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On the local Artin conductor $\mathfrak{f}_{\mathrm{Artin}}(\chi)$ of a character χ of Gal(E/K) – II: Main results for the metabelian case

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Abstract. This paper which is a continuation of [2], is essentially expository in nature, although some new results are presented. Let K be a local field with finite residue class field κ_K. We first define (cf. Definition 2.4) the conductor $\mathfrak{f}(E/K)$ of an arbitrary finite Galois extension E/K in the sense of non-abelian local class field theory as

$$\mathfrak{f}(E/K) = \mathfrak{p}_K^{[[n_G]]+1},$$

where n_G is the break in the upper ramification filtration of $G = \mathrm{Gal}(E/K)$ defined by $G^{n_G} \neq G^{n_G+\delta} = 1, \forall \delta \in \mathbb{R}_{\gneq 0}$. Next, we study the basic properties of the ideal $\mathfrak{f}(E/K)$ in O_K in case E/K is a metabelian extension utilizing Koch–de Shalit metabelian local class field theory (cf. [8]).

After reviewing the Artin character $a_G : G \to \mathbb{C}$ of $G := \mathrm{Gal}(E/K)$ and Artin representations $A_G : G \to GL(V)$ corresponding to $a_G : G \to \mathbb{C}$, we prove that (Proposition 3.2 and Corollary 3.5)

$$\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho) = \mathfrak{p}_K^{\dim_{\mathbb{C}}(V)[n_{G/\ker(\rho)}+1]},$$

where $\chi_\rho : G \to \mathbb{C}$ is the character associated to an irreducible representation $\rho : G \to GL(V)$ of G (over $\mathbb{C}$). The first main result (Theorem 1.2) of the paper states that, if in particular, $\rho : G \to GL(V)$ is an irreducible representation of G (over $\mathbb{C}$) with metabelian image, then

$$\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho) = \mathfrak{p}_K^{[E^{\ker(\rho)^\bullet}:K](n_{G/\ker(\rho)}+1)},$$

where $\mathrm{Gal}(E^{\ker(\rho)}/E^{\ker(\rho)^\bullet})$ is any maximal abelian normal subgroup of $\mathrm{Gal}(E^{\ker(\rho)}/K)$ containing $\mathrm{Gal}(E^{\ker(\rho)}/K)'$, and the break $n_{G/\ker(\rho)}$ in the upper ramification filtration of $G/\ker(\rho)$ can be computed and located by metabelian local class field theory. The proof utilizes Basmaji's theory on the structure of irreducible faithful representations of finite metabelian groups (cf. [1]) and on metabelian local class field theory (cf. [8]).

We then discuss the application of Theorem 1.2 on a problem posed by Weil on the construction of a 'natural' A_G of G over $\mathbb{C}$ (Problem 1.3). More precisely, we prove in Theorem 1.4 that if E/K is a metabelian extension with Galois group G, then

$$A_G \simeq \sum_N [(E^N)^\bullet : K](n_{G/N}+1) \times \sum_{[\omega]_\sim \in \mathcal{V}_N} \mathrm{Ind}^G_{\pi_N^{-1}((G/N)^\bullet)} \left(\omega \circ \pi_N \mid_{\pi_N^{-1}((G/N)^\bullet)}\right),$$

where N runs over all normal subgroups of G, and for such an N, $\mathcal{V}_N$ denotes the collection of all $\sim$-equivalence classes $[\omega]_\sim$, where '$\sim$' denotes the equivalence relation on the set of all representations $\omega : (G/N)^\bullet \to \mathbb{C}^\times$ satisfying the conditions

$$\mathrm{Inert}(\omega) = \{\delta \in G/N : \omega_\delta = \omega\} = (G/N)^\bullet$$

and

$$\bigcap_\delta \ker(\omega_\delta) = \langle 1_{G/N} \rangle,$$

where δ runs over $\mathcal{R}((G/N)^\bullet \backslash (G/N))$, a fixed given complete system of representatives of $(G/N)^\bullet \backslash (G/N)$, by declaring that $\omega_1 \sim \omega_2$ if and only if $\omega_1 = \omega_{2,\delta}$ for some $\delta \in \mathcal{R}((G/N)^\bullet \backslash (G/N))$.

Finally, we conclude our paper with certain remarks on Problem 1.1 and Problem 1.3.

Keywords. Local fields; higher-ramification groups; local Artin conductor; metabelian local class field theory; non-abelian local class field theory; local Langlands correspondence for $GL(n)$.

1. Introduction

This paper is the natural continuation of [2]. However, for the sake of completeness, we have included all the necessary results from our previous companion article, so that one can directly study this work without assuming any background material from [2].

Let K be a local field with finite residue class field $O_K/\mathfrak{p}_K =: \kappa_K$ of q_K elements, where as usual, O_K stands for the ring of integers in K with the unique maximal ideal $\mathfrak{p}_K$. Let $\boldsymbol{\nu} : K \to \mathbb{Z} \cup \{\infty\}$ denote the corresponding normalized exponential valuation on K (normalized by $\boldsymbol{\nu}(K^\times) = \mathbb{Z}$), and $\widetilde{\boldsymbol{\nu}}$ the unique extension of $\boldsymbol{\nu}$ to a fixed seperable closure K^{sep} of K. For any sub-extension L/K in K^{sep}/K, let $\widetilde{\boldsymbol{\nu}}_L$ be the normalized form of the valuation $\widetilde{\boldsymbol{\nu}}\,|_L$ on L.

Let E be a finite Galois extension over K, and $\rho : \mathrm{Gal}(E/K) \to GL(V)$ an irreducible and finite-dimensional representation (over $\mathbb{C}$) of the Galois group $\mathrm{Gal}(E/K)$ of the extension E/K. Let $E^{\ker(\rho)} = \{x \in E : \sigma(x) = x, \forall \sigma \in \ker(\rho)\}$ be the fixed-field of $\ker(\rho)$, and for the time being, assume that $E^{\ker(\rho)}$ is an abelian extension over K (equivalently, $\mathrm{im}(\rho)$ is an abelian subgroup of $GL(V)$ by the fundamental Galois duality). Observe that, under this assumption, the representation $\rho : \mathrm{Gal}(E/K) \to GL(V)$ is one-dimensional. In fact, the representation (ρ, V) of $\mathrm{Gal}(E/K)$ factors through

$$\begin{tikzcd}
\mathrm{Gal}(E/K) \arrow[rr, "\rho"] \arrow[dr, "\pi"'] & & GL(V) \\
& \mathrm{Gal}(E/K)/\ker(\rho) \arrow[ur, "\widetilde{\rho}"'] &
\end{tikzcd}$$

with an isomorphism $\mathrm{Gal}(E/K)/\ker(\rho) \xrightarrow{\sim} \mathrm{Gal}(E^{\ker(\rho)}/K)$ defined by the fundamental Galois duality, where $\pi : \mathrm{Gal}(E/K) \to \mathrm{Gal}(E/K)/\ker(\rho)$ is the canonical mapping, and $\widetilde{\rho} : \mathrm{Gal}(E/K)/\ker(\rho) \to GL(V)$ is defined by $\widetilde{\rho}(\sigma\ker(\rho)) = \rho(\sigma)$ for every $\sigma \in \mathrm{Gal}(E/K)$. Note that, the homomorphism $\widetilde{\rho} : \mathrm{Gal}(E/K)/\ker(\rho) \to GL(V)$ is an irreducible representation of the group $\mathrm{Gal}(E/K)/\ker(\rho)$ in V over $\mathbb{C}$, as (ρ, V) is an irreducible representation of $\mathrm{Gal}(E/K)$. Recall that, $E^{\ker(\rho)}/K$ is assumed to be an abelian extension. Thus, $\dim_{\mathbb{C}}(V_{\widetilde{\rho}}) = 1$, since an irreducible representation of an abelian group should be one-dimensional. Hence, (ρ, V) is a one-dimensional representation of $\mathrm{Gal}(E/K)$, as the representation spaces V_ρ and $V_{\widetilde{\rho}}$ coincide.

It is then well-known that (cf. Proposition 11.6 of chapter VII of [9], or §3)

$$\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho) = \mathfrak{f}(E^{\ker(\rho)}/K), \tag{1}$$

where $\mathfrak{f}(E^{\ker(\rho)}/K)$ is the conductor of the abelian extension $E^{\ker(\rho)}/K$ defined in the sense of abelian local class field theory (which will be reviewed in the next section), and $\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho)$ is the local Artin conductor of the character $\chi_\rho : \mathrm{Gal}(E/K) \to \mathbb{C}$ associated to the representation (ρ, V) of $\mathrm{Gal}(E/K)$ (which will be reviewed in §3).

This identity is very important. In fact, we note that, this identity apart from its own beauty, plays the key role in proving the Artin conjecture in the abelian case of Artin L-functions and in establishing the functional equation satisfied by Artin L-functions.

It is then natural to ask, more generally, the following problem:

Problem 1.1. How can we describe class field theoretically the local Artin conductor $\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho)$ of the character $\chi_\rho : \mathrm{Gal}(E/K) \to \mathbb{C}$ associated to an irreducible representation (ρ, V) of the Galois group $\mathrm{Gal}(E/K)$ of the extension E/K without imposing any condition on the representation (ρ, V) of $\mathrm{Gal}(E/K)$?

Of course, in this generality, full solution of Problem 1.1 does not seem to be possible for the time being, but utilizing a result of Henniart and Tunnell, we can partially answer Problem 1.1. That is, we can describe the local Artin conductor $\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho)$ of the character $\chi_\rho : \mathrm{Gal}(E/K) \to \mathbb{C}$ associated to an irreducible representation $\rho : \mathrm{Gal}(E/K) \to GL(V)$ in terms of the ‘conductor’ $\mathfrak{f}(E^{\ker(\rho)}/K)$ of the Galois extension $E^{\ker(\rho)}/K$ (defined in §2 as Definition 2.4 following an idea of Sen in [10]) as

$$\left[\left[\frac{1}{\dim_{\mathbb{C}}(V)}\mathrm{ord}_{\mathfrak{p}_K}(\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho))\right]\right] = \mathrm{ord}_{\mathfrak{p}_K}\left(\mathfrak{f}(E^{\ker(\rho)}/K)\right),$$

where, for $x \in \mathbb{R}$, $[[x]]$ denotes the integer part of the number x (cf. Corollary 3.5). This equation, however, has two gaps: First of all, the above equation does not give any explicit information about the dimension of the irreducible representation (ρ, V) of $\mathrm{Gal}(E/K)$ in terms of the extension E/K. What is also missing in this equation is the class field theoretic information hidden in the ‘conductor’ $\mathfrak{f}(E^{\ker(\rho)}/K)$ of the Galois extension $E^{\ker(\rho)}/K$. In fact, the definition of $\mathfrak{f}(E^{\ker(\rho)}/K)$ involves the integer part of a certain break $n_{\mathrm{Gal}(E/K)/\ker(\rho)}$ of the upper ramification filtration of the Galois group $\mathrm{Gal}(E^{\ker(\rho)}/K)$ (cf. Definition 2.4). Since $E^{\ker(\rho)}/K$ is an n-abelian extension (cf. §4) for some $1 \leq n \in \mathbb{Z}$, in order to compute and locate $n_{\mathrm{Gal}(E/K)/\ker(\rho)}$, we need Hasse–Arf-type theorems for n-abelian extensions, for every $1 \leq n \in \mathbb{Z}$. If, on the other hand, the representation (ρ, V) of $\mathrm{Gal}(E/K)$ has metabelian image, that is, if the image $\mathrm{im}(\rho) \subset GL(V)$ of the representation (ρ, V) of $\mathrm{Gal}(E/K)$ is a metabelian group (i.e., the 2nd-commutator subgroup $\mathrm{im}(\rho)^{(2)}$ of $\mathrm{im}(\rho)$ is trivial, or equivalently, if $E^{\ker(\rho)}/K$ is a metabelian extension); utilizing Koch–de Shalit theory (cf. [8]) and the representation theory of finite metabelian groups developed by Basmaji (cf. [1]), it is possible to fill these two gaps, and hence it is possible to describe $\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho)$ in terms of the ‘conductor’ $\mathfrak{f}(E^{\ker(\rho)}/K)$ of the metabelian extension $E^{\ker(\rho)}/K$ defined in the sense of Koch–de Shalit metabelian local class field theory (which is the specialization of Definition 2.4 to metabelian extensions and which is defined explicitly in §5).

More precisely, the first main result of this paper is the following:

Theorem 1.2. *Let* E *be a finite Galois extension over* K, *and* $\rho : \mathrm{Gal}(E/K) \to GL(V)$ *be an irreducible and finite-dimensional representation* (*over* $\mathbb{C}$) *of the Galois group* $\mathrm{Gal}(E/K)$ *of the extension* E/K.

If $E^{\ker(\rho)}/K$ *is a metabelian extension, then the Artin conductor* $\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho)$ *is given explicitly by*

$$\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho) = \mathfrak{p}_K^{[E^{\ker(\rho)^\bullet}:K](n_{\mathrm{Gal}(E^{\ker(\rho)}/K)}+1)}, \tag{2}$$

where $n_{\mathrm{Gal}(E^{\ker(\rho)}/K)} =: n$ *is the break in the upper ramification filtration of* $\mathrm{Gal}(E^{\ker(\rho)}/K)$ *defined by* $\mathrm{Gal}(E^{\ker(\rho)}/K)^n \neq \mathrm{Gal}(E^{\ker(\rho)}/K)^{n+\delta} = 1$, $\forall \delta \in \mathbb{R}_{>0}$, *which can be located by Koch–de Shalit metabelian local class field theory, and* $\mathrm{Gal}(E^{\ker(\rho)}/E^{\ker(\rho)^\bullet})$ *is any maximal abelian normal subgroup of* $\mathrm{Gal}(E^{\ker(\rho)}/K)$ *containing* $\mathrm{Gal}(E^{\ker(\rho)}/K)'$.

In order to prove this theorem, we have organized our paper as follows: Sections 2 and 3 are dedicated to a quick review of the higher ramification groups and the Artin representation $(A_{\mathrm{Gal}(E/K)}, V_{A_{\mathrm{Gal}(E/K)}})$ of the Galois group $\mathrm{Gal}(E/K)$ of the extension E/K. In particular, following an idea of Sen in [10], we introduce the conductor $\mathfrak{f}(E/K)$ of any finite Galois extension E/K in Definition 2.4. We then review, in §4, the metabelian local class field theory following [8]. In §5, we give the interpretation of Definition 2.4 in the case of a finite metabelian (= 2-abelian) extension E/K in the sense of metabelian local class field theory, following the main theorem of higher-ramification theory in metabelian local class field theory (cf. Lemma 5.5), and in Proposition 5.6, again assuming E/K is finite metabelian, give an explicit description of the exponent of the conductor $\mathfrak{f}(E/K)$ in terms of the exponent of the conductor $\mathfrak{f}(E/E^\bullet)$, where $G^\bullet = \mathrm{Gal}(E/E^\bullet)$ is any maximal abelian normal subgroup of $G = \mathrm{Gal}(E/K)$ containing G'. In §6, we compute the Artin conductor $\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho)$ of the character $\chi_\rho : \mathrm{Gal}(E/K) \to \mathbb{C}$ associated to a given irreducible representation (ρ, V) of $\mathrm{Gal}(E/K)$ under the assumption that $E^{\ker(\rho)}/K$ is a 2-abelian extension using Basmaji's theory (cf. [1]) and the basic properties of Artin conductors, and get eq. (2), which completes the proof of Theorem 1.2. In §7, as an application of Theorem 1.2, we partially answer the following open problem posed by Weil.

Problem 1.3 (Weil). Determine a 'natural' $A_{\mathrm{Gal}(E/K)}$ of $\mathrm{Gal}(E/K)$ over $\mathbb{C}$.

In fact, we prove the following theorem in §7.

Theorem 1.4. *Let* E *be a finite metabelian extension over* K. *For any* $N \lhd \mathrm{Gal}(E/K)$, *let* $\mathrm{Gal}(E^N/K)^\bullet = \mathrm{Gal}(E^N/(E^N)^\bullet)$ *denote a maximal abelian normal subgroup of* $\mathrm{Gal}(E^N/K)$ *containing the* 1*st commutator subgroup* $\mathrm{Gal}(E^N/K)'$ *of* $\mathrm{Gal}(E^N/K)$. *Let* $\mathcal{I}(\mathrm{Gal}(E^N/K)^\bullet)^\circ$ *denote the set of all one-dimensional representations*

$$\omega : \mathrm{Gal}(E^N/K)^\bullet \to \mathbb{C}^\times$$

satisfying

$$\mathrm{Inert}(\omega) = \mathrm{Gal}(E^N/K)^\bullet$$

and

$$\bigcap_{\delta \in \mathrm{Gal}((E^N)^\bullet/K)} \ker(\omega_\delta) = \langle 1_{\mathrm{Gal}(E^N/K)} \rangle,$$

where $\omega_\delta : \mathrm{Gal}(E^N/K)^\bullet \to \mathbb{C}^\times$ *is the representation defined by* $\omega_\delta : x \mapsto \omega(\delta x \delta^{-1})$ *for* $x \in \mathrm{Gal}(E^N/K)^\bullet$ *and* $\delta \in \mathrm{Gal}((E^N)^\bullet/K) \xrightarrow{\sim} \mathrm{Gal}(E^N/K)/\mathrm{Gal}(E^N/K)^\bullet$. *Now, define an equivalence relation '$\sim$' on* $\mathcal{I}(\mathrm{Gal}(E^N/K)^\bullet)^\mathrm{o}$ *by declaring* $\omega_1 \sim \omega_2$ *in case* $\omega_1 = \omega_{2,\delta}$ *for some* $\delta \in \mathrm{Gal}((E^N)^\bullet/K)$. *Let* $\mathcal{V}_N$ *denote the set of* $\sim$*-equivalence classes* $[\mathcal{I}(\mathrm{Gal}(E^N/K)^\bullet)^\mathrm{o}/\sim]$.

Let

$$A_{\mathrm{Gal}(E/K)} : \mathrm{Gal}(E/K) \to GL(V_{A_{\mathrm{Gal}(E/K)}})$$

be an Artin representation of $\mathrm{Gal}(E/K)$ *over* $\mathbb{C}$. *Then,*

$$A_{\mathrm{Gal}(E/K)} \simeq \sum_N [(E^N)^\bullet : K](n_{\mathrm{Gal}(E^N/K)} + 1) \\ \times \sum_{[\omega]_\sim \in \mathcal{V}_N} \mathrm{Ind}^G_{\pi_N^{-1}(\mathrm{Gal}(E^N/K)^\bullet)} \left(\omega \circ \pi_N \mid_{\pi_N^{-1}(\mathrm{Gal}(E^N/K)^\bullet)}\right),$$

where N *runs over all normal subgroups of* $\mathrm{Gal}(E/K)$. *Here,* $n_{\mathrm{Gal}(E^N/K)}$ *is the number defined in Theorem* 1.2. *Hence it can be computed by metabelian local class field theory*; $\pi_N : \mathrm{Gal}(E/K) \to \mathrm{Gal}(E^N/K)$ *is the canonical mapping, and* $[\omega]_\sim$ *runs over all* $\mathcal{V}_N$, *where* $[\omega]_\sim$ *denotes the* $\sim$*-equivalence class of* $\omega \in \mathcal{I}(\mathrm{Gal}(E^N/K)^\bullet)^\mathrm{o}$.

The proof utilizes Theorem 1.2 and Basmaji's theory. Note that, if E/K is in particular an abelian extension in Theorem 1.4, then we get the well-known identity that $A_{\mathrm{Gal}(E/K)} \simeq \sum_\chi (n_{\mathrm{Gal}(E^{\ker(\chi)}/K)} + 1)\chi$, where $\chi : \mathrm{Gal}(E/K) \to \mathbb{C}^\times$ runs over all one-dimensional representations of G over $\mathbb{C}$.

Finally, in §8, we conclude our paper with a set of complementary remarks on Problem 1.1 and Problem 1.3.

2. Review of higher-ramification groups

Main references for this section are [7, 9, 10, 12].

For a finite seperable extension L/K, and for any $\sigma \in \mathrm{Hom}_K(L, K^{\mathrm{sep}})$, introduce

$$i_{L/K}(\sigma) := \min_{y \in O_L} \{\widetilde{\boldsymbol{\nu}}_L(\sigma(y) - y)\},$$

put

$$\gamma_t := \#\left\{\sigma \in \mathrm{Hom}_K(L, K^{\mathrm{sep}}) : i_{L/K}(\sigma) \geq t + 1\right\}$$

for $-1 \leq t \in \mathbb{R}$, and define the function $\varphi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ (the Hasse–Herbrand transition function of the extension L/K) by

$$\varphi_{L/K}(u) := \begin{cases} \int_0^u \frac{\gamma_t}{\gamma_0} \mathrm{d}t, & 0 \leq u \in \mathbb{R}, \\ u, & -1 \leq u \leq 0. \end{cases}$$

It is well-known that, $\varphi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ is a monotone-increasing and piecewise-linear function, and induces a homeomorphism $\mathbb{R}_{\geq -1} \xrightarrow{\approx} \mathbb{R}_{\geq -1}$. Let $\psi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ be the mapping which is inverse to the function $\varphi_{L/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$.

From now on, unless otherwise stated, assume that E is a non-trivial finite Galois extension over K with Galois group $\mathrm{Gal}(E/K) =: G$. Under this assumption, O_E is monogenic over O_K, that is, $O_E = O_K[x]$ for some $x \in O_E$ (cf. Lemma 10.4 of chapter II of [9]). Therefore

$$i_{E/K}(\sigma) = \widetilde{\boldsymbol{\nu}}_E\,(\sigma(x) - x)$$

for every $\sigma \in G$. In fact, it suffices to prove that, for any $\sigma \in \mathrm{Gal}(E/K)$,

$$\widetilde{\boldsymbol{\nu}}_E\,(\sigma(x) - x) \leq \widetilde{\boldsymbol{\nu}}_E\,(\sigma(y) - y)$$

for every $y \in O_E$. Now, let $y = \sum_{i=0}^{s} \alpha_i x^i$ with $\alpha_i \in O_K$ for $i = 0, \ldots, s$. Then

$$\sigma(y) - y = \sigma\left(\sum_{i=0}^{s} \alpha_i x^i\right) - \sum_{i=0}^{s} \alpha_i x^i = \sum_{i=1}^{s} \alpha_i (\sigma(x)^i - x^i),$$

and as $\sigma(x)^i - x^i = (\sigma(x) - x) \sum_{\substack{0 \leq j,k \in \mathbb{Z} \\ j+k=i-1}} \sigma(x)^j x^k$, for $1 \leq i \in \mathbb{Z}$, it follows that

$$\begin{aligned} \sigma(y) - y &= \sum_{i=1}^{s} \alpha_i \left[(\sigma(x) - x) \sum_{\substack{0 \leq j,k \in \mathbb{Z} \\ j+k=i-1}} \sigma(x)^j x^k \right] \\ &= (\sigma(x) - x)\, z \end{aligned}$$

for some $z \in O_E$, which proves the desired inequality.

Remark 2.1. Note that, $i_{E/K} : \mathrm{Gal}(E/K) \to \mathbb{Z} \cup \{\infty\}$ does not depend on the choice of the generator $x \in O_E$ over O_K. □

The subgroup G_u of G defined by

$$G_u = \{\sigma \in G : \; i_{E/K}(\sigma) \geq u + 1\}$$

for $-1 \leq u \in \mathbb{R}$ is called the uth ramification group of G (in the lower numbering), and has order γ_u. The family $\{G_u\}_{u \in \mathbb{R}_{\geq -1}}$ induces a filtration on G, called the lower ramification filtration of G. A break in the filtration $\{G_u\}_{u \in \mathbb{R}_{\geq -1}}$ of G is defined to be any number $u \in \mathbb{R}_{\geq -1}$ satisfying $G_u \neq G_{u+\varepsilon}$ for every $0 < \varepsilon \in \mathbb{R}$. Observe that

$$G_{j+\varepsilon} = G_{j+1} \tag{3}$$

for $-1 \leq j \in \mathbb{Z}$ and $0 < \varepsilon \leq 1$. In fact

$$\begin{aligned} G_{j+\varepsilon} &= \{\sigma \in G \mid i_{E/K}(\sigma) \geq (j + \varepsilon) + 1\} \\ &= \{\sigma \in G \mid i_{E/K}(\sigma) \geq j + 2\} = G_{j+1} \end{aligned}$$

for $-1 \leq j \in \mathbb{Z}$ and $0 < \varepsilon \leq 1$, since $i_{E/K}(\sigma) \in \mathbb{Z} \cup \{\infty\}$ for every $\sigma \in G$. It is also well-known that, for $0 \leq j \in \mathbb{Z}$, there exists an injection

$$G_j/G_{j+1} \hookrightarrow U^j(E)/U^{j+1}(E) \tag{4}$$

defined by

$$\sigma \mapsto \frac{\sigma(\pi_E)}{\pi_E}$$

for $\sigma \in G_j$, where π_E is a fixed prime element in E. This embedding does not depend on the choice of the prime element π_E in E.

In this case, Hasse–Herbrand transition function $\varphi_{E/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ of the Galois extension E/K is defined by

$$\varphi_{E/K}(u) = \begin{cases} \int_0^u \frac{1}{(G_0:G_t)}\mathrm{d}t, & 0 \leq u \in \mathbb{R}, \\ u, & -1 \leq u \leq 0. \end{cases}$$

Note that, for $0 < u \in \mathbb{R}$,

$$\varphi_{E/K}(u) = \frac{1}{\gamma_0}\left[\sum_{i=1}^{m} \gamma_i + (u-m)\gamma_{m+1}\right], \tag{5}$$

where $m \leq u \leq m+1$ for a unique $0 < m \in \mathbb{Z}$, by eq. (3).

The function $\psi_{E/K} = \varphi_{E/K}^{-1} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ induces the upper ramification filtration $\{G^v\}_{v\in\mathbb{R}_{\geq -1}}$ on G by setting

$$G^v := G_{\psi_{E/K}(v)},$$

or equivalently,

$$G^{\varphi_{E/K}(u)} = G_u$$

for $-1 \leq v, u \in \mathbb{R}$, where G^v is called the vth upper ramification group of G. A break in the upper filtration $\{G^v\}_{v\in\mathbb{R}_{\geq -1}}$ of G is defined to be any number $v \in \mathbb{R}_{\geq -1}$ satisfying $G^v \neq G^{v+\delta}$ for every $0 < \delta \in \mathbb{R}$.

Remark 2.2. Observe that

(i) If $u_0 \in \mathbb{R}_{\geq -1}$ is not a break in the lower ramification filtration $\{G_u\}_{u\in\mathbb{R}_{\geq -1}}$ of G, that is, if $G_{u_0} = G_{u_0+\varepsilon_0}$ for some $0 < \varepsilon_0 \in \mathbb{R}$; then $G_{u_0} = G_{u_0+\varepsilon}$ for every $0 \leq \varepsilon \leq \varepsilon_0$. Since $\varphi_{E/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ is a monotone-increasing and piecewise-linear function, $\varphi_{E/K}([u, u+\varepsilon_0]) = [v_0, v_0+\delta_0]$, where $\varphi_{E/K}(u_0) = v_0$ and $\varphi_{E/K}(u_0+\varepsilon_0) = v_0+\delta_0$ for some $0 < \delta_0 \in \mathbb{R}$, and therefore $G^{v_0} = G^{v_0+\delta}$ for every $0 \leq \delta \leq \delta_0$. So, $\varphi_{E/K}(u_0) = v_0$ is not a break in the upper ramification filtration $\{G^v\}_{v\in\mathbb{R}_{\geq -1}}$ of G.

(ii) Note that, following the same lines of reasoning, the converse of (i) is also true: That is, if $v_0 \in \mathbb{R}_{\geq -1}$ is not a break in the upper ramification filtration $\{G^v\}_{v\in\mathbb{R}_{\geq -1}}$ of G, then $\psi_{E/K}(v_0) \in \mathbb{R}_{\geq -1}$ is not a break in the lower ramification filtration $\{G_u\}_{u\in\mathbb{R}_{\geq -1}}$ of G. Thus, combining (i) and (ii):

(iii) $v_0 = \varphi_{E/K}(u_0) \in \mathbb{R}_{\geq -1}$ is a break in the upper ramification filtration $\{G^v\}_{v \in \mathbb{R}_{\geq -1}}$ of G iff $\psi_{E/K}(v_0) = u_0 \in \mathbb{R}_{\geq -1}$ is a break in the lower ramification filtration $\{G_u\}_{u \in \mathbb{R}_{\geq -1}}$ of G. □

Basic properties of lower and upper ramification filtrations on G. Suppose that $K \subseteq F \subseteq E$ is a sub-extension of E/K, let $\mathrm{Gal}(E/F) = H$.

(i) The lower numbering on G passess well to the subgroup H of G in the sense that

$$H_u = H \cap G_u$$

for $-1 \leq u \in \mathbb{R}$,

(ii) and if furthermore, $H \lhd G$, the upper numbering on G passes well to the quotient G/H as

$$(G/H)^v = G^v H/H$$

for $-1 \leq v \in \mathbb{R}$; but the lower numbering on G passes via '$\varphi_{E/F}$-action' to the quotient G/H as

$$(G/H)_{\varphi_{E/F}(u)} = G_u H/H$$

for $-1 \leq u \in \mathbb{R}$,

(iii) (transitivity of the Hasse–Herbrand function). If F/K is a Galois sub-extension of E/K, then

$$\varphi_{E/K} = \varphi_{F/K} \circ \varphi_{E/F}$$

and

$$\psi_{E/K} = \psi_{E/F} \circ \psi_{F/K}.$$

Let $\mathcal{B}^u_{E/K}$ (resp. $\mathcal{B}^\ell_{E/K}$) be the set of all numbers $v \in \mathbb{R}_{\geq -1}$ which occur as breaks in the upper (resp. lower) ramification filtration of G. Then, by eq. (3),

(iv) $\mathcal{B}^\ell_{E/K}$ is a finite subset of $\mathbb{Z} \cap \mathbb{R}_{\geq -1}$, and by Remark 2.2,

(v) $$\psi_{E/K}(\mathcal{B}^u_{E/K}) \subseteq \mathcal{B}^\ell_{E/K} \quad \text{and} \quad \varphi_{E/K}(\mathcal{B}^\ell_{E/K}) \subseteq \mathcal{B}^u_{E/K}.$$

The description of the set $\mathcal{B}^u_{E/K}$ is far more interesting, and as we will observe in §4, much more involved:

(vi) (Hasse–Arf theorem). If E/K is an abelian extension, then $\mathcal{B}^u_{E/K}$ is a finite subset of $\mathbb{Z} \cap \mathbb{R}_{\geq -1}$. □

Remark 2.3. Note that, if E/K is an abelian extension, and $0 \leq n \in \mathbb{Z}$, then $G^v = G^n$ for every $n - 1 < v \leq n$.

Proof. To prove this, assume that there exists $v_0 \in (n-1, n)$ such that $G^{v_0} \neq G^n$. By Hasse–Arf theorem, this chosen v_0 is not a break in the upper ramification filtration $\{G^v\}_{v \in \mathbb{R}_{\geq -1}}$ of G. So $G^{v_0} = G^{v_0 + \delta_0}$ for some $0 < \delta_0 \in \mathbb{R}$. Now, by Remark 2.2(ii), the number δ_0 should satisfy $v_0 + \delta_0 < n$. Thus, the set $S = \{v \in \mathbb{R}_{\geq v_0} \mid G^{v_0} = G^v\}$ is non-empty and bounded from above (bounded from above strictly by n). So, there exists the least upper bound v_1 of the set S. Note that, $v_1 \in S$. In fact, if $v_1 \notin S$, then $G^{v_0} \neq G^{v_1}$.

So by eq. (3), there exists an integer k such that $\psi_{E/K}(v_0) \lneq k \lneq \psi_{E/K}(v_1)$ (Reason: Note that, $G_{\psi_{E/K}(v_0)} \neq G_{\psi_{E/K}(v_1)}$. So, if the open interval $\big(\psi_{E/K}(v_0), \psi_{E/K}(v_1)\big)$ is a subset of some closed interval $[k, k+1]$ for some integer k, then by eq. (3), $G_{\psi_{E/K}(v_0)} = G_{\psi_{E/K}(v_1)}$, a contradiction.), which must be a break in the lower ramification filtration of G since $G_{\psi_{E/K}(v_0)} \neq G_{\psi_{E/K}(v_1)}$. Thus, $\varphi_{E/K}(k)$ must be a break in the upper ramification filtration of G by Remark 2.2(iii). Since G is assumed to be abelian, $\varphi_{E/K}(k)$ is an integer, contradicting the fact that $(v_0, v_1) \subset (n-1, n)$. Thus, v_1 must be in S. Therefore, $n - 1 < v_1 < n$. This chosen number $v_1 \in S$ is a break in the upper ramification filtration $\{G^v\}_{v \in \mathbb{R}_{\geq -1}}$ of G, because $G^{v_1} \neq G^{v_1+\delta}$ for every $0 < \delta \in \mathbb{R}$, as $v_1 = l.u.b(S)$ and $v_1 \in S$. Thus, by Hasse–Arf theorem, $v_1 \in \mathbb{Z} \cap \mathbb{R}_{\geq -1}$, a contradiction arising from the assumption that $G^{v_0} \neq G^n$ for some $v_0 \in (n-1, n)$. So, $G^v = G^n$ for every $n - 1 < v < n$. □

Recall that, for the abelian extension E/K, the norm-residue symbol

$$(?, E/K) : K^\times \to \mathrm{Gal}(E/K)$$

of abelian local class field theory maps the higher-unit group $U^n(K)$ (which is defined by $U^n(K) = \{x \in K^\times : x \equiv 1 \pmod{\mathfrak{p}_K^n}\}$ for $1 \leq n \in \mathbb{Z}$ and $U^0(K) = U(K)$) onto the nth upper ramification group G^n of G for $0 \leq n \in \mathbb{Z}$. Hence, combining with Remark 2.3, for $x \in K^\times$,

$$x \in N(E/K)U^n(K) \Longleftrightarrow (x, E/K) \in G^v$$

for every $n - 1 < v \leq n$. Thus, it follows that, $G^v = 1$ for every $n - 1 < v \leq n$ if and only if $U^n(K) \subseteq N(E/K)$.

The conductor $\mathfrak{f}(E/K)$ of a finite abelian extension E/K in the sense of abelian local class field theory is defined to be the ideal

$$\mathfrak{f}(E/K) = \mathfrak{p}_K^{m_G}$$

in O_K, where $0 \leq m_G \in \mathbb{Z}$ is the minimal power of $\mathfrak{p}_K$ satisfying $U^{m_G}(K) \subseteq N(E/K)$, or equivalently, $0 \leq m_G \in \mathbb{Z}$ is the minimal integer satisfying $G^{m_G} = 1$.

DEFINITION 2.4 (Conductor $\mathfrak{f}(E/K)$ of a finite Galois extension E/K)

Let E/K be a finite Galois extension (with Galois group G). Define the real number n_G by

$$n_G = \inf\{x \in \mathbb{R}_{\geq -1} : G^{x+\delta} = 1, \forall \delta > 0\}.$$

The conductor $\mathfrak{f}(E/K)$ of E/K (in the sense of 'non-abelian' local class field theory[1]) is defined to be the ideal

$$\mathfrak{f}(E/K) = \mathfrak{p}_K^{[[n_G]]+1}$$

in O_K, where $[[n_G]]$ denotes the 'integer-part' of the number n_G.

---

[1] Note that, E/K is a finite n-abelian extension. So, modulo n-abelian local class field theories ($1 \leq n \in \mathbb{Z}$) in the sense of Koch–de Shalit (yet to be constructed for $3 \leq n \in \mathbb{Z}$!), we can locate and compute n_G by n-abelian Hasse–Arf theorem. This is the reason for us to call $\mathfrak{f}(E/K)$ as the conductor of the extension E/K in the sense of 'non-abelian' local class field theory.

Remark 2.5. Observe that, for the extension E/K, the number n_G introduced in Definition 2.4 is a jump in the ramification filtration of E/K in upper numbering, and $0 \leq [[n_G]] + 1 \in \mathbb{Z}$ is the minimal integer satisfying $G^{[[n_G]]+1} = 1$. If, in particular, E/K is abelian, then n_G is, by Hasse–Arf theorem, the maximal integer satisfying $G^{n_G} \neq 1$. Thus, $[[n_G]] + 1 = n_G + 1$ is the minimal integer satisfying $G^{[[n_G]]+1} = 1$, that is $[[n_G]] + 1 = m_G$. So, Definition 2.4 coincides with the well-known definition of conductor of E/K in case E/K is abelian. □

Lemma 2.6. *G^{n_G} is an abelian normal subgroup of G.*

Proof. In fact, since $G^{n_G} = G_{\psi_{E/K}(n_G)}$ and n_G is a jump in the upper ramification filtration of E/K, it follows that $\psi_{E/K}(n_G)$ is a jump in the lower ramification filtration of E/K and $\psi_{E/K}(n_G) \in \mathbb{Z}_{\geq -1}$ by Remark 2.2(iii) and eq. (3). Now, assume that $\psi_{E/K}(n_G) \geq 0$ (that is, equivalently, assume that $n_G \geq 0$). Then by eq. (4), there exists an injection

$$G^{n_G}/G_{\psi_{E/K}(n_G)+1} \hookrightarrow U^{\psi_{E/K}(n_G)}(E)\big/U^{\psi_{E/K}(n_G)+1}(E),$$

which is defined by

$$\sigma \mapsto \frac{\sigma(\pi_E)}{\pi_E}$$

for $\sigma \in G^{n_G}$, where π_E is any chosen and fixed prime element in E. The assertion now follows. In fact, $\varphi_{E/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ is a piecewise-linear, strictly increasing function. So

$$\varphi_{E/K}([\psi_{E/K}(n_G), \psi_{E/K}(n_G) + 1]) = [n_G, n_G + \delta_0]$$

for a unique $0 < \delta_0 \in \mathbb{R}$. Therefore, $G_{\psi_{E/K}(n_G)+1} = G^{n_G+\delta_0} = 1$, which proves that G^{n_G} is an abelian subgroup of G. If $\psi_{E/K}(n_G) = -1$, then necessarily $n_G = -1$ and therefore the inertia G^0 is trivial, proving that $G \xrightarrow{\sim} \mathrm{Gal}(\kappa_E/\kappa_K)$, which is a cyclic group. □

Basic properties of lower and upper ramification filtration on G (continuation). If E/K is an infinite Galois extension with Galois group $\mathrm{Gal}(E/K) = G$ (which is a topological group under the Krull topology), define the upper ramification filtration $\{G^v\}_{v \in \mathbb{R}_{\geq -1}}$ on G by the projective limit

$$G^v := \varprojlim_{K \subseteq F \subset E} \mathrm{Gal}(F/K)^v$$

defined over the transition morphisms

$$\begin{tikzcd}
(G/\mathrm{Gal}(E/F))^v \arrow[d, equal] & \big(G/\mathrm{Gal}(E/F')\big)^v \arrow[l, "t_F^{F'}"'] \arrow[d, equal] \\
G^v\mathrm{Gal}(E/F)/\mathrm{Gal}(E/F) & G^v\mathrm{Gal}(E/F')/\mathrm{Gal}(E/F') \arrow[l, "\text{can.}"']
\end{tikzcd}$$

induced from (ii), as $K \subseteq F \subseteq F' \subseteq E$ runs over all finite Galois extensions F and F' over K inside E. Observe that

(vii) $G^{-1} = G$ and G^0 is the inertia group of G,

(viii) $\bigcap_{v \in \mathbb{R}_{\geq -1}} G^v = \langle 1 \rangle$,

(ix) G^v is a closed subgroup of G (with respect to the Krull topology) for $-1 \leq v \in \mathbb{R}$.

In this setting, a number $-1 \leq v \in \mathbb{R}$ is said to be a break in the upper ramification filtration $\{G^v\}_{v \in \mathbb{R}_{\geq -1}}$ of G, if v is a break in the upper filtration of some finite quotient G/H for some $H \lhd G$. As introduced previously, let $\mathcal{B}^u_{E/K}$ be the set of all numbers $v \in \mathbb{R}_{\geq -1}$, which occur as breaks in the upper ramification filtration of G. Then,

(x) (Hasse–Arf theorem). $\mathcal{B}^u_{K^{ab}/K} \subseteq \mathbb{Z} \cap \mathbb{R}_{\geq -1}$, and the final important result, in the spirit of Koch–de Shalit local class field theory, is

(xi) $\mathcal{B}^u_{K^{\text{sep}}/K} \subseteq \mathbb{Q} \cap \mathbb{R}_{\geq -1}$. □

3. Artin representation $A_{\mathrm{Gal}(E/K)}$ of Gal(*E*/*K*)

Main references for this section are [9, 11, 12].

Now, again assume that E/K is a finite Galois extension with Galois group $\mathrm{Gal}(E/K) = G$. Introduce a complex-valued function $a_G : G \to \mathbb{C}$ on G by

$$a_G(\sigma) = \begin{cases} -f(E/K) i_{E/K}(\sigma), & \sigma \neq 1 \\ \\ f(E/K) \sum\limits_{\substack{\tau \in G \\ \tau \neq 1}} i_{E/K}(\tau), & \sigma = 1 \end{cases}$$

for every $\sigma \in G$. Note that $a_G : G \to \mathbb{C}$ is a class function on G, that is, $a_G(\delta\sigma\delta^{-1}) = a_G(\sigma)$ for every $\delta, \sigma \in G$ since $i_{E/K}(\delta\sigma\delta^{-1}) = i_{E/K}(\sigma)$ whenever $\delta\sigma\delta^{-1} \neq 1$. In fact

$$\begin{aligned} i_{E/K}\left(\delta\sigma\delta^{-1}\right) &= \widetilde{\boldsymbol{\nu}}_E((\delta\sigma\delta^{-1})(x) - x) \\ &= \widetilde{\boldsymbol{\nu}}_E(\delta(\sigma(\delta^{-1}(x)) - \delta^{-1}(x))), \end{aligned}$$

and since $\widetilde{\boldsymbol{\nu}}_E\,(\delta(y)) = \widetilde{\boldsymbol{\nu}}_E(y)$ for every $y \in O_E$,

$$\begin{aligned} i_{E/K}(\delta\sigma\delta^{-1}) &= \widetilde{\boldsymbol{\nu}}_E(\sigma(\delta^{-1}(x)) - \delta^{-1}(x)) \\ &= i_{E/K}(\sigma), \end{aligned}$$

as $O_K[x] = O_E = \delta^{-1} O_E = O_K[\delta^{-1}(x)]$, and by Remark 2.1. Note that, this argument also proves that $G_u \lhd G$, for every $-1 \leq u \in \mathbb{R}$.

Let $X(G)$ be the vector space of all class functions $G \to \mathbb{C}$ on G. The mapping $(?, ?)_G : X(G) \times X(G) \to \mathbb{C}$ defined by

$$(f, g)_G = \frac{1}{[E : K]} \sum_{\sigma \in G} f(\sigma)\overline{g(\sigma)}$$

for every $f, g \in X(G)$ is a Hermitian inner-product on $X(G)$ (note that $\#(G) = [E : K]$ in this particular case), and the set $\mathcal{B}(G)$ of all irreducible characters $\chi : G \to \mathbb{C}$ of G forms an orthonormal basis of the vector space $X(G)$. Hence, the dimension of $X(G)$ over $\mathbb{C}$ is equal to the number of conjugacy classes $r(G)$ in G. Any class function $g : G \to \mathbb{C}$ on G is a $\mathbb{C}$-linear combination

$$g = \sum_{\chi} f_\chi(g)\chi$$

of irreducible characters $\chi : G \to \mathbb{C}$ of G, where the uniquely defined χ-coordinate $f_\chi(g) \in \mathbb{C}$ of the function $g : G \to \mathbb{C}$, which is called the Fourier coefficient of $g : G \to \mathbb{C}$ at $\chi : G \to \mathbb{C}$, is given by

$$f_\chi(g) = (g, \chi)_G.$$

In the remaining of this section we will sketch the proof of the fact that $f_\chi(a_G) \in \mathbb{Z}_{\geq 0}$, for every irreducible character $\chi : G \to \mathbb{C}$ of G. Thus, the class function $a_G : G \to \mathbb{C}$ is in fact a character, called the 'Artin character' of G, of a representation $A_G : G \to GL(V_{A_G})$, called an 'Artin representation' of G over $\mathbb{C}$.

Digression: Regular representation $\mathbf{R}_G$ *of* G. Recall that, the regular representation $\mathbf{R}_G : G \to GL(\mathbb{C}[G])$ is defined by the G-module $\mathbb{C}[G] = \{\sum_{g \in G} x_g g \mid x_g \in \mathbb{C}, \forall g \in G\}$, where the action of G on $\mathbb{C}[G]$ is defined by the left-multiplication $(h, \sum_{g \in G} x_g g) \mapsto \sum_{g \in G} x_g (hg)$ for every $h \in G$ and $\sum_{g \in G} x_g g \in \mathbb{C}[G]$. So, $\mathbf{R}_G : G \to GL(\mathbb{C}[G])$ is defined explicitly by

$$\mathbf{R}_G : h \mapsto T_h,$$

where $T_h : \mathbb{C}[G] \to \mathbb{C}[G]$ is the linear isomorphism defined by

$$T_h : \sum_{g \in G} x_g g \mapsto \sum_{g \in G} x_g (hg)$$

for every $\sum_{g \in G} x_g g \in \mathbb{C}[G]$, and for every $h \in G$. Note that G is the canonical $\mathbb{C}$-basis of $\mathbb{C}[G]$. So $\mathbb{C}[G]$ is a $\#(G) = \gamma_{-1} =: \gamma$-dimensional vector space over $\mathbb{C}$. Fixing an ordering on the group G, for any $h \in G$, the matrix $M_h = [T_h]_G \in GL_\gamma(\mathbb{C})$ corresponding to $T_h \in GL(\mathbb{C}[G])$ is $M_h = (M_{h;g,t})_{g,t \in G}$, where the (g, t) component $M_{h;g,t}$ of M_h is $M_{h;g,t} = \delta_{h,gt^{-1}}$ for every $g, t \in G$. Thus, the character $\chi_{\mathbf{R}_G} = \mathbf{r}_G : G \to \mathbb{C}$ of G associated to the regular representation $\mathbf{R}_G : G \to GL(\mathbb{C}[G])$ is

$$\mathbf{r}_G(h) = \begin{cases} \#(G) = \gamma, & \text{if } h = 1_G; \\ 0, & \text{if } h \neq 1_G \end{cases}$$

for every $h \in G$. Note that

$$\mathbb{C}[G] = \mathbb{C} \sum_{g \in G} g \bigoplus \left\{ \sum_{g \in G} x_g g \mid \sum_{g \in G} x_g = 0 \right\},$$

where both $\mathbb{C}[G]_1 = \mathbb{C} \sum_{g \in G} g$ and $\mathbb{C}[G]_0 = \{\sum_{g \in G} x_g g \mid \sum_{g \in G} x_g = 0\}$ are G-invariant subspaces of $\mathbb{C}[G]$. Thus, the regular representation $\mathbf{R}_G : G \to GL(\mathbb{C}[G])$ of G is the direct sum of the trivial representation $\mathbf{1}_G : G \to GL(\mathbb{C}[G]_1)$ and the sub-representation $\mathbf{U}_G : G \to GL(\mathbb{C}[G]_0)$, called the augmentation representation of G, that is,

$$\mathbf{R}_G = \mathbf{1}_G \oplus \mathbf{U}_G.$$

The character $\mathbf{r}_G : G \to \mathbb{C}$ associated to the regular representation $(\mathbf{R}_G, \mathbb{C}[G])$ of G is then given by the sum

$$\mathbf{r}_G = \chi_{\mathbf{1}_G} + \chi_{\mathbf{U}_G},$$

where $\chi_{\mathbf{1}_G} : G \to \mathbb{C}$ is the character associated to the trivial representation $\mathbf{1}_G : G \to GL\,(\mathbb{C}[G]_1)$ of G and $\chi_{\mathbf{U}_G} = \mathbf{u}_G : G \to \mathbb{C}$ is the character associated to the augmentation representation $\mathbf{U}_G : G \to GL\,(\mathbb{C}[G]_0)$ of G. Hence, $\mathbf{r}_G(h) = \chi_{\mathbf{1}_G}(h) + \mathbf{u}_G(h)$ for every $h \in G$, proving that

$$\mathbf{u}_G(h) = \begin{cases} \#(G) - 1 = \gamma - 1, & \text{if } h = 1_G \\ -1, & \text{if } h \neq 1_G \end{cases}$$

for every $h \in G$. □

Let G_i be the ith-ramification group of G (in the lower numbering), $\mathbf{u}_{G_i} : G_i \to \mathbb{C}$ the character associated to the augmentation representation $\mathbf{U}_{G_i} : G_i \to GL\,(\mathbb{C}[G_i]_0)$ of G_i over $\mathbb{C}$ for $-1 \leq i \in \mathbb{Z}$. It is then well-known that

$$a_G = \sum_{0 \leq i \in \mathbb{Z}} \frac{\gamma_i}{\gamma_0} \operatorname{Ind}_{G_i}^{G}(\mathbf{u}_{G_i}), \tag{6}$$

where $\gamma_i = \#(G_i)$ for every $0 \leq i \in \mathbb{Z}$; and for any subgroup H of G,

$$\operatorname{Res}_H(a_G) = \boldsymbol{\nu}_K(\mathfrak{d}(E^H/K))\mathbf{r}_H + f(E^H/K)a_H, \tag{7}$$

where $\mathfrak{d}(E^H/K)$ denotes the discriminant of the extension E^H/K.

For any class function $g : G \to \mathbb{C}$ on G, set $(a_G, g)_G =: f_{\text{Artin}}(g)$. The complex number $f_{\text{Artin}}(g)$ is then given by

$$f_{\text{Artin}}(g) = \sum_{0 \leq i \in \mathbb{Z}} \frac{\gamma_i}{\gamma_0} \left(g(\langle 1_G \rangle) - g(G_i) \right), \tag{8}$$

where $\gamma_i = \#(G_i)$ and $g(G_i) = \frac{1}{\gamma_i} \sum_{h \in G_i} g(h)$ for $0 \leq i \in \mathbb{Z}$. In fact, by the Frobenius reciprocity law

$$\begin{aligned}(g, \operatorname{Ind}_{G_i}^{G}(\mathbf{u}_{G_i}))_G &= (\operatorname{Res}_{G_i}(g), \mathbf{u}_{G_i})_{G_i} \\ &= \frac{1}{\gamma_i} \sum_{h \in G_i} \operatorname{Res}_{G_i}(g)(h)\bar{\mathbf{u}}_{G_i}(h),\end{aligned}$$

and since

$$\mathbf{u}_{G_i}(h) = \begin{cases} \gamma_i - 1, & \text{if } h = 1_{G_i} \\ -1, & \text{if } h \neq 1_{G_i} \end{cases}$$

it follows that

$$\begin{aligned}(g, \operatorname{Ind}_{G_i}^{G}(\mathbf{u}_{G_i}))_G &= \frac{1}{\gamma_i} \left[\operatorname{Res}_{G_i}(g)(1_{G_i})(\gamma_i - 1) - \sum_{h \neq 1_{G_i}} \operatorname{Res}_{G_i}(g)(h) \right] \\ &= \operatorname{Res}_{G_i}(g)(1_{G_i}) - \frac{1}{\gamma_i} \sum_{h \in G_i} \operatorname{Res}_{G_i}(g)(h) \\ &= g(\langle 1_G \rangle) - g(G_i),\end{aligned}$$

which yields the desired equality. In particular, if $\chi_\rho : G \to \mathbb{C}$ is the character of a representation (ρ, V) of G over $\mathbb{C}$, then

$$\begin{aligned} f_{\mathrm{Artin}}(\chi_\rho) &= \sum_{0\le i\in\mathbb{Z}} \frac{\gamma_i}{\gamma_0}(\chi_\rho(\langle 1_G\rangle) - \chi_\rho(G_i)) \\ &= \sum_{0\le i\in\mathbb{Z}} \frac{\gamma_i}{\gamma_0}(\dim_{\mathbb{C}}(V) - \dim_{\mathbb{C}}(V^{G_i})). \end{aligned}$$

Since $\chi_\rho(\langle 1_G\rangle) = \dim_{\mathbb{C}}(V)$ and $\chi_\rho(G_i) = \dim_{\mathbb{C}}(V^{G_i})$, where

$$V^{G_i} = \{v \in V \mid \rho(g)(v) = v, \forall g \in G_i\}$$

is a G-invariant subspace of V for $0 \le i \in \mathbb{Z}$, which follows from the fact that $G_i \lhd G$ for $-1 \le i \in \mathbb{Z}$. Observe that

$$V^{G_i} \subseteq V^{G_{i'}} \tag{9}$$

if $i \le i'$ with $0 \le i, i' \in \mathbb{Z}$.

The following proposition generalizes a theorem of Henniart and Tunnell (cf. p. 103 of [12e], and for the one-dimensional case, Proposition 11.3 of chapter VII of [9]). However, before stating the proposition, we first introduce some notation.

Notation 3.1. Let $\rho : G \to GL(V)$ be a non-trivial representation of G over $\mathbb{C}$. Let $-1 \le j \in \mathbb{Z}$ be maximal among all integers $-1 \le i$ satisfying

$$G_i \nsubseteq \ker(\rho).$$

Then, the set of integers $I(\rho) = [0, j] \cap \mathbb{Z} = \{0, \dots, j\}$ has a disjoint decomposition

$$I(\rho) = \bigsqcup_{1\le k\le s} I(\rho)_k$$

defined by a sequence $-1 = i_0 \lneq i_1 \lneq \cdots \lneq i_s = j$ so that $I(\rho)_k = [i_{k-1}+1, i_k] \cap \mathbb{Z} = \{i_{k-1}+1, \dots, i_k\}$ for $1 \le k \le s$ is the maximal interval satisfying

$$V^{G_{i_{k-1}+1}} = \cdots = V^{G_{i_k}}.$$

Note that, the existence of such a disjoint collection of intervals $I(\rho)_k$ for $1 \le k \le s$ follows from eq. (9). Therefore, for the representation (ρ, V) of G over $\mathbb{C}$, there exists an s-step flag

$$V^{G_{i_0+1}} = \cdots = V^{G_{i_1}} \subsetneq \cdots \subsetneq V^{G_{i_{s-1}+1}} = \cdots = V^{G_{i_s}}$$

of V^{G_j} constructed as above. Let $d_k = \dim_{\mathbb{C}}(V^{G_i})$ for $i \in I(\rho)_k$, $1 \le k \le s$. □

PROPOSITION 3.2

Let $\chi_\rho : G \to \mathbb{C}$ be the character associated to a non-trivial representation (ρ, V) of G over $\mathbb{C}$. Let $-1 \le j \in \mathbb{Z}$ be maximal among all integers $-1 \le i$ satisfying

$$G_i \nsubseteq \ker(\rho).$$

Then, following Notation 3.1,

$$f_{\mathrm{Artin}}(\chi_\rho) = \dim_{\mathbb{C}}(V)\left[\varphi_{E/K}(j)+1\right] - \sum_{1\le k\le s} d_k[\varphi_{E/K}(i_k) - \varphi_{E/K}(i_{k-1})].$$

Moreover,

(i) $\varphi_{E/K}(j)$ *is a jump in the ramification filtration of* $E^{\ker(\rho)}/K$ *in upper numbering and* $\varphi_{E/K}(j) = n_{G/\ker(\rho)}$ (*cf. Definition* 2.4),

(ii) $\varphi_{E/K}(i_k)$ *is a jump in the ramification filtration of* $E^{\ker(\rho_{i_k+1})}/K$ *in upper numbering for* $1 \le k \le s$ (*here*, $\rho_{i_k+1} : G \to GL(V^{G_{i_k+1}})$ *is the subrepresentation of* (ρ, V) *corresponding to the G-invariant subspace* $V^{G_{i_k+1}}$ *in V for* $1 \le k \le s$) *and* $\varphi_{E/K}(i_k) = n_{G/\ker(\rho_{i_k+1})}$.

Furthermore, if (ρ, V) *is an irreducible representation of G over* $\mathbb{C}$, *then we get the result of Henniart and Tunnell, namely*

$$f_{\mathrm{Artin}}(\chi_\rho) = \dim_{\mathbb{C}}(V)[\varphi_{E/K}(j)+1].$$

Proof. If $j < i \in \mathbb{Z}$, then $G_i \subseteq \ker(\rho)$, and therefore $\rho(g)(v) = v$ for every $g \in G_i$ and $v \in V$, that is, $V^{G_i} = V$. Thus

$$\begin{aligned} f_{\mathrm{Artin}}(\chi_\rho) &= \sum_{0\le i\in\mathbb{Z}} \frac{\gamma_i}{\gamma_0}(\chi_\rho(\langle 1_G\rangle) - \chi_\rho(G_i)) \\ &= \sum_{0\le i\le j} \frac{\gamma_i}{\gamma_0}(\dim_{\mathbb{C}}(V) - \dim_{\mathbb{C}}(V^{G_i})). \end{aligned}$$

Now, following Notation 3.1, since $I(\rho) = \bigsqcup_{1\le k\le s} I(\rho)_k$,

$$\begin{aligned} f_{\mathrm{Artin}}(\chi_\rho) &= \sum_{0\le i\le j} \frac{\gamma_i}{\gamma_0}(\dim_{\mathbb{C}}(V) - \dim_{\mathbb{C}}(V^{G_i})) \\ &= \sum_{1\le k\le s}\sum_{i\in I(\rho)_k} \frac{\gamma_i}{\gamma_0}(\dim_{\mathbb{C}}(V) - \dim_{\mathbb{C}}(V^{G_i})) \\ &= \dim_{\mathbb{C}}(V)\left[\varphi_{E/K}(j)+1\right] - \sum_{1\le k\le s} d_k \sum_{i\in I(\rho)_k} \frac{\gamma_i}{\gamma_0}, \end{aligned}$$

by eq. (5). Since $\sum_{i\in I(\rho)_k}(\gamma_i/\gamma_0) = \sum_{0\le i\le i_k}(\gamma_i/\gamma_0) - \sum_{0\le i\le i_{k-1}}(\gamma_i/\gamma_0)$, for $2 \le k$, it follows that

$$\begin{aligned} f_{\mathrm{Artin}}(\chi_\rho) &= \dim_{\mathbb{C}}(V)[\varphi_{E/K}(j)+1] \\ &\quad - \sum_{1\lneq k\le s} d_k \left(\sum_{0\le i\le i_k} \frac{\gamma_i}{\gamma_0} - \sum_{0\le i\le i_{k-1}} \frac{\gamma_i}{\gamma_0}\right) - d_1 \sum_{0\le i\le i_1} \frac{\gamma_i}{\gamma_0} \\ &= \dim_{\mathbb{C}}(V)[\varphi_{E/K}(j)+1] - \sum_{1\le k\le s} d_k[\varphi_{E/K}(i_k) - \varphi_{E/K}(i_{k-1})] \end{aligned}$$

again by eq. (5), which completes the proof of the desired equality.

(i) Now, let $E' = E^{\ker(\rho)}$ be the fixed field of $\ker(\rho) = H$. As usual, let $\widetilde{\rho} : G/H \to GL(V)$ be the non-trivial faithful representation of G/H defined by $\rho : G \xrightarrow{\text{can.}} G/H \xrightarrow{\widetilde{\rho}} GL(V)$. By Herbrand's theorem

$$G_j H/H = (G/H)_{\varphi_{E/E'}(j)}.$$

As $G_j = G^{\varphi_{E/K}(j)}$ and $(G/H)_{\varphi_{E/E'}(j)} = (G/H)^{\varphi_{E'/K}(\varphi_{E/E'}(j))} = (G/H)^{\varphi_{E/K}(j)}$, in terms of the upper ramification groups:

$$G^{\varphi_{E/K}(j)} H/H = (G/H)^{\varphi_{E/K}(j)}.$$

Now, $\widetilde{\rho}(G_j H/H) \neq 1$ and $\widetilde{\rho}(G_{j+\varepsilon} H/H) = \widetilde{\rho}(G_{j+1} H/H) = 1$ for every $0 < \varepsilon \in \mathbb{R}$ by the choice of the integer j and by eq. (3). Thus

$$G_j H/H \neq G_{j+\varepsilon} H/H,$$

for every $0 < \varepsilon \in \mathbb{R}$. Thus, $\varphi_{E/K}(j)$ is a jump in the ramification filtration of E'/K in upper numbering, since

$$G^{\varphi_{E/K}(j)} H/H = (G/H)^{\varphi_{E/K}(j)} \neq (G/H)^{\varphi_{E/K}(j)+\delta} = G^{\varphi_{E/K}(j)+\delta} H/H,$$

for every $0 < \delta \in \mathbb{R}$, as $\varphi_{E/K} : \mathbb{R}_{\geq -1} \to \mathbb{R}_{\geq -1}$ is a continuous and strictly increasing function. Moreover, since $\widetilde{\rho}(G_{j+\varepsilon} H/H) = \widetilde{\rho}(G_{j+1} H/H) = 1$ for every $0 < \varepsilon \in \mathbb{R}$ by the choice of the integer j and by eq. (3), it follows that $G_{j+\varepsilon} \subseteq H$ for every $0 < \varepsilon \in \mathbb{R}$. That is, $G_{j+\varepsilon} H/H = \langle 1_{G/H} \rangle$ for every $0 < \varepsilon \in \mathbb{R}$, proving that

$$(G/H)^{\varphi_{E/K}(j)+\delta} = \langle 1_{G/H} \rangle$$

for every $0 < \delta \in \mathbb{R}$. Hence, $\varphi_{E/K}(j) = n_{G/H}$ following the notation of Definition 2.4.

(ii) Consider the subrepresentation $\rho_{i_k+1} : G \to GL(V^{G_{i_k+1}})$ of the representation (ρ, V) of G. Clearly $G_{i_k+1} \subseteq \ker(\rho_{i_k+1})$ and $G_{i_k} \nsubseteq \ker(\rho_{i_k+1})$ since $V^{G_{i_k}} \subsetneq V^{G_{i_k+1}}$. Thus i_k is the maximal integer satisfying $G_{i_k} \nsubseteq \ker(\rho_{i_k+1})$. Now, the proof follows from the same lines of reasoning of part (i).

Finally, since $G_u \lhd G$ for every $-1 \leq u \in \mathbb{R}$, V^{G_u} is a G-invariant subspace of V. Therefore, $d_k = 0$ for $k = 1, \ldots, s$, since (ρ, V) is assumed to be an irreducible representation of G over $\mathbb{C}$. Now, the result of Henniart and Tunnell follows. □

For a representation (ρ, V) of G over $\mathbb{C}$, in order to prove that $f_{\text{Artin}}(\chi_\rho) \in \mathbb{Z}_{\geq 0}$, it suffices to prove this integrality result for any one-dimensional representation $\chi : G \to \mathbb{C}^\times$ of G over $\mathbb{C}$. In fact, by Brauer induction theorem, there exist certain type of subgroups (called the elementary subgroups) H_i of G and one-dimensional representations $\chi_i : H_i \to \mathbb{C}^\times$ of H_i over $\mathbb{C}$ such that the character $\chi_\rho : G \to \mathbb{C}$ of G is a certain linear combination of the form

$$\chi_\rho = \sum_i n_i \operatorname{Ind}_{H_i}^G(\chi_i),$$

where $n_i \in \mathbb{Z}$, and i runs over a finite set. Thus, by Frobenius reciprocity law

$$\begin{aligned} f_{\text{Artin}}(\chi_\rho) &= \left(a_G, \sum_i n_i \operatorname{Ind}_{H_i}^G(\chi_i) \right)_G = \sum_i n_i (a_G, \operatorname{Ind}_{H_i}^G(\chi_i))_G \\ &= \sum_i n_i (\operatorname{Res}_{H_i}(a_G), \chi_i)_{H_i}, \end{aligned}$$

and by eq. (7),

$$\begin{aligned} f_{\mathrm{Artin}}(\chi_\rho) &= \sum_i n_i (\boldsymbol{\nu}(\mathfrak{d}(E^{H_i}/K))\mathbf{r}_{H_i} + f(E^{H_i}/K)a_{H_i}, \chi_i)_{H_i} \\ &= \sum_i n_i \boldsymbol{\nu}(\mathfrak{d}(E^{H_i}/K))(\mathbf{r}_{H_i}, \chi_i)_{H_i} + \sum_i n_i f(E^{H_i}/K)(a_{H_i}, \chi_i)_{H_i} \\ &= \sum_i n_i \boldsymbol{\nu}(\mathfrak{d}(E^{H_i}/K))\chi_i(1_{H_i}) + \sum_i n_i f(E^{H_i}/K) f_{\mathrm{Artin}}(\chi_i). \end{aligned}$$

So, if $f_{\mathrm{Artin}}(\chi_i) \in \mathbb{Z}_{\geq 0}$ for every i, then $f_{\mathrm{Artin}}(\chi_\rho) \in \mathbb{Z}$. Moreover, $\gamma_0 f_{\mathrm{Artin}}(\chi_\rho) \geq 0$ by eq. (6) and so $f_{\mathrm{Artin}}(\chi_\rho) \geq 0$.

Thus, let $\chi : G \to \mathbb{C}^\times$ be a one-dimensional representation of G over $\mathbb{C}$. The fact that $f_{\mathrm{Artin}}(\chi) \in \mathbb{Z}_{\geq 0}$ is then a direct consequence of Proposition 3.2 stated as follows:

COROLLARY 3.3

Let $\chi : G \to \mathbb{C}^\times$ be a non-trivial one-dimensional representation of G over $\mathbb{C}$, $-1 \leq j \in \mathbb{Z}$ the maximal integer satisfying $G_j \not\subseteq \ker(\chi)$. Then

$$f_{\mathrm{Artin}}(\chi) = \varphi_{E/K}(j) + 1 = \mathrm{ord}_{\mathfrak{p}_K}(\mathfrak{f}(E^{\ker(\chi)}/K)),$$

where $\varphi_{E/K}(j) = n_{G/\ker(\chi)}$ is a jump in the ramification filtration of $E^{\ker(\chi)}/K$ in upper numbering, and $\varphi_{E/K}(j) \in \mathbb{Z}_{\geq -1}$.

Proof. The proof of the first equality follows from the main identity of Proposition 3.2. In fact, following Notation 3.1,

$$f_{\mathrm{Artin}}(\chi) = \varphi_{E/K}(j) + 1$$

since in this case $\dim(V_\chi) = 1$ and any $i \in I(\chi) = \{0, \ldots, j\}$ satisfies $V^{G_i} = V^{G_j} = 0$ by eq. (9). The second equality follows from part (i) of Proposition 3.2 and by Remark 2.5 applied to the abelian extension $E^{\ker(\rho)}/K$. □

Remark 3.4. If $\chi : G \to \mathbb{C}^\times$ is the trivial representation of G over $\mathbb{C}$, then the main equality of Corollary 3.3 remains true by setting $j = -1$. In fact, $f_{\mathrm{Artin}}(\chi) = (\chi, a_G)_G = 0$ by eq. (8). So $f_{\mathrm{Artin}}(\chi) = \varphi_{E/K}(-1) + 1$. On the other hand, $E^{\ker(\chi)} = K$, since $\ker(\chi) = G$. Thus, the main equality in Corollary 3.3 follows immediately. □

The following corollary can be viewed as a partial answer (i.e., answer modulo n-abelian local class field theories ($1 \leq n \in \mathbb{Z}$), and modulo explicit information on the dimensions of irreducible representations of G over $\mathbb{C}$) to Problem 1.1.

COROLLARY 3.5

Let $\rho : G \to GL(V)$ be an irreducible representation of G over $\mathbb{C}$. Then,

$$\left[\left[\frac{1}{\dim_{\mathbb{C}}(V)} f_{\mathrm{Artin}}(\chi_\rho)\right]\right] = \mathrm{ord}_{\mathfrak{p}_K}(\mathfrak{f}(E^{\ker(\rho)}/K)).$$

Proof. Let $-1 \leq j \in \mathbb{Z}$ be the maximal integer satisfying $G_j \not\subseteq \ker(\rho)$. Then, by Henniart–Tunnell's theorem

$$f_{\mathrm{Artin}}(\chi_\rho) = \dim_{\mathbb{C}}(V)\left[\varphi_{E/K}(j) + 1\right],$$

where $\varphi_{E/K}(j) = n_{G/\ker(\rho)}$, by part (i) of Proposition 3.2. Combining with Definition 2.4, the proof now follows. □

However, in case $E^{\ker(\rho)}/K$ is a metabelian extension, utilizing the results of [1] and [8], we have a complete and detailed answer to Problem 1.1.

4. Metabelian local class field theory

Main references for this section are: [3, 4, 7, 8].

Digression: n-abelian extensions over a field F. The main reference for this part is §2 of [4]. Recall that, a group G is called n-abelian (resp. 'strict' n-abelian), if the nth-commutator subgroup $G^{(n)} := [G^{(n-1)}, G^{(n-1)}]$ of G is $\langle 1_G \rangle$ (resp. $G^{(n)} = \langle 1_G \rangle$ and $G^{(m)} \neq \langle 1_G \rangle$ for every integer $0 \leq m \lneq n$), that is, G is a solvable group with derived length $\mathrm{d}l(G) \leq n$ (resp. $\mathrm{d}l(G) = n$) for $1 \leq n \in \mathbb{Z}$. (Notational convention: $G^{(0)} = G$.)

Now, an extension Q over any field F is called n-abelian (resp. 'strict' n-abelian), if it is a Galois extension with an n-abelian (resp. 'strict' n-abelian) Galois group $\mathrm{Gal}(Q/F)$. Note that, the derived series for $\mathrm{Gal}(Q/F) =: G$ is then given by

$$\langle 1 \rangle = G^{(n)} \leq G^{(n-1)} \leq \cdots \leq G^{(1)} \leq G^{(0)} = G,$$

and passing to the fixed field $Q^{G^{(i)}} = Q_i$ of $G^{(i)}$ for $i = 0, \ldots, n$, the derived series for G transforms to the chain of sub-extensions in Q/F,

$$Q_n = Q \supseteq Q_{n-1} \supseteq \cdots \supseteq Q_1 \supseteq Q_0 = F$$

with Q_i/Q_{i-1} abelian extension for $i = 1, \ldots, n$, called the 'canonical chain of sub-extensions in Q/F' throughout the text. Moreover, if Q/F is 'strict' n-abelian, then the derived series for G is of the form

$$\langle 1 \rangle = G^{(n)} \lneq G^{(n-1)} \lneq \cdots \lneq G^{(1)} \lneq G^{(0)} = G,$$

where the inclusions are strict in this case, because if $G^{(i)} = G^{(i-1)}$ for some $i \in \{1, \ldots, n-1\}$, then $G^{(i)} = [G^{(i-1)}, G^{(i-1)}] = [G^{(i)}, G^{(i)}] = G^{(i+1)}$, and hence $G^{(n-1)} = \langle 1_G \rangle$, which contradicts the assumption that $G^{(n-1)} \neq \langle 1_G \rangle$. The corresponding chain of sub-extensions in Q/F obtained by Galois duality is then of the form

$$Q_n = Q \supsetneq Q_{n-1} \supsetneq \cdots \supsetneq Q_1 \supsetneq Q_0 = F$$

with Q_i/Q_{i-1} non-trivial abelian extension ($i = 1, \ldots, n$). Note that, any m-abelian extension over F is n-abelian for $m \leq n$.

Fix a seperable closure F^{sep} of F (when it is convenient, instead of F^{sep}, sometimes the notation $\overline{F}$ will be used to denote the separable closure of F), and let $F^{(ab)^n}$ denote the maximal n-abelian extension of F inside F^{sep} (which exists! cf. [4]).

This concludes the digression on n-abelian extensions over F. □

In [8], Koch and de Shalit have constructed class field theory for metabelian (that is, 2-abelian) extensions of a local field K, which induces the abelian local class field theory when restricted to the abelian extensions of K. Koch–de Shalit local class field theory is indeed an 'approximation' of the non-abelian local class field theory[2].

---

[2] The local Langlands correspondence for $GL(n)$ (cf. [14]) and Koch–de Shalit local class field theory are closely related to each other. We plan to investigate the relationship between the local Langlands correspondence for $GL(n)$ and Koch–de Shalit local class field theory in a seperate paper.

The main idea in [8] is the following: as before, let K be a local field with finite residue field κ_K of q_K elements. Let $\overline{\kappa}_K$ be a fixed separable closure of κ_K. Fix an extension $\phi \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ of the Frobenius automorphism ϕ_K over K. That is, fix a Lubin–Tate splitting ϕ over K. It is then well-known that, depending on the choice of ϕ, there exists a unique norm-compatible set of primes

$$\mathcal{L}_\phi^0 = \{\pi_L \in L : \ K \subseteq L \subset K_\phi \text{s.t} \ [L : K] < \infty\},$$

where K_ϕ denotes the fixed field of ϕ in K^{sep}, which has a canonical extension to a norm-compatible set of primes

$$\mathcal{L}_\phi = \{\pi_L \in \widetilde{L^{nr}} : K \subseteq L \subset K^{\mathrm{sep}} \text{s.t} \ [L : K] < \infty\},$$

called the Lubin–Tate labelling over K attached to the Lubin–Tate splitting ϕ over K. Here, $\widetilde{L^{nr}}$ denotes the completion of L^{nr}. For a finite extension L over K which is pointwise fixed by ϕ, there exists a unique Lubin–Tate formal power series $f_L \in O_L[[X]]$ (belonging to π_L for $\widetilde{L^{nr}}$, where π_L chosen from $\mathcal{L}_\phi^0$) satisfying certain properties (cf. 0.3 in [8]), and a unique Lubin–Tate formal group law $F_L \in O_{\widetilde{L^{nr}}}[[X, Y]]$ attached to f_L. Let $[u]_{f_L} : F_L \to F_L$ be the unique endomorphism of F_L over $O_{\widetilde{L^{nr}}}$ of the form $[u]_{f_L} = uX +$ (higher-degree terms) $\in X O_{\widetilde{L^{nr}}}[[X]]$ for $u \in O_L$, and let

$$\{u\}_{f_L} = [u]_{f_L} \pmod{\pi_L}.$$

Let $\widehat{K^\times}$ denote the profinite completion of $K^\times$, and fix the isomorphism $\widehat{K^\times} \xrightarrow{\sim} U(K) \times \widehat{\mathbb{Z}}$ defined by $a = u_a \pi_K^{v_a} \mapsto (u_a, v_a)$ for $a \in \widehat{K^\times}$, where π_K is the prime element in K chosen uniquely from $\mathcal{L}_\phi^0$. For $1 \leq d \in \mathbb{Z}$, consider the topological group

$$\mathfrak{G}_d^{[2]}(K, \phi) := \left\{ (a, \xi) \in \widehat{K^\times} \times \overline{\kappa}_K[[X]]^\times : \frac{\xi^{\phi^d}}{\xi} = \frac{\{u_a\}_{f_K}}{X} \right\}$$

under the law of composition defined by

$$(a, \xi)(b, \psi) = (ab, \xi(\psi^{\phi^{-v_a}} \circ \{u_a\}_{f_K}))$$

for $(a, \xi), (b, \psi) \in \mathfrak{G}_d^{[2]}(K, \phi)$, and with a basis of neighborhoods of the underlying topology

$$\mathfrak{G}_d^{[2]}(K, \phi)^{(i,j)} = \{(a, \xi) \in \mathfrak{G}_d^{[2]}(K, \phi) : a \in U^i(K), \ \ \xi \equiv 1 \pmod{X^j}\}$$

for $0 \leq i, j \in \mathbb{Z}$. Note that, $\mathfrak{G}_d^{[2]}(K, \phi)$ is a non-trivial topological group for every $1 \leq d \in \mathbb{Z}$, by [6].

Let $\mathfrak{G}^{[2]}(K, \phi) := \varprojlim_d \mathfrak{G}_d^{[2]}(K, \phi)$, where the projective limit is defined over the transition morphisms $\mathfrak{G}_{d'}^{[2]}(K, \phi) \to \mathfrak{G}_d^{[2]}(K, \phi)$ for every $1 \leq d, d' \in \mathbb{Z}$ with $d | d'$, which is defined by

$$(a, \xi) \mapsto \left(a, \prod_{0 \leq i \lneq \frac{d'}{d}} \xi^{\phi^{di}} \right)$$

for $(a,\xi) \in \mathfrak{G}^{[2]}_{d'}(K,\phi)$. Note that, these transition morphisms send $\mathfrak{G}^{[2]}_{d'}(K,\phi)^{(i,j)}$ to $\mathfrak{G}^{[2]}_{d}(K,\phi)^{(i,j)}$ for every $1 \leq d, d' \in \mathbb{Z}$ with $d|d'$, which in return defines a subgroup $\mathfrak{G}^{[2]}(K,\phi)^{(i,j)} = \varprojlim_{d} \mathfrak{G}^{[2]}_{d}(K,\phi)^{(i,j)}$ of $\mathfrak{G}^{[2]}(K,\phi)$.

Let L be a ϕ-compatible extension over K, $\phi' = \phi^{f(L/K)}$ (cf. 0.4 in [8]), and

$$\mathfrak{G}^{[2]}(L,\phi') \xrightarrow{M_{\phi,L/K}} \mathfrak{G}^{[2]}(K,\phi)$$

the '2-abelian' norm map which is a canonical morphism defined via 'Coleman theory' (cf. 1.5 in [8]). As a notation, put $M_\phi(L/K) := M_{\phi,L/K}(\mathfrak{G}^{[2]}(L,\phi'))$, which is a closed subgroup of finite index in $\mathfrak{G}^{[2]}(K,\phi)$, and for an infinite extension F/K which is a union of ϕ-compatible extensions L over K (such an F will be called as an infinite ϕ-compatible extension over K), define $M_\phi(F/K) := \bigcap_{K\subseteq L\subset F} M_\phi(L/K)$ where L runs over all ϕ-compatible sub-extensions in F/K (so $M_\phi(F/K)$ is a closed subgroup of $\mathfrak{G}^{[2]}(K,\phi)$). Define the map

$$\left\{ \begin{matrix} \phi\text{-compatible extensions} \\ \text{over } K \end{matrix} \right\} \xrightarrow{M_\phi} \left\{ \begin{matrix} \text{closed subgroups} \\ \text{of } \mathfrak{G}^{[2]}(K,\phi) \end{matrix} \right\}$$

by

$$F/K \mapsto M_\phi(F/K)$$

for every ϕ-compatible extension F/K.

The metabelian local class field theory of Koch–de Shalit states the following:

Theorem 4.1. (*Metabelian local class field theory*). *Let K be a local field with finite residue field κ_K of q_K elements. Fix an extension $\phi \in \mathrm{Gal}(K^{\mathrm{sep}}/K)$ of the Frobenius automorphism $\phi_K \in \mathrm{Gal}(K^{nr}/K)$ over K, that is, fix a Lubin–Tate splitting ϕ over K.*

(i) (*2-abelian reciprocity map*). *There exists a topological isomorphism*

$$\mathfrak{G}^{[2]}(K,\phi) \xrightarrow[\sim]{(?,K)_\phi} \mathrm{Gal}(K^{(ab)^2}/K)$$

called the '2-abelian' local Artin map in this text, which depends only on the choice of ϕ,

(ii) (*Existence*). *There exists an order-preserving bijection*

$$\left\{ \begin{matrix} 2\text{-abelian extensions} \\ \text{over } K \end{matrix} \right\} \xrightarrow{M_\phi} \left\{ \begin{matrix} \text{closed subgroups} \\ \text{of } \mathfrak{G}^{[2]}(K,\phi) \end{matrix} \right\}$$

defined by

$$E/K \mapsto M_\phi(E/K) = \left(\mathrm{Gal}(K^{(ab)^2}/E), K\right)_\phi^{-1}$$

for any 2-abelian extension E/K. Note that, the closed subgroup $M_\phi(E/K)$ of $\mathfrak{G}^{[2]}(K,\phi)$ is of finite index if and only if E/K is a finite metabelian extension, and if this is the case, then

$$[E:K] = \left(\mathfrak{G}^{[2]}(K,\phi) : M_\phi(E/K)\right),$$

(iii) (2*-abelian reciprocity map – Continuation –). For any* 2*-abelian extension* E *over* K*, the surjective morphism*

$$\begin{tikzcd}
\mathfrak{G}^{[2]}(K,\phi) \arrow[r, "(?,K)_\phi", "\sim"'] \arrow[rr, bend right, two heads, "(?,K)_\phi|_E"'] & \mathrm{Gal}(K^{(ab)^2}/K) \arrow[r, two heads, "\mathrm{res}._E^{K^{(ab)^2}}"] & \mathrm{Gal}(E/K)
\end{tikzcd}$$

induces the canonical topological isomorphism

$$\mathfrak{G}^{[2]}(K,\phi)/M_\phi(E/K) \xrightarrow[\sim]{(?,E/K)_\phi} \mathrm{Gal}(E/K),$$

where $M_\phi(E/K)$ *is the closed normal subgroup of* $\mathfrak{G}^{[2]}(K,\phi)$ *defined as in part* (ii),

(iv) (*Functoriality*). *If* K' *is a finite* ϕ*-compatible extension over* K, *then the square*

$$\begin{tikzcd}
\mathfrak{G}^{[2]}(K',\phi') \arrow[r, "(?,K')_{\phi'}"] \arrow[d, "M_{\phi,K'/K}"'] & \mathrm{Gal}(K'^{(ab)^2}/K') \arrow[d, "\mathrm{res}_K"] \\
\mathfrak{G}^{[2]}(K,\phi) \arrow[r, "(?,K)_\phi"] & \mathrm{Gal}(K^{(ab)^2}/K)
\end{tikzcd}$$

is commutative. Therefore, *the* 2*-abelian norm map*

$$\left\{\begin{matrix}\phi\text{-}\textit{compatible extensions}\\ \textit{over } K\end{matrix}\right\} \xrightarrow{M_\phi} \left\{\begin{matrix}\textit{closed subgroups}\\ \textit{of } \mathfrak{G}^{[2]}(K,\phi)\end{matrix}\right\},$$

defined via Coleman theory, *has a natural extension to*

$$\left\{\begin{matrix}2\text{-}\textit{abelian extensions}\\ \textit{over } K\end{matrix}\right\} \xrightarrow{M_\phi} \left\{\begin{matrix}\textit{closed subgroups}\\ \textit{of } \mathfrak{G}^{[2]}(K,\phi)\end{matrix}\right\},$$

defined as in part (ii), *in the sense that the triangle*

$$\begin{tikzcd}
\left\{\begin{matrix}2\text{-}\textit{abelian extensions}\\ \textit{over } K\end{matrix}\right\} \arrow[dr, hook, two heads, "M_\phi"] & \\
& \left\{\begin{matrix}\textit{closed subgroups}\\ \textit{of } \mathfrak{G}^{[2]}(K,\phi)\end{matrix}\right\} \\
\left\{\begin{matrix}\phi\text{-}\textit{compatible extensions}\\ \textit{over } K\end{matrix}\right\} \arrow[uu, rightsquigarrow] \arrow[ur, "M_\phi"'] &
\end{tikzcd}$$

is commutative, *where the vertical arrow is defined by* $K' \rightsquigarrow \left(K' \cap K^{(ab)^2}\right)^{\text{Gal-cl}}$, *which is known to be a* 2*-abelian extension over* K (*cf.* [4]), *for any* ϕ*-compatible extension* K' *over* K,

(v) (*Ramification theory*). *The breaks in the upper numbering of the ramification groups for* 2*-abelian extensions over* K *occur at* $r = 0$ *and at rational numbers of the form*

$$u_{i,j} = i - \frac{q_K^i - 1 - j}{q_K^i - q_K^{i-1}}$$

with $1 \le i \in \mathbb{Z}$ *and* $j \in \mathbb{Z}$ *such that* $q_K^{i-1} \le j < q_K^i$. *If* E/K *is a finite* 2*-abelian extension, and*

$$i - 1 \le u_{i,j-1} < r \le u_{i,j} \le i,$$

then

$$\alpha \in M_\phi(E/K)\mathfrak{G}^{[2]}(K,\phi)^{(i,j)} \Longleftrightarrow (\alpha, E/K)_\phi \in \mathrm{Gal}(E/K)^r.$$

The 2-abelian local class field theory is related with the abelian local class field theory as follows: if E/K is a 2-abelian extension, then

$$pr_1\left(M_\phi(E/K)\right) = N(E/K). \tag{10}$$

The abelian extension over K, which is class field to $N(E/K)$ is the maximal abelian sub-extension of E/K. Let $K = E_0 \subseteq E_1 \subseteq E_2 = E$ be the 'canonical chain of sub-extensions in E/K' as introduced in the beginning of this section. Note that, E_1/K is the maximal abelian sub-extension of E/K, that is, $E_1 = E \cap K^{ab}$. In fact, to simplify the notation, setting $G = \mathrm{Gal}(E/K)$ as usual, $G' = \mathrm{Gal}(E/E_1)$. If F/K is any abelian sub-extension of E/K, and $N = \mathrm{Gal}(E/F)$, which is a closed and normal subgroup of G, then $G/N = \mathrm{Gal}(F/K)$ is an abelian group, proving that $G' \subseteq N$. Thus, passing to the fixed fields, $E_1 \supseteq F$, which completes the proof of the maximality of E_1/K among all abelian sub-extensions F/K in E/K, which also shows that $E_1 = E \cap K^{ab}$. Therefore,

$$pr_1\left(M_\phi(E/K)\right) = N(E_1/K). \tag{11}$$

Lemma 4.2. (Koch–de Shalit)

$$pr_1(\mathfrak{G}^{[2]}(K,\phi)^{(i,j)}) = U^i(K)$$

for $1 \le i \in \mathbb{Z}$ *and* $j \in \{q_K^{i-1}, \dots, q_K^i - 1\}$.

So, combining eq. (11) and Lemma 4.2,

Lemma 4.3. *Let* E/K *be a finite* 2*-abelian extension.*

(i) *If* $\alpha \in M_\phi(E/K)\mathfrak{G}^{[2]}(K,\phi)^{(i,j)}$ *for* $1 \le i \in \mathbb{Z}$ *and* $j \in \{q_K^{i-1}, \dots, q_K^i - 1\}$, *then* $pr_1(\alpha) \in N(E/K)U^i(K) = N(E_1/K)U^i(K)$ *for* $1 \le i \in \mathbb{Z}$.

(ii) *If* $(\alpha, E/K)_\phi \in \mathrm{Gal}(E/K)^r$ *for* $u_{i,j-1} < r \le u_{i,j}$ *with* $1 \le i \in \mathbb{Z}$, $j \in \{q_K^{i-1}, \dots, q_K^i - 1\}$, *then* $(pr_1(\alpha), E_1/K) \in \mathrm{Gal}(E_1/K)^v$ *for* $i - 1 < v \le i$.

Remark 4.4. Note that, the metabelian local class field theory depends on a choice of Lubin–Tate splitting ϕ over K. Let $\mathcal{LT}_K$ denote the set of all Lubin–Tate splittings over K. Then, there exists a bijection between $\mathcal{LT}_K$ and I_K (the inertia group $\mathrm{Gal}(K^{\mathrm{sep}}/K^{nr})$

of K). We can define such a bijection by fixing a Lubin–Tate splitting ϕ_c over K, and defining the map $\mathcal{LT}_K \to I_K$ by $\phi \mapsto \phi \circ \phi_c^{-1}$ for every $\phi \in \mathcal{LT}_K$. Now, consider the set $\Gamma_c := \left\{\gamma\phi_c\gamma^{-1} : \gamma \in I_K\right\}$. Clearly, $\Gamma_c \subseteq \mathcal{LT}_K$. Moreover, there exists a bijection between I_K and Γ_c defined by $\gamma \mapsto \gamma\phi_c\gamma^{-1}$ for every $\gamma \in I_K$. Note that, this defines a bijection between I_K and Γ_c, since if $\gamma_1\phi_c\gamma_1^{-1} = \gamma_2\phi_c\gamma_2^{-1}$ for some $\gamma_1, \gamma_2 \in I_K$, then $\gamma_2^{-1}\gamma_1 = \phi_c^n$ for some $n \in \mathbb{Z}$ as the normalizer of $\langle\phi_c\rangle$ in $\mathrm{Gal}(K^{\mathrm{sep}}/K)$ is $\langle\phi_c\rangle$ itself (cf. 0.2 in [8]). Thus, there exist bijections $\mathcal{LT}_K \leftrightarrow I_K \leftrightarrow \Gamma_c$. Observe that, even though Γ_c and $\mathcal{LT}_K$ have the same cardinality and $\Gamma_c \subseteq \mathcal{LT}_K$, it may happen in general that $\Gamma_c \neq \mathcal{LT}_K$. However, there exists a collection $\{\phi_i\}_{i\in I}$ of Lubin–Tate splittings over K (indexed over a set I), and a partition of $\mathcal{LT}_K$ as

$$\mathcal{LT}_K = \bigsqcup_i \Gamma_i,$$

where i runs over this indexing set I.

Now, if $\phi, \phi_0 \in \Gamma_i$ for some $i \in I$, that is, if $\phi_0 = \widetilde{\sigma}\phi\widetilde{\sigma}^{-1}$ for some $\widetilde{\sigma} \in I_K$; then in view of [3],

$$\left(\widetilde{\sigma}^+(\alpha), K\right)_{\phi_0} = \widetilde{\sigma} \mid_{K^{(ab)^2}} (\alpha, K)_\phi \, \widetilde{\sigma}^{-1} \mid_{K^{(ab)^2}}$$

for every $\alpha \in \mathfrak{G}^{[2]}(K, \phi)$, where $\widetilde{\sigma}^+ : \mathfrak{G}^{[2]}(K, \phi) \to \mathfrak{G}^{[2]}(K, \phi_0)$ is a certain natural isomorphism of topological groups, which depends on $\widetilde{\sigma}$ (cf. Lemma 5.2 of [3]). It is now easy to verify that the following statements are equivalent for a finite 2-abelian extension E/K and for a given $r \in \mathbb{R}$ satisfying $i - 1 \leq u_{i,j-1} < r \leq u_{i,j} \leq i$:

(i) $\alpha \in M_\phi(E/K)\mathfrak{G}^{[2]}(K, \phi)^{(i,j)}$,
(ii) $(\alpha, E/K)_\phi \in \mathrm{Gal}(E/K)^r$,
(iii) $\widetilde{\sigma}^+(\alpha) \in M_{\phi_0}(E/K)\mathfrak{G}^{[2]}(K, \phi_0)^{(i,j)}$,
(iv) $\left(\widetilde{\sigma}^+(\alpha), K\right)_{\phi_0} \in \mathrm{Gal}(E/K)^r$.

The equivalence of these statements follows from Lemmas 5.2 and 5.3 of [3].

Now, if $\phi \in \Gamma_i$ and $\phi_0 \in \Gamma_j$ for some $i, j \in I$ with $i \neq j$, then consider the canonical topological isomorphism $\mathfrak{i}_{\phi,\phi_0} : \mathfrak{G}^{[2]}(K, \phi) \xrightarrow{\sim} \mathfrak{G}^{[2]}(K, \phi_0)$, which makes the following triangle

$$\begin{tikzcd}
\mathfrak{G}^{[2]}(K, \phi) \arrow[rr, "\mathfrak{i}_{\phi,\phi_0}"] \arrow[dr, "(?, K)_\phi"'] & & \mathfrak{G}^{[2]}(K, \phi_0) \arrow[dl, "(?, K)_{\phi_0}"] \\
& \mathrm{Gal}(K^{(ab)^2}/K) &
\end{tikzcd}$$

commutative. Then,

$$(\mathfrak{i}_{\phi,\phi_0}(\alpha), K)_{\phi_0} = (\alpha, K)_\phi,$$

for every $\alpha \in \mathfrak{G}^{[2]}(K, \phi)$. It is now easy to verify the equivalence of the following statements for a finite 2-abelian extension E/K and for a given $r \in \mathbb{R}$ satisfying $i - 1 \leq u_{i,j-1} < r \leq u_{i,j} \leq i$:

(i) $\alpha \in M_\phi(E/K)\mathfrak{G}^{[2]}(K, \phi)^{(i,j)}$,

(ii) $(\alpha, E/K)_{\phi} \in \mathrm{Gal}(E/K)^{r}$,
(iii) $\mathrm{i}_{\phi,\phi_0}(\alpha) \in M_{\phi_0}(E/K)\mathfrak{G}^{[2]}(K, \phi_0)^{(i,j)}$,
(iv) $\left(\mathrm{i}_{\phi,\phi_0}(\alpha), K\right)_{\phi_0} \in \mathrm{Gal}(E/K)^{r}$.

So, ramification theory of metabelian extensions does not depend on the choice of Lubin–Tate splitting over K. $\square$

5. Conductor $\mathfrak{f}(E/K)$ of a finite metabelian extension E/K

Main reference for this section is [2].

Let E/K be a finite Galois extension with Galois group G, and H a normal subgroup of G. Let $H = \mathrm{Gal}(E/F)$, where F/K is the Galois sub-extension of E/K defined as the fixed field of H.

Lemma 5.1. *For any* $-1 \leq u, t \in \mathbb{R}$, *if* $\mathrm{Gal}(E/K)_u \subseteq \mathrm{Gal}(E/F)$ *and* $\mathrm{Gal}(E/F)_t = 1$, *then* $\mathrm{Gal}(E/K)_{\max\{u,t\}} = 1$.

Proof. If $\sigma \in G_{\max\{u,t\}}$, then $i_{E/K}(\sigma) \geq \max\{u, t\} + 1 \geq u + 1$, and therefore $\sigma \in H$, since $G_u \subseteq H$. Moreover,

$$i_{E/F}(\sigma) = \min_{y \in O_E} \{\widetilde{\boldsymbol{\nu}}_E\,(\sigma(y) - y)\} = i_{E/K}(\sigma) \geq \max\{u, t\} + 1 \geq t + 1.$$

Thus, $\sigma \in H_t$, which proves that $G_{\max\{u,t\}} = 1$. $\square$

PROPOSITION 5.2

For any $-1 \leq v, w, z \in \mathbb{R}$,

(i) $\mathrm{Gal}(E/K)^{v} = 1 \Longrightarrow \mathrm{Gal}(F/K)^{v} = 1$,
(ii) $\mathrm{Gal}(E/K)^{v} = 1 \Longrightarrow \mathrm{Gal}(E/F)^{\psi_{F/K}(v)} = 1$,
(iii) $\mathrm{Gal}(F/K)^{w} = 1$ *and* $\mathrm{Gal}(E/F)^{z} = 1 \Longrightarrow \mathrm{Gal}(E/K)^{\max\{w,\varphi_{F/K}(z)\}} = 1$.

Proof.

(i) For any $-1 \leq v \in \mathbb{R}$, $(G/H)^{v} = G^{v}H/H = 1$, if $G^{v} = 1$.
(ii) For any $-1 \leq v \in \mathbb{R}$, if $G^{v} = G_{\psi_{E/K}(v)} = 1$, then $H_{\psi_{E/K}(v)} = H \cap G_{\psi_{E/K}(v)} = 1$, and therefore $H_{\psi_{E/K}(v)} = H_{\psi_{E/F}(\psi_{F/K}(v))} = H^{\psi_{F/K}(v)} = 1$.
(iii) For any $-1 \leq w \in \mathbb{R}$, if $(G/H)^{w} = G^{w}H/H = 1$, then $G^{w} = G_{\psi_{E/K}(w)} \subseteq H$. For $-1 \leq z \in \mathbb{R}$, $H^{z} = H_{\psi_{E/F}(z)} = 1$. Thus, by Lemma 5.1, $G_{\max\{\psi_{E/K}(w), \psi_{E/F}(z)\}} = 1$. Now, note that

$$\begin{aligned} \varphi_{E/K}(\max\{\psi_{E/K}(w), \psi_{E/F}(z)\}) &= \max\{\varphi_{E/K} \circ \psi_{E/K}(w), \\ &\qquad \varphi_{E/K} \circ \psi_{E/F}(z)\} \\ &= \max\{w, \varphi_{F/K} \circ \varphi_{E/F} \circ \psi_{E/F}(z)\} \\ &= \max\{w, \varphi_{F/K}(z)\} \end{aligned}$$

by the transitivity of the Hasse–Herbrand functions and by the fact that Hasse–Herbrand functions are monotone increasing. Thus, $G^{\max\{w,\varphi_{F/K}(z)\}} = 1$. $\square$

From now on, assume that E is a finite 2-abelian extension over K. Let $E_0 = K \subseteq E_1 \subseteq E_2 = E$ be the 'canonical chain of sub-extensions in E/K' as introduced in the beginning of §4. Hence, $G = \mathrm{Gal}(E/K)$ and $G' = H = \mathrm{Gal}(E/E_1)$ with $F = E_1$. As an immediate consequence of Proposition 5.2, we have the following:

COROLLARY 5.3

For any $-1 \leq v \in \mathbb{R}$,

$$\mathrm{Gal}(E/K)^v = 1 \Longrightarrow v \gneq \max\{\mathrm{ord}_{\mathfrak{p}_K}(\mathfrak{f}(E_1/K)) - 1,\\ \varphi_{E_1/K}(\mathrm{ord}_{\mathfrak{p}_{E_1}}(\mathfrak{f}(E/E_1)) - 1)\}.$$

Proof. By Proposition 5.2 parts (i) and (ii), for any $-1 \leq v \in \mathbb{R}$, if $G^v = 1$, then $(G/H)^v = 1$ and $H^{\psi_{E_1/K}(v)} = 1$. Therefore, $v \gneq \mathrm{ord}_{\mathfrak{p}_K}(\mathfrak{f}(E_1/K)) - 1$ and $\psi_{E_1/K}(v) \gneq \mathrm{ord}_{\mathfrak{p}_{E_1}}(\mathfrak{f}(E/E_1)) - 1$ by Remark 2.3 and by the definition of the conductor of an abelian extension. Thus

$$v \gneq \max\{\mathrm{ord}_{\mathfrak{p}_K}(\mathfrak{f}(E_1/K)) - 1, \varphi_{E_1/K}(\mathrm{ord}_{\mathfrak{p}_{E_1}}(\mathfrak{f}(E/E_1)) - 1)\},$$

which completes the proof. □

Recall that, the conductor $\mathfrak{f}(E/K)$ (in the sense of 'non-abelian' local class field theory) is defined in §2 (look at Definition 2.4) to be the ideal $\mathfrak{p}_K^{[[n_G]]+1}$ in O_K, where n_G is the real number defined by

$$n_G = \inf\{x \in \mathbb{R}_{\geq -1} : G^{x+\delta} = 1, ^\forall \delta > 0\}.$$

By Remark 2.5, it follows that, the number n_G is a jump in the ramification filtration of E/K in upper numbering, and $0 \leq [[n_G]] + 1 \in \mathbb{Z}$ is the minimal integer satisfying $G^{[[n_G]]+1} = 1$. As E/K is in particular a finite 2-abelian extension, it follows from Koch–de Shalit theory (look at Theorem 4.1(v)) that, either $n_G = 0$ or

$$n_G = i - \frac{q_K^i - 1 - j}{q_K^i - q_K^{i-1}} = u_{i,j}$$

for some $1 \leq i \in \mathbb{Z}$ and $j \in \{q_K^{i-1}, \dots, q_K^i - 1\}$.

Remark 5.4. Observe that, for $0 < r \in \mathbb{R}$, there exists a unique $1 \leq i(r) \in \mathbb{Z}$ satisfying $i(r) - 1 < r \leq i(r)$. Let I_r denote the half-open unit interval $(i(r) - 1, i(r)]$ in $\mathbb{R}_{\geq 0}$. Partition I_r into $q_K^{i(r)} - q_K^{i(r)-1}$ sub-intervals of length $(q_K^{i(r)} - q_K^{i(r)-1})^{-1}$ as

$$I_r = \overbrace{\left(u_{i(r)-1,q_K^{i(r)-1}-1}, u_{i(r),q_K^{i(r)-1}}\right]}^{I_r^1} \bigsqcup \cdots \bigsqcup \overbrace{\left(u_{i(r),q_K^{i(r)}-2}, u_{i(r),q_K^{i(r)}-1}\right]}^{I_r^{q_K^{i(r)}-q_K^{i(r)-1}}}.$$

Then, there exists a unique $j(r) \in \{q_K^{i(r)-1}, \dots, q_K^{i(r)} - 1\}$ such that $r \in I_r^{j(r)-q_K^{i(r)-1}+1}$. That is,

$$r \in \begin{cases} I_r^1 = (u_{i(r)-1,q_K^{i(r)-1}-1}, u_{i(r),q_K^{i(r)-1}}], & \text{if } j(r) = q_K^{i(r)-1}; \\ \\ I_r^{j(r)-q_K^{i(r)-1}+1} = (u_{i(r),j(r)-1}, u_{i(r),j(r)}], & \text{if } q_K^{i(r)-1} \lneq j(r) \leq q_K^{i(r)} - 1. \end{cases}$$

□

Following Remark 5.4, there is the following reformulation of Definition 2.4 for finite metabelian extensions.

Lemma 5.5. *Let* (i_0, j_0) *be the minimal pair in the set* $\{(i, j) \in \mathbb{Z}\times\mathbb{Z} : q_K^{i-1} \leq j \leq q_K^{i}-1\}$ *endowed with lexicographic ordering* (*induced from that of* $\mathbb{Z} \times \mathbb{Z}$) *with the property*

$$\mathfrak{G}^{[2]}(K, \phi)^{(i,j)} \subseteq M_\phi(E/K).$$

(*Note that, the pair* (i_0, j_0) *does not depend on the choice of the Lubin–Tate splitting* ϕ *over* K.) *Then,*

$$n_G = \begin{cases} u_{i_0, j_0-1}, & \text{if } q_K^{i_0-1} \lneq j_0 \leq q_K^{i_0} - 1, \\ \\ u_{i_0-1, q_K^{i_0-1}-1}, & \text{if } q_K^{i_0-1} = j_0. \end{cases}$$

The conductor $\mathfrak{f}(E/K)$ *of the finite metabelian extension* E/K (*in the sense of metabelian local class field theory*) *is then given by the ideal*

$$\mathfrak{f}(E/K) = \mathfrak{p}_K^{i_0}$$

in O_K.

Proof. In fact, let (i_0, j_0) be the minimal pair in the set $\{(i, j) \in \mathbb{Z} \times \mathbb{Z} : \ q_K^{i-1} \leq j \leq q_K^{i} - 1\}$ endowed with lexicographic ordering (induced from that of $\mathbb{Z}\times\mathbb{Z}$) with the property

$$\mathfrak{G}^{[2]}(K, \phi)^{(i,j)} \subseteq M_\phi(E/K).$$

Then, by metabelian local class field theory (cf. part (v) of Theorem 4.1), $G^r = 1$ for $u_{i_0,j_0-1} \lneq r$, while $G^{u_{i_0,j_0-1}} \neq 1$, whenever $q_K^{i_0-1} \lneq j_0 \leq q_K^{i_0} - 1$. In case $j_0 = q_K^{i_0-1}$, then $G^r = 1$ for $u_{i_0-1,q_K^{i_0-1}-1} \lneq r$ and $G^{u_{i_0-1,q_K^{i_0-1}-1}} \neq 1$. Therefore,

$$n_G = \begin{cases} u_{i_0, j_0-1}, & \text{if } q_K^{i_0-1} \lneq j_0 \leq q_K^{i_0} - 1, \\ \\ u_{i_0-1, q_K^{i_0-1}-1}, & \text{if } q_K^{i_0-1} = j_0. \end{cases}$$

Now, following Remark 5.4,

$$i(n_G) = \begin{cases} i_0, & \text{if } q_K^{i_0-1} \lneq j_0 \leq q_K^{i_0} - 1, \\ \\ i_0 - 1, & \text{if } q_K^{i_0-1} = j_0, \end{cases}$$

where $n_G \lneq i(n_G)$ in the first case, while $n_G = i(n_G)$ in the second. Therefore, $[[n_G]] = i_0 - 1$, proving that $[[n_G]] + 1 = i_0$ is the minimal integer satisfying $G^{i_0} = 1$. Thus,

$$\mathfrak{f}(E/K) = \mathfrak{p}_K^{i_0}.$$

Note that, by Remark 4.4, (i_0, j_0) (and thereby $n_G, [[n_G]], \ldots$) does not depend on the choice of the Lubin–Tate splitting ϕ over K. □

Observe that, if the pair (i_0, j_0) is chosen as in Lemma 5.5, then by Lemma 4.3, $U^{i_0}(K) \subseteq N(E_1/K)$. Therefore, $\mathrm{Gal}(E_1/K)^{i_0} = 1$, which is a special case of Proposition 5.2(i).

Now, assume that E/K is a 'strict' 2-abelian extension with Galois group G as introduced in the beginning of §4. Let $G^\bullet$ be a maximal abelian normal subgroup of G with $G' \subseteq G^\bullet$. As further notation, let $E^\bullet = E^{G^\bullet}$ and $E^{n_G} = E^{G^{n_G}}$.

$$\begin{tikzcd}
 & E \arrow[d, dash] & \\
 & E^\bullet E^{n_G} \arrow[dl, dash] \arrow[dr, dash] & \\
E^\bullet \arrow[uur, bend left, leftrightarrow, "\text{abelian ext.}"] \arrow[dr, dash] \arrow[ddr, bend right, leftrightarrow, "\text{abelian ext.}"'] & & E^{n_G}. \arrow[dl, dash] \\
 & E^\bullet \cap E^{n_G} \arrow[d, dash] & \\
 & K &
\end{tikzcd}$$

Note that, $E^\bullet E^{n_G} \subsetneq E$. In fact, if $E^\bullet E^{n_G} = E$, then the extensions $E\big/(E^\bullet \cap E^{n_G})$ and $(E^\bullet \cap E^{n_G})\big/K$ are abelian, and therefore $\mathrm{Gal}(E\big/(E^\bullet \cap E^{n_G}))$ is an abelian normal subgroup of G containing G' and $G^\bullet$. Now, the maximality of $G^\bullet$ yields that $G^\bullet = \mathrm{Gal}(E\big/(E^\bullet \cap E^{n_G}))$. Hence, $E^\bullet \cap E^{n_G} = E^\bullet$ and so $E^\bullet \subseteq E^{n_G}$. This leads to a contradiction, since $E = E^{n_G}$ and $G^{n_G} = 1$. Therefore, $E^\bullet E^{n_G} \subsetneq E$. This result also proves that $\mathrm{Gal}(E/E^\bullet E^{n_G})$ is a non-trivial group. So, $G^\bullet \cap G^{n_G}$ is a non-trivial intersection.

PROPOSITION 5.6

Keeping the above notation,

$$\psi_{E^\bullet/K}(n_G) = n_{G^\bullet},$$

where $n_{G^\bullet} \in \mathbb{Z}$.

Proof. Note that

$$\begin{aligned}(G^\bullet)^\mu = (G^\bullet)_{\psi_{E/E^\bullet}(\mu)} &= G^\bullet \cap G_{\psi_{E/E^\bullet}(\mu)} \\ &= G^\bullet \cap G^{\varphi_{E/K}(\psi_{E/E^\bullet}(\mu))} = G^\bullet \cap G^{\varphi_{E^\bullet/K}(\mu)}.\end{aligned}$$

Now, let $\mu^\bullet \in \mathbb{R}_{\geq -1}$ such that $\varphi_{E^\bullet/K}(\mu^\bullet) = n_G$. Then, $(G^\bullet)^{\mu^\bullet} = G^\bullet \cap G^{n_G} \neq 1$ and for any $0 < \delta_0 \in \mathbb{R}$, there exists $0 < \delta \in \mathbb{R}$ such that $(G^\bullet)^{\mu^\bullet + \delta_0} = G^\bullet \cap G^{n_G + \delta} = 1$, proving that $\mu^\bullet = \psi_{E^\bullet/K}(n_G) = n_{G^\bullet}$. Finally, since $G^\bullet$ is abelian, by Hasse–Arf theorem, $n_{G^\bullet} \in \mathbb{Z}$. □

6. Computation of $\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho)$ in case $E^{\ker(\rho)}/K$ is a metabelian extension

Main references for this section are [1] and [2].

Now, we are in a position to prove the main theorem (Theorem 1.2). As before, let E/K be a finite Galois extension, and $\rho : \mathrm{Gal}(E/K) \to GL(V)$ be an irreducible representation of the Galois group $\mathrm{Gal}(E/K) =: G$ of the extension E/K. Note that, the representation

$$\widetilde{\rho} : \mathrm{Gal}(E^{\ker}(\rho)/K) \xrightarrow{\sim} G/\ker(\rho) \to GL(V)$$

is an irreducible faithful representation of $\mathrm{Gal}(E^{\ker(\rho)}/K)$. Now, if $\widetilde{g} : G/\ker(\rho) \to \mathbb{C}$ is any class function on $G/\ker(\rho)$, and g is the corresponding class function on G defined by

$$g : G \xrightarrow{\text{canonical map}} G/\ker(\rho) \xrightarrow{\widetilde{g}} \mathbb{C},$$

then $f_{\mathrm{Artin}}(\widetilde{g}) = f_{\mathrm{Artin}}(g)$. In particular, for $g = \chi_\rho$, we have $\widetilde{g} = \chi_{\widetilde{\rho}}$. Therefore,

$$f_{\mathrm{Artin}}(\chi_\rho) = f_{\mathrm{Artin}}(\chi_{\widetilde{\rho}}).$$

Hence, in order to compute explicitly the local Artin conductor $\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho)$ of an irreducible representation $\rho : G \to GL(V)$ in the case $E^{\ker(\rho)}/K$ is a metabelian extension, we have to study irreducible faithful representations of finite metabelian groups. This is the content of the following digression.

Digression: Representation theory of finite metabelian groups. The main reference for this part is [1].

Theorem. (*Basmaji*). *Let M be a finite metabelian group, and assume that M has an irreducible faithful representation. Let $M^\bullet$ denote a maximal abelian normal subgroup of M containing M'. Then,*

(i) *all irreducible faithful representations $\rho : M \to GL(V)$ have the same degree $d = (M : M^\bullet)$. Let $\rho : M \to GL(V)$ be an irreducible faithful representation of M in V (over $\mathbb{C}$). Then,*

(ii) *$\rho : M \to GL(V)$ is equivalent to the representation* $\mathrm{Ind}_{M^\bullet}^{M}\omega : M \to GL(\mathrm{Ind}_{M^\bullet}^{M} V_\omega)$ *induced from a one-dimensional representation $\omega : M^\bullet \to \mathbb{C}^\times$ of $M^\bullet$ in $\mathbb{C} = V_\omega$.*

Of course, the assumption on the existence of faithful irreducible representations of M seems to be a strong assumption, but there are many metabelian groups with this property. In fact, if M is metacyclic, then M has faithful irreducible representations (over $\mathbb{C}$) iff $Z(M)$ is a cyclic group. □

Now, for simplicity, fix the following notation: let $F = E^{\ker(\rho)}$, $M = \mathrm{Gal}(F/K)$, and $H = \mathrm{Gal}(F/F_1)$ where $F_1 = F \cap K^{ab}$, the maximal abelian sub-extension in F over K. Since M is assumed to be a metabelian group, $H = M'$.

Consider the irreducible faithful representation $\widetilde{\rho} : M \to GL(V)$. Then, there exists a one-dimensional representation $\omega : M^\bullet \to \mathbb{C}^\times$, such that

$$\widetilde{\rho} \simeq \mathrm{Ind}_{M^\bullet}^{M}\omega,$$

where $M^\bullet$ is a maximal abelian normal subgroup of M containing M'. Set $M^\bullet = \mathrm{Gal}(F/F^\bullet)$, where $F^\bullet/K$ is a minimal abelian sub-extension of F/K. Thus, by Frobenius reciprocity law

$$\begin{aligned} f_{\mathrm{Artin}}(\chi_{\widetilde{\rho}}) &= (\chi_{\widetilde{\rho}}, a_M)_M = (\chi_{\mathrm{Ind}_{M^\bullet}^{M}\omega}, a_M)_M \\ &= (\mathrm{Ind}_{M^\bullet}^{M}\chi_\omega, a_M)_M = (\chi_\omega, \mathrm{Res}_{M^\bullet}(a_M))_{M^\bullet}, \end{aligned}$$

and by eq. (7),

$$\begin{aligned} f_{\mathrm{Artin}}(\chi_{\tilde{\rho}}) &= (\chi_\omega, \boldsymbol{\nu}_K(\mathfrak{d}(F^\bullet/K))r_{M^\bullet} + f(F^\bullet/K)a_{M^\bullet})_{M^\bullet} \\ &= \boldsymbol{\nu}_K(\mathfrak{d}(F^\bullet/K)) + f(F^\bullet/K) f_{\mathrm{Artin}}(\chi_\omega), \end{aligned}$$

where $\mathfrak{d}(F^\bullet/K)$ is the discriminant and $f(F^\bullet/K)$ the residue-class degree of the extension $F^\bullet/K$. Note that, $\boldsymbol{\nu}_K(\mathfrak{d}(F^\bullet/K)) = f(F^\bullet/K)\boldsymbol{\nu}_{F^\bullet}(\mathfrak{D}(F^\bullet/K))$, where $\mathfrak{D}(F^\bullet/K)$ denotes the different of the extension $F^\bullet/K$. Thus,

$$f_{\mathrm{Artin}}(\chi_{\tilde{\rho}}) = f(F^\bullet/K)[\boldsymbol{\nu}_{F^\bullet}(\mathfrak{D}(F^\bullet/K)) + f_{\mathrm{Artin}}(\chi_\omega)]. \tag{12}$$

It is well-known that (for instance, see §2 of chapter III of [9]),

$$\boldsymbol{\nu}_{F^\bullet}(\mathfrak{D}(F^\bullet/K)) = \sum_{i=0}^{\infty}(\#(M/M^\bullet)_i - 1).$$

Thus, combining with eq. (12)

$$f_{\mathrm{Artin}}(\chi_{\tilde{\rho}}) = f(F^\bullet/K)\left[\sum_{i=0}^{\infty}(\#(M/M^\bullet)_i - 1) + f_{\mathrm{Artin}}(\chi_\omega)\right].$$

By Corollary 3.3 and Proposition 5.6,

$$f_{\mathrm{Artin}}(\chi_\omega) = \mathrm{ord}_{\mathfrak{p}_{F^\bullet}}(\mathfrak{f}(F/F^\bullet)) = n_{M^\bullet} + 1 = \psi_{F^\bullet/K}(n_M) + 1,$$

since $\mathrm{Res}_{M^\bullet}(\rho) = \mathrm{Res}_{M^\bullet}\left(\mathrm{Ind}_{M^\bullet}^{M}\omega\right) = \sum_{\delta\in\mathcal{R}(M^\bullet\backslash M)}\omega_\delta$ (where $\omega_\delta : M^\bullet \to \mathbb{C}^\times$ is the one-dimensional representation of $M^\bullet$ defined by $\omega_\delta : x \mapsto \omega(\delta x \delta^{-1})$ for $x \in M^\bullet$ and for $\delta \in \mathcal{R}(M^\bullet\backslash M)$ a fixed complete system of representatives of $M^\bullet\backslash M$) is an injection. Moreover, $(M/M^\bullet)^v = (M/M^\bullet)_{\psi_{F^\bullet/K}(v)} = 1$ if $v \gneq n_M$ by Proposition 5.2(i). Therefore (note that, $\psi_{F^\bullet/K}(n_M) \in \mathbb{Z}$ by Proposition 5.6),

$$\begin{aligned} f_{\mathrm{Artin}}(\chi_{\tilde{\rho}}) = f(F^\bullet/K)\Bigg[&\sum_{i=0}^{\psi_{F^\bullet/K}(n_M)} \#(M/M^\bullet)_i \\ &\qquad -(\psi_{F^\bullet/K}(n_M) + 1) + f_{\mathrm{Artin}}(\chi_\omega)\Bigg] \\ = f(F^\bullet/K)&\sum_{i=0}^{\psi_{F^\bullet/K}(n_M)} \#(M/M^\bullet)_i. \end{aligned}$$

Now, by eq. (5),

$$\begin{aligned} f_{\mathrm{Artin}}(\chi_{\tilde{\rho}}) &= f(F^\bullet/K)\#(M/M^\bullet)_0\left[\varphi_{F^\bullet/K}\left(\psi_{F^\bullet/K}(n_M)\right) + 1\right] \\ &= [F^\bullet : K](n_M + 1), \end{aligned}$$

since $(M/M^\bullet)_0$ is the inertia group of $M/M^\bullet$. Thus, we have the desired result, namely

$$\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho) = \mathfrak{p}_K^{[F^\bullet:K](n_M+1)},$$

which, combining with the results of §5 and Corollary 3.5, completes the proof of Theorem 1.2.[3]

Remark 6.1. Note that, by Corollary 3.3, $f_{\text{Artin}}(\chi_\rho) \in \mathbb{Z}_{\geq 0}$ for any given representation $\rho : \text{Gal}(E/K) \to GL(V)$. Thus, following the notation of Theorem 1.2, the exponent $[E^{\ker(\rho)^\bullet} : K](n_{\text{Gal}(E^{\ker(\rho)}/K)} + 1)$ of the Artin conductor $\mathfrak{f}_{\text{Artin}}(\chi_\rho)$ of the representation (ρ, V) of $\text{Gal}(E/K)$ is a non-negative integer. Therefore, setting

$$n_{\text{Gal}(E^{\ker(\rho)}/K)} = \frac{\text{num}(n_{\text{Gal}(E^{\ker(\rho)}/K)})}{\text{den}(n_{\text{Gal}(E^{\ker(\rho)}/K)})},$$

where $\text{num}(n_{\text{Gal}(E^{\ker(\rho)}/K)}), \text{den}(n_{\text{Gal}(E^{\ker(\rho)}/K)}) \in \mathbb{Z}$, $0 < \text{den}(n_{\text{Gal}(E^{\ker(\rho)}/K)})$, and are relatively prime, it then follows that $\text{den}(n_{\text{Gal}(E^{\ker(\rho)}/K)}) \mid [E^{\ker(\rho)^\bullet} : K]$. That is, the denominator of $n_{\text{Gal}(E^{\ker(\rho)}/K)}$ divides $\dim_{\mathbb{C}}(V)$. □

7. Artin representation $A_{\text{Gal}(E/K)}$ of Gal(*E/K*) (continuation)

Main reference for this section is [11].

From now on, unless otherwise stated (as in Notation 7.1, in the following digression and in Remark 7.2), we assume that E/K is a finite metabelian extension of the local field K with Galois group $\text{Gal}(E/K) = G$. In this section, we prove Theorem 1.4. That is, we construct a 'natural' Artin representation $A_G : G \to GL(V_{A_G})$ of G over $\mathbb{C}$, utilizing Theorem 1.2, which settles Problem 1.3 in the case G is a metabelian Galois group.

Notation 7.1. For a finite group G, let $\mathcal{B}(G) = \{\chi\}$ denote the set of all irreducible characters of G, which is a finite set of $r(G)$ elements, where $r(G)$ denotes the number of conjugacy classes in G. If there is no fear of confusion, we let $\mathcal{B}(G) = \mathcal{B}$. Let ρ_χ be an irreducible representation of G corresponding to χ (so in our notation, $\chi = \chi_{\rho_\chi}$), and $V_{\rho_\chi} =: V_\chi$ the representation space of G corresponding to the representation ρ_χ of G over $\mathbb{C}$, for $\chi \in \mathcal{B}$. □

The irreducible representation $\rho_\chi : G \to GL(V_\chi)$ of G has metabelian image for $\chi \in \mathcal{B}$ since G is assumed to be metabelian. Let $E_\chi = E^{\ker(\rho_\chi)}$, $M_\chi = G/\ker(\rho_\chi) = \text{Gal}(E_\chi/K)$ which is a metabelian group, and let $M_\chi^\bullet$ denotes any maximal abelian normal subgroup of M_χ containing the 1st commutator subgroup M'_χ of M_χ for $\chi \in \mathcal{B}$. Then, the triangle

$$\begin{tikzcd}
G \arrow[rr, "\rho_\chi"] \arrow[dr, "\pi_\chi"'] & & GL(V_\chi) \\
& M_\chi \arrow[ur, "\widetilde{\rho}_\chi"'] &
\end{tikzcd}$$

---

[3] Let π denote a prime element of $O_{K_d^{nr}}$, where K_d^{nr} denotes the unique unramified extension of degree d over K. Consider the abelian extension $K_{d,i}$ of K which is class field to $\langle N_{K_d^{nr}/K}(\pi)\rangle U^i(K)$, and let $K_d^{(i,j)}$ be the abelian extension of $K_{d,i}$ which is class field to $U^j(K_{d,i})$. Note that, if E/K is a finite metabelian extension, then there exists an $i \in \mathbb{Z}_{\geq 0}$ and an integer $0 \leq j < q_K^i$ such that $E \subseteq K_d^{(i,j)}$ (cf. §3.2 of [8]), which can be viewed as the metabelian Kronecker–Weber theorem. Now, following the notation of Theorem 1.2, $E^{\ker(\rho)}/K$ is a metabelian extension, and so $E^{\ker(\rho)} \subseteq K_d^{(i,j)}$ as in the above discussion. If, in particular, $E^{\ker(\rho)} = K_d^{(i,j)}$, then $n_{\text{Gal}(E^{\ker(\rho)}/K)} = \max\left(i-1, i - \frac{q^i - j}{q^i - q^{i-1}}\right)$ (cf. §3.3 of [8]), which yields a more explicit description of $f_{\text{Artin}}(\chi_\rho)$.

is commutative, where $\pi_\chi : G \to M_\chi$ is the canonical mapping and $\widetilde{\rho}_\chi : M_\chi \to GL(V_\chi)$ is the irreducible faithful representation of M_χ over $\mathbb{C}$ attached to the irreducible representation $\rho_\chi : M_\chi \to GL(V_\chi)$. So, by Basmaji's theory

$$\widetilde{\rho}_\chi \simeq \mathrm{Ind}_{M_\chi^\bullet}^{M_\chi} \omega_\chi,$$

where $\omega_\chi : M_\chi^\bullet \to \mathbb{C}^\times$ is a one-dimensional representation of $M_\chi^\bullet$ satisfying

$$\mathrm{Inert}(\omega_\chi) = \{\delta \in M_\chi : \omega_{\chi,\delta} = \omega_\chi\} = M_\chi^\bullet,$$

by Mackey irreducibility criterion and

$$\bigcap_{\delta \in \mathcal{R}(M_\chi^\bullet \backslash M_\chi)} \ker(\omega_{\chi,\delta}) = \langle 1_{M_\chi^\bullet} \rangle,$$

by the fact that $\ker(\mathrm{Ind}_{M_\chi^\bullet}^{M_\chi} \omega_\chi) = \bigcap_{\delta \in \mathcal{R}(M_\chi^\bullet \backslash M_\chi)} \ker(\omega_{\chi,\delta})$.

Digression: On induced representations. Here, all through this digression, G denotes an arbitrary group, and N a normal subgroup of G. For simplicity, we set $\overline{G} := G/N$.

(i) Let $\xi : N \to GL(V_\xi)$ be any finite-dimensional representation of N over $\mathbb{C}$. Recall that, the G-module $\mathrm{Ind}_N^G V_\xi$ is defined by $\mathrm{Ind}_N^G V_\xi = \{f : G \to V_\xi : f(\tau x) = \xi(\tau) f(x), \forall \tau \in N, \forall x \in G\}$, and the G-action on $\mathrm{Ind}_N^G V_\xi$ by $(\sigma, f) \mapsto \underline{\sigma}(f)$, where $\underline{\sigma}(f)(x) := f(x\sigma)$ for every $\sigma, x \in G$ and $f \in \mathrm{Ind}_N^G V_\xi$.

Now, assume that $(G : N) < \infty$. Let $\mathcal{R}(N\backslash G) = \{\delta_1 = 1_G, \ldots, \delta_m\}$ be a complete system of representatives of $N\backslash G$, and $B = \{\varepsilon_1, \ldots, \varepsilon_n\}$ a basis of V_ξ over $\mathbb{C}$. Note that, any $f \in \mathrm{Ind}_N^G V_\xi$ is completely determined by the values $f(\delta_1), \ldots, f(\delta_m)$ in V_ξ. In fact, for $x \in G$, there exists a unique $\tau \in N$ and a unique $\delta \in \mathcal{R}(N\backslash G)$ satisfying $x = \tau\delta$. Thus, $f(x) = f(\tau\delta) = \xi(\tau) f(\delta)$, proving that f is completely determined by the values $f(\delta_1), \ldots, f(\delta_m)$. For $1 \le i \le m$ and $1 \le j \le n$, define $f_{ij} \in \mathrm{Ind}_N^G V_\xi$ by

$$f_{ij}(\delta_i) = 0; \ldots; f_{ij}(\delta_i) = \varepsilon_j; \ldots; f_{ij}(\delta_m) = 0.$$

Then, $\{f_{ij} : 1 \le i \le m; 1 \le j \le n\}$ is a basis of $\mathrm{Ind}_N^G V_\xi$ over $\mathbb{C}$. So,

$$\dim_{\mathbb{C}}(\mathrm{Ind}_N^G V_\xi) = mn = (G : N)\dim_{\mathbb{C}} V_\xi.$$

(ii) Assume that $(G : N) < \infty$. Then, for any finite-dimensional representation $\xi : N \to GL(V_\xi)$ of N over $\mathbb{C}$,

$$\ker\left(\mathrm{Ind}_N^G \xi\right) = \bigcap_{s \in \mathcal{R}(N\backslash G)} \ker(\xi_s).$$

In fact, if $\sigma \in G$ such that $(\mathrm{Ind}_N^G \xi)(\sigma) = \underline{\sigma} = 1_{\mathrm{Ind}_N^G V_\xi}$, then $\underline{\sigma}(f) = f$ for every $f \in \mathrm{Ind}_N^G V_\xi$. Hence, in particular, $\underline{\sigma}(f_{ij}) = f_{ij}$ for every $1 \le i \le (G : N) =: m$ and $1 \le j \le \dim_{\mathbb{C}} V_\xi =: n$. Therefore, in view of part (i), $\underline{\sigma}(f_{ij})(\delta_k) = f_{ij}(\delta_k)$ for every $1 \le k \le m$. Now,

$$\underline{\sigma}(f_{ij})(\delta_k) = f_{ij}(\delta_k \sigma),$$

and setting $\delta_k\sigma = \sigma_k\delta_{k'}$ for $\sigma_k \in N$ and $\delta_{k'} \in \mathcal{R}(N\backslash G)$ (both of them unique!),

$$\begin{aligned}\underline{\sigma}(f_{ij})(\delta_k) &= f_{ij}(\delta_k\sigma) = f_{ij}(\sigma_k\delta_{k'})\\ &= \xi(\sigma_k)f_{ij}(\delta_{k'}) = \begin{cases}\xi(\sigma_k)(\varepsilon_j), & \text{if } k' = i,\\ \xi(\sigma_k)(0) = 0, & \text{if } k' \neq i.\end{cases}\end{aligned}$$

On the other hand,

$$f_{ij}(\delta_k) = \begin{cases}\varepsilon_j, & \text{if } k = i,\\ 0, & \text{if } k \neq i.\end{cases}$$

Thus, $i' = i$ and $\xi(\sigma_i)(\varepsilon_j) = \varepsilon_j$. Hence, $\delta_i\sigma\delta_i^{-1} = \sigma_i$, and $\xi(\delta_i\sigma\delta_i^{-1})(\varepsilon_j) = \varepsilon_j$. Thus, $\xi_{\delta_i}(\sigma) = \xi(\delta_i\sigma\delta_i^{-1}) = 1_{V_\xi}$, proving that $\sigma \in \bigcap_{1\leq i\leq m}\ker(\xi_{\delta_i})$. For the reverse inclusion, if $\sigma \in \ker(\omega_{\delta_i})$ for every $1 \leq i \leq m$, then $\omega(\delta_i\sigma\delta_i^{-1})(\varepsilon_j) = \varepsilon_j$ for every $1 \leq i \leq m$ and $1 \leq j \leq n$. Let $\sigma_i = \delta_i\sigma\delta_i^{-1}$, for $1 \leq i \leq m$. Then, $\delta_i\sigma = \sigma_i\delta_i$ and

$$\underline{\sigma}(f_{ij})(\delta_i) = f_{ij}(\delta_i\sigma) = f_{ij}(\sigma_i\delta_i) = \xi(\sigma_i)\varepsilon_j = \varepsilon_j = f_{ij}(\delta_i),$$

and for $k \neq i$,

$$\underline{\sigma}(f_{ij})(\delta_k) = f_{ij}(\delta_k\sigma) = f_{ij}(\sigma_k\delta_k) = \xi(\sigma_k)0 = 0 = f_{ij}(\delta_k),$$

for every $1 \leq i \neq k \leq m$ and $1 \leq j \leq n$. So, $\underline{\sigma}(f_{ij}) = f_{ij}$ for every i and j, proving that $\sigma \in \ker(\mathrm{Ind}_N^G\xi)$.

(iii) Now, consider a representation $\bar{\rho} : \bar{G} \to GL(V)$ of $\bar{G}$ in V over $\mathbb{C}$. Then, there exists a representation $\rho : G \to GL(V)$ of G in V over $\mathbb{C}$, which is defined by $\rho : G \xrightarrow{\pi} \bar{G} \xrightarrow{\bar{\rho}} GL(V)$, where $\pi : G \to \bar{G}$ denotes the canonical map (so $N \subseteq \ker(\rho)$). If

$$\bar{\rho} \simeq \mathrm{Ind}_{\bar{H}}^{\bar{G}}\omega$$

for some normal subgroup $\bar{H}$ of $\bar{G}$ satisfying $(\bar{G} : \bar{H}) < \infty$ and for some representation $\omega : \bar{H} \to GL(V_\omega)$ of $\bar{H}$ in V_ω over $\mathbb{C}$, then

$$\rho \simeq \mathrm{Ind}_{\pi^{-1}(\bar{H})}^{G}(\omega \circ \pi \mid_{\pi^{-1}(\bar{H})}).$$

In fact, it suffices to prove that $(\mathrm{Ind}_{\bar{H}}^{\bar{G}}\omega) \circ \pi \simeq \mathrm{Ind}_{\pi^{-1}(\bar{H})}^{G}(\omega \circ \pi \mid_{\pi^{-1}(\bar{H})})$. Note that, the canonical map $\pi : G \to \bar{G}$ defines a linear isomorphism $\pi_* : \mathrm{Ind}_{\bar{H}}^{\bar{G}}V_\omega \xrightarrow{\sim} \mathrm{Ind}_{\pi^{-1}(\bar{H})}^{G}V_{\omega\circ\pi|_{\pi^{-1}(\bar{H})}}$ defined by $\pi_* : f \mapsto f\circ\pi$ for every $f \in \mathrm{Ind}_{\bar{H}}^{\bar{G}}V_\omega = \{f : \bar{G} \to V_\omega : f(\tau x) = \omega(\tau)f(x), \forall\tau \in \bar{H}, \forall x \in \bar{G}\}$. Note that, surjectivity follows from the assumption that $(\bar{G} : \bar{H}) < \infty$ and by observing that $(G : \pi^{-1}(\bar{H})) = (\bar{G} : \bar{H})$. Moreover,

$$\pi_* \circ [(\mathrm{Ind}_{\bar{H}}^{\bar{G}}\omega) \circ \pi](g) = [\mathrm{Ind}_{\pi^{-1}(\bar{H})}^{G}(\omega \circ \pi \mid_{\pi^{-1}(\bar{H})})](g) \circ \pi_*$$

for every $g \in G$. In fact, choose $f \in \mathrm{Ind}_{\bar{H}}^{\bar{G}}V_\omega$. Then, computing the left-hand side

$$\{\pi_* \circ [(\mathrm{Ind}_{\bar{H}}^{\bar{G}}\omega) \circ \pi](g)\}(f) = \{[(\mathrm{Ind}_{\bar{H}}^{\bar{G}}\omega) \circ \pi](g)(f)\} \circ \pi,$$

and therefore

$$\begin{aligned}\{[(\mathrm{Ind}_{\bar{H}}^{\bar{G}}\omega)\circ\pi](g)(f)\circ\pi\}(x) &= [(\mathrm{Ind}_{\bar{H}}^{\bar{G}}\omega)\circ\pi](g)(f)(\pi(x))\\ &= f(\pi(x)\pi(g))\end{aligned}$$

for every $x \in G$. Now, computing the right-hand side

$$\begin{aligned}&\{[\mathrm{Ind}_{\pi^{-1}(\bar{H})}^{G}(\omega\circ\pi \mid_{\pi^{-1}(\bar{H})})](g)\circ\pi_*\}(f)\\ &\quad= [\mathrm{Ind}_{\pi^{-1}(\bar{H})}^{G}(\omega\circ\pi \mid_{\pi^{-1}(\bar{H})})](g)(f\circ\pi),\end{aligned}$$

and therefore

$$\begin{aligned}\{[\mathrm{Ind}_{\pi^{-1}(\bar{H})}^{G}(\omega\circ\pi \mid_{\pi^{-1}(\bar{H})})](g)(f\circ\pi)\}(x) &= (f\circ\pi)(xg)\\ &= f(\pi(x)\pi(g))\end{aligned}$$

for every $x \in G$. Thus

$$\begin{aligned}&\{\pi_*\circ[(\mathrm{Ind}_{\bar{H}}^{\bar{G}}\omega)\circ\pi](g)\}(f)\\ &\quad= \{[\mathrm{Ind}_{\pi^{-1}(\bar{H})}^{G}(\omega\circ\pi \mid_{\pi^{-1}(\bar{H})})](g)\circ\pi_*\}(f)\end{aligned}$$

for every $f \in \mathrm{Ind}_{\bar{H}}^{\bar{G}} V_\omega$. That is

$$(\mathrm{Ind}_{\bar{H}}^{\bar{G}}\omega)\circ\pi \simeq \mathrm{Ind}_{\pi^{-1}(\bar{H})}^{G}(\omega\circ\pi \mid_{\pi^{-1}(\bar{H})}),$$

which completes the proof. □

Hence,

$$\rho_\chi = \widetilde{\rho}_\chi\circ\pi_\chi \simeq \mathrm{Ind}_{\pi_\chi^{-1}(M_\chi^\bullet)}^{G}(\omega_\chi\circ\pi_\chi \mid_{\pi_\chi^{-1}(M_\chi^\bullet)}) \tag{13}$$

for $\chi \in \mathcal{B}$.

Remark 7.2. For an arbitrary group G, let $\mathrm{Irr}(G)$ (resp. $\mathrm{Irr}(G, N)$ for a normal subgroup N of G) denote the set of all irreducible finite-dimensional representations of G over $\mathbb{C}$ (resp. the set of all irreducible finite-dimensional representations of G over $\mathbb{C}$ with kernel N). Now, let $\mathcal{I}(G)$ denote the set $[\mathrm{Irr}(G)/\simeq]$ of $\simeq$-equivalence classes $[\rho]$ of $\rho \in \mathrm{Irr}(G)$. Therefore, $\mathcal{I}(G) = \{[\rho_\chi] : \chi \in \mathcal{B}(G)\}$. Observe that, for $\rho, \psi \in \mathrm{Irr}(G)$, if $\rho \simeq \psi$, then $\ker(\rho) = \ker(\psi)$. Let $\ker([\rho]) = \ker(\rho)$. For a normal subgroup N of G, let $\mathcal{I}(G, N) = [\mathrm{Irr}(G, N)/\simeq]$. So, the set $\mathcal{I}(G, N)$ consists of those $[\rho]$ in $\mathcal{I}(G)$ with kernel N. In particular, $\mathcal{I}(G, \langle 1_G\rangle)$ is the set of $\simeq$-equivalence classes $[\rho]$ of $\rho \in \mathrm{Irr}(G, \langle 1_G\rangle)$ (= the set of all irreducible faithful representations of G over $\mathbb{C}$). Note that

$$\mathrm{Irr}(G) = \bigsqcup_N \mathrm{Irr}(G, N), \tag{14}$$

and

$$\mathcal{I}(G) = \bigsqcup_N \mathcal{I}(G, N), \tag{15}$$

where N runs over all normal subgroups of G. Note that, it may turn out that $\mathrm{Irr}(G, N) = \varnothing$ and $\mathcal{I}(G, N) = \varnothing$ for some $N \lhd G$.

Observe that, for any $N \lhd G$, the canonical map $\pi_N : G \to G/N$ induces a bijection

$$\pi_{N,*} : \mathcal{I}(G/N, \langle 1_{G/N} \rangle) \to \mathcal{I}(G, N)$$

defined by

$$\pi_{N,*} : [\widetilde{\rho}] \mapsto [\widetilde{\rho} \circ \pi_N]$$

for every $\widetilde{\rho} \in \mathcal{I}(G/N, \langle 1_{G/N} \rangle)$. □

For $N \lhd G$, the quotient group G/N is a metabelian group. Let $(G/N)^{\bullet}$ denote a (fixed) maximal abelian normal subgroup of G/N containing the 1st commutator subgroup $(G/N)'$ of G/N. Then, since irreducible representations of an abelian group are all one-dimensional,

$$\mathrm{Irr}((G/N)^{\bullet}) = \{\omega : (G/N)^{\bullet} \to \mathbb{C}^{\times}\},$$

and

$$\mathcal{I}((G/N)^{\bullet}) = \{\omega : (G/N)^{\bullet} \to \mathbb{C}^{\times}\},$$

as for one-dimensional representations $(\omega_1, \mathbb{C})$ and $(\omega_2, \mathbb{C})$ of $(G/N)^{\bullet}$, $\omega_1 \simeq \omega_2$ if and only if $\omega_1 = \omega_2$. Now, consider the subset $\mathcal{I}((G/N)^{\bullet})^0$ of $\mathcal{I}((G/N)^{\bullet})$ defined by

$$\mathcal{I}((G/N)^{\bullet})^0 = \left\{ \omega : (G/N)^{\bullet} \to \mathbb{C}^{\times} : \mathrm{Inert}(\omega) = (G/N)^{\bullet}, \right.$$
$$\left. \bigcap_{\delta \in \mathcal{R}(M^{\bullet} \backslash M)} \ker(\omega_{\delta}) = \langle 1_{G/N} \rangle \right\},$$

where M denotes here the quotient group G/N and $\mathcal{R}(M^{\bullet} \backslash M)$ denotes a complete system of representatives of $M^{\bullet} \backslash M$. For $\delta \in \mathcal{R}(M^{\bullet} \backslash M)$, the representation $\omega_{\delta} : M^{\bullet} \to \mathbb{C}^{\times}$ of $M^{\bullet}$ is defined by $\omega_{\delta} : x \mapsto \omega(\delta x \delta^{-1})$ for $x \in M^{\bullet}$.

Lemma 7.3. *The mapping*

$$\mathrm{Ind}_{(G/N)^{\bullet}}^{G/N} : \mathcal{I}((G/N)^{\bullet})^0 \to \mathcal{I}(G/N, \langle 1_{G/N} \rangle) \tag{16}$$

defined by

$$\mathrm{Ind}_{(G/N)^{\bullet}}^{G/N} : \omega \mapsto \left[\mathrm{Ind}_{(G/N)^{\bullet}}^{G/N} \omega \right],$$

for every $\omega \in \mathcal{I}((G/N)^{\bullet})^0$ *is a surjection. Define an equivalence relation '$\sim$' on* $\mathcal{I}((G/N)^{\bullet})^0$ *by* $\omega_1 \sim \omega_2$ *if and only if* $\omega_1 = \omega_{2,\delta}$ *for some* $\delta \in \mathcal{R}(M^{\bullet} \backslash M)$, *where* $M = G/N$. *Then,*

(i) *for* $\omega_1, \omega_2 \in \mathcal{I}((G/N)^{\bullet})^0$,

$$\mathrm{Ind}_{(G/N)^{\bullet}}^{G/N} \omega_1 \simeq \mathrm{Ind}_{(G/N)^{\bullet}}^{G/N} \omega_2 \Longleftrightarrow \omega_1 \sim \omega_2,$$

(ii) *the surjection eq.* (16) *induces a well-defined bijection*

$$[\mathcal{I}((G/N)^{\bullet})^0/\sim] =: \mathcal{V}_N \to \mathcal{I}(G/N, \langle 1_{G/N}\rangle)$$

defined by

$$[\omega]_{\sim} \mapsto \left[\mathrm{Ind}_{(G/N)^{\bullet}}^{G/N}\omega\right],$$

where $[\omega]_{\sim}$ *denotes the* $\sim$*-equivalence class of* $\omega \in \mathcal{I}((G/N)^{\bullet})^0$.

Proof. Note that, the surjectivity of this map follows immediately from Basmaji's theory. It is also straightforward to observe that '$\sim$' is an equivalence relation on $\mathcal{I}((G/N)^{\bullet})^0$.

(i) The proof of this part follows from the Frobenius reciprocity law. In fact, for $\omega_1, \omega_2 \in \mathcal{I}((G/N)^{\bullet})^0$, note that

$$\mathrm{Ind}_{(G/N)^{\bullet}}^{G/N}\omega_1 \simeq \mathrm{Ind}_{(G/N)^{\bullet}}^{G/N}\omega_2,$$

if and only if

$$\mathrm{Ind}_{(G/N)^{\bullet}}^{G/N}\chi_{\omega_1} = \chi_{\mathrm{Ind}_{(G/N)^{\bullet}}^{G/N}\omega_1} = \chi_{\mathrm{Ind}_{(G/N)^{\bullet}}^{G/N}\omega_2} = \mathrm{Ind}_{(G/N)^{\bullet}}^{G/N}\chi_{\omega_2}. \tag{17}$$

Since both $\mathrm{Ind}_{(G/N)^{\bullet}}^{G/N}\omega_1$ and $\mathrm{Ind}_{(G/N)^{\bullet}}^{G/N}\omega_2$ are irreducible representations of G/N, by Mackey's irreducibility criterion, eq. (17) is equivalent to

$$\left(\mathrm{Ind}_{(G/N)^{\bullet}}^{G/N}\chi_{\omega_1}, \mathrm{Ind}_{(G/N)^{\bullet}}^{G/N}\chi_{\omega_2}\right)_{G/N} = 1.$$

Now, by the Frobenius reciprocity law, eq. (17) is equivalent to

$$\begin{aligned}
1 &= \left(\chi_{\omega_1}, \mathrm{Res}_{(G/N)^{\bullet}}\mathrm{Ind}_{(G/N)^{\bullet}}^{G/N}\chi_{\omega_2}\right)_{(G/N)^{\bullet}} \\
&= \left(\chi_{\omega_1}, \chi_{\mathrm{Res}_{(G/N)^{\bullet}}\mathrm{Ind}_{(G/N)^{\bullet}}^{G/N}\omega_2}\right)_{(G/N)^{\bullet}} \\
&= \left(\chi_{\omega_1}, \chi_{\sum\limits_{\delta\in\mathcal{R}(M^{\bullet}\backslash M)}\omega_{2,\delta}}\right)_{(G/N)^{\bullet}} = \sum_{\delta\in\mathcal{R}(M^{\bullet}\backslash M)}(\chi_{\omega_1}, \chi_{\omega_{2,\delta}})_{(G/N)^{\bullet}},
\end{aligned}$$

which is the case if and only if $(\chi_{\omega_1}, \chi_{\omega_{2,\delta}})_{(G/N)^{\bullet}} = 1$, that is, $\omega_1 = \omega_{2,\delta}$, for some $\delta \in \mathcal{R}(M^{\bullet}\backslash M)$, completing the proof.

(ii) The proof follows immediately from part (i). □

So, there exists the following chain of bijections

$$[\mathcal{I}((G/N)^{\bullet})^0/\sim] = \mathcal{V}_N \xrightarrow{\mathrm{Ind}_{(G/N)^{\bullet}}^{G/N}} \mathcal{I}(G/N, \langle 1_{G/N}\rangle) \xrightarrow{\pi_{N,*}} \mathcal{I}(G, N) \tag{18}$$

defined by

$$\begin{aligned}
[\omega]_{\sim} &\mapsto [\mathrm{Ind}_{(G/N)^{\bullet}}^{G/N}\omega] \mapsto [\mathrm{Ind}_{(G/N)^{\bullet}}^{G/N}\omega \circ \pi_N] \\
&= [\mathrm{Ind}_{\pi_N^{-1}((G/N)^{\bullet})}^{G}(\omega \circ \pi_N\,|_{\pi_N^{-1}((G/N)^{\bullet})})],
\end{aligned} \tag{19}$$

for $[\omega]_\sim \in [\mathcal{I}((G/N)^\bullet)^0/\sim]$. Note that, the equality in the right-hand side follows from eq. (13).

Now, we are in a position to prove Theorem 1.4. An Artin representation A_G of G over $\mathbb{C}$ can be defined generally by

$$A_G = \sum_{[\rho]\in\mathcal{I}(G)} \mathrm{ord}_{\mathfrak{p}_K}(\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho))\rho.$$

So, by eq. (15),

$$A_G = \sum_N \sum_{[\rho]\in\mathcal{I}(G,N)} \mathrm{ord}_{\mathfrak{p}_K}(\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho))\rho, \tag{20}$$

where N runs over all normal subgroups of G. Thus, combining eq. (20) with Theorem 1.2,

$$\begin{aligned} A_G &= \sum_N \sum_{[\rho]\in\mathcal{I}(G,N)} [(E^N)^\bullet : K](n_{G/N}+1)\rho \\ &= \sum_N [(E^N)^\bullet : K](n_{G/N}+1) \sum_{[\rho]\in\mathcal{I}(G,N)} \rho, \end{aligned}$$

where N runs over all normal subgroups of G. Observe that

$$\sum_{[\rho]\in\mathcal{I}(G,N)} \rho \simeq \sum_{[\omega]_\sim\in\mathcal{V}_N} \mathrm{Ind}^G_{\pi_N^{-1}((G/N)^\bullet)}(\omega\circ\pi_N \mid_{\pi_N^{-1}((G/N)^\bullet)}), \tag{21}$$

where $\mathcal{V}_N = [\mathcal{I}((G/N)^\bullet)^0/\sim]$ for every $N \lhd G$, by eqs. (18) and (19). Hence,

$$\begin{aligned} A_G &\simeq \sum_N [(E^N)^\bullet : K](n_{G/N}+1) \\ &\quad\times \sum_{[\omega]_\sim\in\mathcal{V}_N} \mathrm{Ind}^G_{\pi_N^{-1}((G/N)^\bullet)}(\omega\circ\pi_N \mid_{\pi_N^{-1}((G/N)^\bullet)}), \end{aligned}$$

where N runs over all normal subgroups of G, which completes the proof of Theorem 1.4.

The following is then an easy consequence of Theorem 1.4.

COROLLARY 7.4 (Dimension formula for A_G)

Let E/K be a finite metabelian extension with Galois group G. Then

$$\dim_{\mathbb{C}}(A_G) = \sum_{[\rho]\in\mathcal{I}(G)} [E^{\ker(\rho)^\bullet} : K]^2(n_{G/\ker(\rho)}+1).$$

Proof. Note that, by eqs. (21) and (22),

$$\begin{aligned} \dim_{\mathbb{C}}(A_G) &= \sum_N [(E^N)^\bullet : K](n_{G/N}+1) \sum_{[\rho]\in\mathcal{I}(G,N)} \dim_{\mathbb{C}}(\rho) \\ &= \sum_N [(E^N)^\bullet : K](n_{G/N}+1) \\ &\quad\times \sum_{[\omega]_\sim\in\mathcal{V}_N} \dim_{\mathbb{C}}(\mathrm{Ind}^G_{\pi_N^{-1}((G/N)^\bullet)}(\omega\circ\pi_N \mid_{\pi_N^{-1}((G/N)^\bullet)})), \end{aligned}$$

where N runs over all normal subgroups of G. Now, by eq. (13), for any $\omega \in [\omega]_{\sim} \in \mathcal{V}_N$,

$$\begin{aligned}\dim_{\mathbb{C}}(\mathrm{Ind}^{G}_{\pi_N^{-1}((G/N)^\bullet)}(\omega \circ \pi_N \mid_{\pi_N^{-1}((G/N)^\bullet)})) &= \dim_{\mathbb{C}}(\mathrm{Ind}^{G/N}_{(G/N)^\bullet}\omega) \\ &= (G/N : (G/N)^\bullet) \\ &= [(E^N)^\bullet : K].\end{aligned}$$

Thus,

$$\begin{aligned}\dim_{\mathbb{C}}(A_G) &= \sum_{N}[(E^N)^\bullet : K](n_{G/N}+1) \sum_{[\omega]_{\sim}\in\mathcal{V}_N} [(E^N)^\bullet : K] \\ &= \sum_{N} \sum_{[\omega]_{\sim}\in\mathcal{V}_N} [(E^N)^\bullet : K]^2(n_{G/N}+1) \\ &= \sum_{N} \sum_{[\rho]\in\mathcal{I}(G,N)} [(E^N)^\bullet : K]^2(n_{G/N}+1) \\ &= \sum_{[\rho]\in\mathcal{I}(G)} [(E^{\ker(\rho)})^\bullet : K]^2(n_{G/\ker(\rho)}+1),\end{aligned}$$

by eqs. (15), (18) and (19), which completes the proof. □

8. Conclusion

We conclude our paper with the following set of remarks.

Remark 8.1. (On problem 1.1). Note that, the general solution of Problem 1.1 of describing the local Artin conductor $\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho)$ of the character $\chi_\rho : \mathrm{Gal}(E/K) \to \mathbb{C}$ associated to an irreducible representation (ρ, V) of $\mathrm{Gal}(E/K)$ requires the construction of non-abelian local class field theory over K à lá Koch–de Shalit (that is, 'n-abelian' local class field theories over K for each $n \geq 1$), and a detailed study of finite-dimensional complex representations of finite solvable groups (i.e., generalized Basmaji theory)[4]. In fact, if $\rho : \mathrm{Gal}(E/K) \to GL(V)$ is any irreducible representation of $\mathrm{Gal}(E/K) =: G$ (over $\mathbb{C}$), then by Proposition 3.2,

$$\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho) = \mathfrak{p}_K^{\dim_{\mathbb{C}}(V)[n_{G/\ker(\rho)}+1]},$$

where $G/\ker(\rho)$ is an n-abelian group, for some $1 \leq n \in \mathbb{Z}$. Thus, assuming that we have generalized Koch–de Shalit theory for Galois extensions over K with corresponding Galois groups having degrees of solvability n, where $3 \leq n \in \mathbb{Z}$ (for instance, for 3-abelian local class field theory, look at [5]), the number $n_{G/\ker(\rho)}$ can be located and computed by this n-abelian local class field theory. Also, assuming that we have generalized Basmaji's theory[5], the dimension of the representation space V can be computed in terms of the extension E/K. Therefore, following the same lines of this paper, we would have solution to Problem 1.1. □

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[4] The study of representation theory of finite solvable groups over $\mathbb{C}$ seems to be closely related with the following theorem of Dade: if S is any solvable group, then there exists an $\mathcal{M}$-group T, such that S can be embedded into T.

[5] The following well-known theorem (cf. [11]) may be a good starting point to investigate finite-dimensional representations of finite n-abelian groups (over $\mathbb{C}$). Let G be a finite n-abelian group. Let $\rho : G \to GL(V)$ be an irreducible representation of G (over $\mathbb{C}$). Then, either (i) there exists a proper subgroup H of G containing G', and an irreducible representation $\sigma : H \to GL(V_\sigma)$ such that $\rho \simeq \mathrm{Ind}_H^G\sigma$; or (ii) the representation $\mathrm{Res}_{G'}(\rho)$ of G' is isotypic.

Remark 8.2 (On Weil's problem). Let E/K be a finite Galois extension with Galois group G.

(i) In the direction of Weil's problem, the first important result is obtained by Serre (cf. [13]), where he proved that there exists a representation A_G^ℓ of G over $\mathbb{Q}_\ell$ (here ℓ is a prime number satisfying $\ell \neq q_K = \mathrm{char}(\kappa_K)$) whose character is the Artin character a_G of G.

(ii) Recall, from §7, that an Artin representation A_G of G over $\mathbb{C}$ can be defined by

$$A_G = \sum_{[\rho]\in\mathcal{I}(G)} \mathrm{ord}_{\mathfrak{p}_K}(\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho))\rho.$$

Now, by eq. (15),

$$A_G = \sum_N \sum_{[\rho]\in\mathcal{I}(G,N)} \mathrm{ord}_{\mathfrak{p}_K}(\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho))\rho,$$

where N runs over all normal subgroups of G.

Let d denote the derived length of the solvable group G. Any subgroup of G is solvable and of derived length bounded from above by d. Now, partition the set $\{N\}$ of all normal subgroups of G as

$$\{N\} = \mathcal{N}_1 \bigsqcup \mathcal{N}_2 \bigsqcup \cdots \bigsqcup \mathcal{N}_d,$$

where $\mathcal{N}_i$ denotes the set of all normal subgroups of G of derived length i for $1 \leq i \leq d$. Thus,

$$\begin{aligned} A_G &= \sum_{1\leq i\leq d} \sum_{N\in\mathcal{N}_i} \sum_{[\rho]\in\mathcal{I}(G,N)} \mathrm{ord}_{\mathfrak{p}_K}(\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho))\rho \\ &= A_G^{[1]} + A_G^{[2]} + \cdots + A_G^{[d]}, \end{aligned}$$

where $A_G^{[i]}$ is the representation of G defined by

$$A_G^{[i]} = \sum_{N\in\mathcal{N}_i} \sum_{[\rho]\in\mathcal{I}(G,N)} \mathrm{ord}_{\mathfrak{p}_K}(\mathfrak{f}_{\mathrm{Artin}}(\chi_\rho))\rho$$

for $1 \leq i \leq d$.

Note that, we can compute $A_G^{[1]}$ and $A_G^{[2]}$ explicitly following Theorem 1.4. However, to compute $A_G^{[i]}$ for $2 \lneq i \leq d$, we need the full solution of Problem 1.1 as well as generalized Basmaji's theory. □

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