hence  $t$ , is in  $\Omega_G$ ,  $|\alpha_i(t)| \geq 1$ , so that  $\alpha_i(t)^{-1} \in \mathbf{O}$ , for  $i = 1, \dots, r$ . For each root  $\alpha \in \Phi(G, T)$ , let  $U_\alpha$  denote the corresponding root subgroup. Fix an isomorphism  $u_\alpha : \mathbf{G}_a \rightarrow U_\alpha$  defined over  $\mathbf{F}_q$ . Then for  $x \in F_\bullet$ , we have

$$tu_\alpha(x) = (tu_\alpha(x)t^{-1})t = u_\alpha(\alpha(t)x)t.$$

Therefore, if we write  $\alpha(t)x = P + h$ , where  $P \in R$  and  $h \in \mathbf{O}$ , then

$$tu_\alpha(x) = tu_\alpha(\alpha(t)^{-1}P)u_\alpha(\alpha(t)^{-1}h) = u_\alpha(P)tu_\alpha(\alpha(t)^{-1}h).$$

Given two positive roots  $\alpha$  and  $\beta$ , the commutator  $[U_\alpha, U_\beta]$  is contained in the product of root subgroups  $U_{\alpha'}$  where the  $\alpha'$  are roots which can be written as positive linear combinations of  $\alpha$  and  $\beta$  and are distinct from either  $\alpha$  or  $\beta$ . Moreover, we may enumerate the positive roots as  $\beta_1, \beta_2, \dots$  so that if  $j > i$ , then  $\beta_i$  cannot be written as a sum of  $\beta_j$  and any other positive roots.

Write  $n$  as  $\prod_i u_{\beta_i}(x_i)$ . Then

$$tn = tu_{\beta_1}(x_1) \prod_{i>1} u_{\beta_i}(x_i).$$

If we write  $\beta_1(t)x_1 = P_1 + h_1$ , where  $P_1 \in \mathbf{F}_q[\pi^{-1}]$  and  $h \in \mathbf{O}$ , then

$$tn = u_{\beta_1}(P_1)tu_{\beta_1}(\beta_1(t)^{-1}h_1) \prod_{i>1} u_{\beta_i}(x_i).$$

Since  $u_{\beta_1}(P_1) \in \Gamma$ ,  $\beta_1(t)^{-1} \in \mathbf{O}$ , and the image of  $u_{\beta_1}$  normalizes all the subsequent root subgroups whose elements appear in the above expression, we may assume for the purpose of proving Theorem 2, that

$$tn = t \prod_{i>1} u_{\beta_i}(x'_i),$$

for  $x'_i \in F_\bullet$ . We may continue in this manner to reduce  $tn$  to  $t$ . It is then easy to see (using the decomposition  $F_\bullet^\times = \pi^{\mathbf{Z}}\mathbf{O}^\times$ ) that  $t$  may be replaced by  $\pi^\eta$  for  $\eta \in X_*(T)$ . Since  $|\alpha_i(\pi^\eta)| \geq 1$ , it follows that  $\eta$  is antidominant, proving the existence part of Theorem 2.

We now prove the uniqueness part of Theorem 2. In order to do this, it suffices to show that if  $\eta$  and  $\nu$  are two dominant co-weights, and  $\pi^\nu = \gamma\pi^\eta k$  for some  $\gamma \in \Gamma$  and  $k \in G(\mathbf{O})$ , then  $\nu = \eta$ . Since the weights  $\mu_1, \dots, \mu_r$  corresponding to the fundamental representations in Theorem 11 generate the vector space  $X^*(T) \otimes \mathbf{Q}$ , it suffices to show that  $\langle \mu_i, \nu \rangle = \langle \mu_i, \eta \rangle$  for each  $i$ . In order to do this, we need the following:

*Lemma 24.* For any non-zero vector  $\mathbf{v} \in V_i(F_\bullet)$  and any antidominant co-weight  $\mu \in X_*(T)$ ,

$$\frac{\|\mathbf{v}\rho_i(\pi^\mu)\|}{\|\mathbf{v}\|} \geq \frac{\|\mathbf{v}_i\rho_i(\pi^\mu)\|}{\|\mathbf{v}_i\|}.$$

*Proof.* Since  $T$  is an  $\mathbf{F}_q$ -split torus and  $\rho_i$  is defined over  $\mathbf{F}_q$ ,  $V$  has a decomposition (over  $\mathbf{F}_q$ ) into root subspaces

$$V = \bigoplus_{\lambda} V_{\lambda},$$

where  $T$  acts on  $V_\lambda$  by the character  $\lambda : T \rightarrow \mathbf{G}_m$ . It is easy to see that  $\mu_i$  is the lowest weight of  $T$  occurring in  $(\rho_i, V_i)$ , so that  $\langle \mu_i, \mu \rangle \geq \langle \lambda, \mu \rangle$  for any weight  $\lambda$  of  $T$  occurring in  $(\rho_i, V_i)$  and any antidominant co-weight  $\mu$ . Given any vector  $\mathbf{v} \in V(F_\bullet)$ , we may write

$$\mathbf{v} = \sum x_j \mathbf{u}_j,$$

where  $x_j \in F_\bullet$  and  $\mathbf{u}_j \in V_{\lambda_j}(F_q)$  for each  $j$  and the  $\lambda_j$ s are not necessarily distinct. Thus

$$\begin{aligned} \|\mathbf{v} \rho_i(\pi^\mu)\| &= \left\| \sum \lambda_j(\pi^\mu) x_j \mathbf{u}_j \right\| \\ &= \sup_j \{ |\lambda_j(\pi^\mu) x_j| \} \\ &= \sup_j \{ q^{-\langle \lambda_j, \mu \rangle} |x_j| \} \\ &\geq q^{-\langle \mu_i, \mu \rangle} \sup_j \{ |x_j| \} \\ &= \|\mathbf{v}_i \rho_i(\pi^\mu)\| \|\mathbf{v}\|. \end{aligned}$$

Since  $\|\mathbf{v}_i\| = 1$ , this completes the proof of Lemma 24.  $\square$

Lemma 24 allows us to compare  $\langle \mu_i, \nu \rangle$  and  $\langle \mu_i, \eta \rangle$ :

$$\begin{aligned} q^{-\langle \mu_i, \eta \rangle} &= \frac{\|\mathbf{v}_i \rho_i(\pi^\eta)\|}{\|\mathbf{v}_i\|} \\ &\leq \frac{\|\mathbf{v}_i \rho_i(\gamma \pi^\eta)\|}{\|\mathbf{v}_i \rho_i(\gamma)\|} \\ &\leq \frac{\|\mathbf{v}_i \rho_i(\gamma \pi^\eta)\|}{\|\mathbf{v}_i\|} \\ &= \frac{\|\mathbf{v}_i \rho_i(\pi^\nu)\|}{\|\mathbf{v}_i\|} \\ &= q^{-\langle \mu_i, \nu \rangle}. \end{aligned}$$

The first inequality is Lemma 24 applied to  $\mathbf{v} = \mathbf{v}_i \rho_i(\gamma)$ . The second inequality follows from Lemma 9 with  $\mathbf{x} = \mathbf{v}_i \rho_i(\gamma)$ . Interchanging the roles of  $\eta$  and  $\nu$  in the above arguments shows that  $\langle \mu_i, \eta \rangle = \langle \mu_i, \nu \rangle$  for each  $i$ . This completes the proof of the uniqueness part of the assertion of Theorem 2.

## 6. Global reduction theory

If  $g = (g_v)_v$  is an element of  $G(\mathbf{A})$  then, since  $g_v \in G(\mathbf{O}_v)$  for all but finitely many places  $v$  of  $F$ , we may assume, for the purpose of proving Theorem 1 that  $g$  is a finite product  $g = g_\infty g_{v_1} g_{v_2} \cdots g_{v_k}$ , with  $g_\infty \in G(F_\infty)$  and  $g_{v_j} \in G(F_{v_j})$ ,  $v_j \neq \infty$ , for  $1 \leq j \leq k$ . By Theorem 2, there is a decomposition

$$g_{v_k} = \gamma_k \pi_{v_k}^{\eta_k} \kappa_k,$$

where  $\gamma_k \in G(\mathbf{F}_q[\pi_{v_k}^{-1}])$ ,  $\eta_k \in X_*(T)$ , and  $\kappa_k \in G(\mathbf{O}_{v_k})$ . Now  $\gamma_k$  and  $\pi_{v_k}^{\eta_k}$  are both contained in  $G(F)$  and in  $G(\mathbf{O}_v)$  for all  $v \neq \infty$ . Therefore, by multiplying  $g$  on the left by  $\pi_{v_k}^{-\eta_k} \gamma^{-1}$  we get an element of the subset

$$G(F_\infty) \times \prod_{j=1}^{k-1} G(F_{v_j}) \times \prod_{\text{all other } v} G(\mathbf{O}_v)$$

of  $G(\mathbf{A})$ .

We have now reduced  $g$  to an element with non-trivial entries only at most  $k - 1$  places and  $\infty$ . We may continue in this manner until the entries at all places except  $\infty$  are trivial. Finally, the use of Theorem 2 to  $v = \infty$  gives us a representative each double coset of type asserted by Theorem 1.

The uniqueness part of the theorem follows from the corresponding assertion in the local situation, because two elements  $g$  and  $h$  of  $G(F_\infty)$  lie in the same double coset if and only if  $g = \gamma h k$ , with  $\gamma \in G(\mathbf{F}_q[t])$  and  $k \in G(\mathbf{O}_\infty)$ .

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