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Some functional equations originating from number theory

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Abstract. We will introduce new functional equations (3) and (4) which are strongly related to well-known formulae (1) and (2) of number theory, and investigate the solutions of the equations. Moreover, we will also study some stability problems of those equations.

Keywords. Functional equation; stability; multiplicative function.

1. Introduction

In 1940, Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems [18]. Among those was the question concerning the stability of homomorphisms:

Let G_1 be a group and let G_2 be a metric group with a metric $d(\cdot\,,\cdot)$. Given any $\varepsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \varepsilon$ for all $x \in G_1$?

If the answer is affirmative, the functional equation for homomorphisms is said to be stable in the sense of Hyers and Ulam because the first result concerning the stability of functional equations was presented by Hyers. Indeed, he has answered the question of Ulam for the case where G_1 and G_2 are assumed to be Banach spaces (see [8]).

We may find a number of papers concerning the stability results of various functional equations (see [1–7, 9–16] and the references cited therein).

According to a well-known theorem in number theory, a positive integer of the form m^2n, where each divisor of n is not a square of the integer, can be represented as a sum of two squares of integer if and only if every prime factor of n is not of the form $4k + 3$. In the proof of this theorem, we make use of the following elementary equalities

$$(x_1^2 + y_1^2)(x_2^2 + y_2^2) = (x_1x_2 + y_1y_2)^2 + (x_1y_2 - y_1x_2)^2 \tag{1}$$

and

$$\begin{aligned}(x_1^2 + y_1^2 + z_1^2 + w_1^2)(x_2^2 + y_2^2 + z_2^2 + w_2^2) &= (x_1x_2 + y_1y_2 + z_1z_2 + w_1w_2)^2\\ &+ (x_1y_2 - y_1x_2 + z_1w_2 - w_1z_2)^2 + (x_1z_2 - y_1w_2 - z_1x_2 + w_1y_2)^2\\ &+ (x_1w_2 + y_1z_2 - z_1y_2 - w_1x_2)^2.\end{aligned} \tag{2}$$

As we know, the above equations explain that the product of any sums of two (four) squares of integer is also a sum of two (four) squares of integer.

These equalities (1) and (2) may be formulated by the following functional equations

$$f(x_1, y_1)f(x_2, y_2) = f(x_1x_2 + y_1y_2, x_1y_2 - y_1x_2) \tag{3}$$

and

$$\begin{aligned} f(x_1, y_1, z_1, w_1)f(x_2, y_2, z_2, w_2) = f(&x_1x_2 + y_1y_2 + z_1z_2 + w_1w_2,\\ &x_1y_2 - y_1x_2 + z_1w_2 - w_1z_2, x_1z_2 - y_1w_2 - z_1x_2 + w_1y_2,\\ &x_1w_2 + y_1z_2 - z_1y_2 - w_1x_2). \end{aligned} \tag{4}$$

In this paper, the solutions and stability problems of the above equations will be investigated.

2. Solutions and stability of (3)

We will first investigate the solutions of the functional equation (3) in the class of functions $f : \mathbb{R}^2 \to \mathbb{R}$.

Theorem 1. *If a function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies the functional equation* (3) *for all $x_1, x_2, y_1, y_2 \in \mathbb{R}$, then there exist a multiplicative function $m : \mathbb{R} \to \mathbb{R}$ and a signum function $\sigma : \mathbb{R}^2 \to \{\pm 1\}$ such that*

$$f(x, y) = \sigma(x, y)\, m\left(\sqrt{x^2 + y^2}\right)$$

for all real numbers x and y.

Proof. Put $y_1 = y_2 = 0$ in (3) to get

$$f(x_1, 0)f(x_2, 0) = f(x_1x_2, 0) \tag{5}$$

for all $x_1, x_2 \in \mathbb{R}$. Replace the x_is by x and the y_is by y in (3) to get

$$f(x, y)f(x, y) = f(x^2 + y^2, 0), \tag{6}$$

for any $x, y \in \mathbb{R}$. Using (6) twice, we have

$$f(y, x)f(y, x) = f(y^2 + x^2, 0) = f(x^2 + y^2, 0) = f(x, y)f(x, y)$$

and hence we may define a function $\sigma_1 : \mathbb{R}^2 \to \{\pm 1\}$ by

$$f(y, x) = \sigma_1(x, y)f(x, y) \tag{7}$$

for all real numbers x and y.

From (3), (6) and (7), it follows that

$$\begin{aligned} f(2xy, x^2 - y^2) &= f(x, y)f(y, x)\\ &= \sigma_1(x, y)f(x, y)f(x, y) = \sigma_1(x, y)f(x^2 + y^2, 0) \end{aligned} \tag{8}$$

for any real numbers x and y.

We notice that for any given $u, v \in \mathbb{R}$, the following system of equations

$$\begin{cases} 2xy = u, \\ x^2 - y^2 = v \end{cases} \tag{9}$$

has solutions $(x(u, v), y(u, v))$ in $\mathbb{R}^2$ as we see in the following:

$$(x(u,v), y(u,v)) = \begin{cases} (\pm\sqrt{v}, 0) & \text{for } u = 0 \text{ and } v \geq 0, \\ (0, \pm\sqrt{-v}\,) & \text{for } u = 0 \text{ and } v < 0, \\ \left(\pm\sqrt{\frac{v+\sqrt{u^2+v^2}}{2}}, \pm\frac{u}{\sqrt{2v+2\sqrt{u^2+v^2}}}\right) & \text{for } u \neq 0. \end{cases}$$

It follows from (9) that $x^2 + y^2 = \sqrt{u^2 + v^2}$. According to (8) and (9), we obtain

$$f(u, v) = \sigma_1(x(u, v), y(u, v)) f\left(\sqrt{u^2 + v^2}, 0\right) \tag{10}$$

for any $u, v \in \mathbb{R}$.

Taking (10) into account, we may introduce another function $\sigma : \mathbb{R}^2 \to \{\pm 1\}$ that satisfies the equality

$$f(u, v) = \sigma(u, v) f\left(\sqrt{u^2 + v^2}, 0\right) \tag{11}$$

for all $u, v \in \mathbb{R}$.

Finally, define a function $m : \mathbb{R} \to \mathbb{R}$ by $m(x) = f(x, 0)$ for each $x \in \mathbb{R}$. Then, (5) and (11) ensure that m is a multiplicative function and that

$$f(x, y) = \sigma(x, y)\, m\left(\sqrt{x^2 + y^2}\,\right)$$

for all real numbers x and y.

We will now investigate some stability problem of the functional equation (3). In view of Theorem 1, we can guess that the stability of (3) is strongly connected with multiplicative functions.

Theorem 2. *Let X be a field and $M_1, M_2, N_1, N_2 : X \to [0, \infty)$ be functions. If a function $f : X^2 \to \mathbb{C}$ satisfies the following inequality*

$$\begin{aligned} &|f(x_1, y_1) f(x_2, y_2) - f(x_1x_2 + y_1y_2, x_1y_2 - y_1x_2)| \\ &\quad \leq \min\{M_1(x_1), M_2(x_2), N_1(y_1), N_2(y_2)\} \end{aligned} \tag{12}$$

for all $x_1, x_2, y_1, y_2 \in X$, then $f(x, 0)$ is either bounded or multiplicative and further it satisfies

$$|f(x, y)^2 - f(x^2 + y^2, 0)| \leq \min\{M_1(x), M_2(x), N_1(y), N_2(y)\}$$

for any $x, y \in X$.

Proof. With $y_1 = y_2 = 0$, (12) implies

$$|f(x_1, 0)f(x_2, 0) - f(x_1x_2, 0)| \le \min\{M_1(x_1), M_2(x_2), N_1(0), N_2(0)\}$$

for $x_1, x_2 \in X$. If we substitute $m(x)$ instead of $f(x, 0)$ in the above inequality, then we have

$$|m(x_1)m(x_2) - m(x_1x_2)| \le \min\{M_1(x_1), M_2(x_2), N_1(0), N_2(0)\}$$

for all $x_1, x_2 \in X$.

Applying a theorem of Székelyhidi [17] (see Corollary 8.4 in [12]), we conclude that m is either bounded or multiplicative.

Finally, put $x_1 = x_2 = x$ and $y_1 = y_2 = y$ in (12) to get

$$|f(x, y)^2 - f(x^2 + y^2, 0)| \le \min\{M_1(x), M_2(x), N_1(y), N_2(y)\}$$

for all $x, y \in X$.

3. Solutions and stability of (4)

We first prove a lemma which turns out to be indispensable for the investigation of solutions of the functional equation (4).

Lemma 3. *For any given $a, b, c, d \in \mathbb{R}$, the system of equations*

$$\begin{cases} (x + z)(y + w) = a, \\ 2xz - y^2 - w^2 = b, \\ (x + z)(w - y) = c, \\ x^2 - z^2 = d \end{cases}$$

has at least one solution (x, y, z, w) in $\mathbb{R}^4$.

Proof.

(a) If $a = c = d = 0$ and $b \le 0$, then $(x, y, z, w) = \left(0, \sqrt{-b/2}, 0, \sqrt{-b/2}\right)$ is a solution of our system of equations.

(b) If $b > 0$ and $d = 0$, set $x = z = \alpha \neq 0$ and we will determine the value of α later. It follows from the first and third equations that

$$y = \frac{a - c}{4\alpha} \quad \text{and} \quad w = \frac{a + c}{4\alpha}.$$

By the second one, we get a biquadratic equation

$$16\alpha^4 - 8b\alpha^2 - a^2 - c^2 = 0,$$

and one of its solutions is

$$\alpha = \frac{\sqrt{b + \sqrt{a^2 + b^2 + c^2}}}{2} > 0.$$

Hence, the system of equations is solvable in $\mathbb{R}^4$ when $b > 0$ and $d = 0$.

(c) For the remaining cases under the condition $d = 0$: either if $a = 0, b \leq 0, c \neq 0$ and $d = 0$, or if $a \neq 0, b \leq 0, c = 0$ and $d = 0$, or if $a \neq 0, b \leq 0, c \neq 0$ and $d = 0$, then we follow the lines in part (b) and find out one solution of our system of equations.

(d) If $d \neq 0$, set $x = \sqrt{d+\alpha}$ and $z = \sqrt{\alpha}$ for some $\alpha \geq \max\{0, -d\}$ (α will be determined later). By the first and third equations, we have

$$y = \frac{a-c}{2\left(\sqrt{d+\alpha}+\sqrt{\alpha}\right)} \quad \text{and} \quad w = \frac{a+c}{2\left(\sqrt{d+\alpha}+\sqrt{\alpha}\right)}.$$

If we substitute those expressions for x, y, z, w in the second one and if we carry out a tedious calculation, then we get a quadratic equation

$$\begin{aligned} q(\alpha) = 16(a^2+c^2+d^2)\alpha^2 + 8(2d(a^2+c^2+d^2) - b(a^2+c^2))\alpha \\ - (a^2+c^2+2bd)^2 = 0. \end{aligned}$$

This equation has one solution α which is not less than 0 and $-d$ because of $q(0) \leq 0$ and $q(-d) = -(a^2+c^2-2bd)^2 \leq 0$. Thus, the system is solvable in $\mathbb{R}^4$ for $d \neq 0$.

In the following theorem, we investigate the solutions of the functional equation (4) by the same idea that was applied to the proof of Theorem 1.

Theorem 4. *If a function* $f : \mathbb{R}^4 \to \mathbb{R}$ *satisfies the functional equation* (4) *for all* $x_i, y_i, z_i, w_i \in \mathbb{R}$ $(i = 1, 2)$, *then there exist a multiplicative function* $m : \mathbb{R} \to \mathbb{R}$ *and a signum function* $\sigma : \mathbb{R}^4 \to \{\pm 1\}$ *such that*

$$f(x,y,z,w) = \sigma(x,y,z,w)\, m\left(\sqrt{x^2+y^2+z^2+w^2}\right)$$

for all real numbers x, y, z, w.

Proof. If we set $y_i = z_i = w_i = 0$ $(i = 1, 2)$ in (4), then

$$f(x_1,0,0,0)f(x_2,0,0,0) = f(x_1x_2,0,0,0) \tag{13}$$

for all $x_1, x_2 \in \mathbb{R}$. If we substitute x, y, z, w for the x_i, y_i, z_i, w_i in (4), then we have

$$f(x,y,z,w)f(x,y,z,w) = f(x^2+y^2+z^2+w^2,0,0,0) \tag{14}$$

for any $x, y, z, w \in \mathbb{R}$. Use eq. (14) twice to get

$$\begin{aligned} f(y,z,w,x)f(y,z,w,x) &= f(y^2+z^2+w^2+x^2,0,0,0) \\ &= f(x^2+y^2+z^2+w^2,0,0,0) \\ &= f(x,y,z,w)f(x,y,z,w). \end{aligned}$$

Therefore, we may define a function $\sigma_1 : \mathbb{R}^4 \to \{\pm 1\}$ by

$$f(y,z,w,x) = \sigma_1(x,y,z,w)f(x,y,z,w) \tag{15}$$

for all $x, y, z, w \in \mathbb{R}$.

It follows from (4), (14) and (15) that

$$\begin{aligned} & f((x+z)(y+w), 2xz - y^2 - w^2, (x+z)(w-y), x^2 - z^2) \\ & \quad = f(x, y, z, w) f(y, z, w, x) \\ & \quad = \sigma_1(x, y, z, w) f(x, y, z, w) f(x, y, z, w) \\ & \quad = \sigma_1(x, y, z, w) f(x^2 + y^2 + z^2 + w^2, 0, 0, 0) \end{aligned} \tag{16}$$

for any real numbers x, y, z, w.

According to Lemma 3, we can easily see that

$$\{((x+z)(y+w), 2xz - y^2 - w^2, (x+z)(w-y), x^2 - z^2) : \\ x, y, z, w \in \mathbb{R}\} = \mathbb{R}^4$$

because the following system of equations

$$\begin{cases} (x+z)(y+w) = a, \\ 2xz - y^2 - w^2 = b, \\ (x+z)(w-y) = c, \\ x^2 - z^2 = d \end{cases} \tag{17}$$

has at least one solution $(x(a,b,c,d), y(a,b,c,d), z(a,b,c,d), w(a,b,c,d))$ for any given $a, b, c, d \in \mathbb{R}$.

It follows from (17) that $x^2 + y^2 + z^2 + w^2 = \sqrt{a^2 + b^2 + c^2 + d^2}$. According to (16) and (17), we obtain

$$f(a, b, c, d) = \sigma_1(x, y, z, w) f\left(\sqrt{a^2 + b^2 + c^2 + d^2}, 0, 0, 0\right) \tag{18}$$

for any $a, b, c, d \in \mathbb{R}$, where we denote the solution of (17) by (x, y, z, w). Taking (18) into account, we may introduce another function $\sigma : \mathbb{R}^4 \to \{\pm 1\}$ that satisfies the equality

$$f(a, b, c, d) = \sigma(a, b, c, d) f\left(\sqrt{a^2 + b^2 + c^2 + d^2}, 0, 0, 0\right) \tag{19}$$

for all $a, b, c, d \in \mathbb{R}$.

Finally, define a function $m : \mathbb{R} \to \mathbb{R}$ by $m(x) = f(x, 0, 0, 0)$ for every $x \in \mathbb{R}$. Then, (13) and (19) ensure that m is a multiplicative function and that

$$f(x, y, z, w) = \sigma(x, y, z, w)\, m\left(\sqrt{x^2 + y^2 + z^2 + w^2}\right)$$

for all real numbers x, y, z, w.

We will now study a stability problem of the functional equation (4). In view of Theorem 4, we can guess that the stability problem of (4) is strongly connected with multiplicative functions.

Theorem 5. *Let X be a field and $K_i, L_i, M_i, N_i : X \to [0, \infty)$ be functions for $i = 1, 2$. If a function $f : X^4 \to \mathbb{C}$ satisfies the following inequality*

$$\begin{aligned}
&|f(x_1, y_1, z_1, w_1)f(x_2, y_2, z_2, w_2)\\
&\quad - f(x_1x_2 + y_1y_2 + z_1z_2 + w_1w_2, x_1y_2 - y_1x_2 + z_1w_2 - w_1z_2,\\
&\qquad x_1z_2 - y_1w_2 - z_1x_2 + w_1y_2,\ x_1w_2 + y_1z_2 - z_1y_2 - w_1x_2)|\\
&\quad \le \min\{K_1(x_1), K_2(x_2), L_1(y_1), L_2(y_2), M_1(z_1), M_2(z_2), N_1(w_1), N_2(w_2)\}
\end{aligned} \tag{20}$$

for all $x_i, y_i, z_i, w_i \in X$, then $f(x, 0, 0, 0)$ is either bounded or multiplicative. Further it satisfies

$$\begin{aligned}
&|f(x, y, z, w)^2 - f(x^2 + y^2 + z^2 + w^2, 0, 0, 0)|\\
&\quad \le \min\{K_1(x), K_2(x), L_1(y), L_2(y), M_1(z), M_2(z), N_1(w), N_2(w)\}
\end{aligned}$$

for any $x, y, z, w \in X$.

Proof. With $y_1 = y_2 = z_1 = z_2 = w_1 = w_2 = 0$, (20) implies

$$\begin{aligned}
&|f(x_1, 0, 0, 0)f(x_2, 0, 0, 0) - f(x_1x_2, 0, 0, 0)|\\
&\quad \le \min\{K_1(x_1), K_2(x_2), L_1(0), L_2(0), M_1(0), M_2(0), N_1(0), N_2(0)\}
\end{aligned}$$

for $x_1, x_2 \in X$. If we substitute $m(x)$ for $f(x, 0, 0, 0)$ in the above inequality, we have

$$\begin{aligned}
&|m(x_1)m(x_2) - m(x_1x_2)|\\
&\quad \le \min\{K_1(x_1), K_2(x_2), L_1(0), L_2(0), M_1(0), M_2(0), N_1(0), N_2(0)\}
\end{aligned}$$

for all $x_1, x_2 \in X$.

Applying a theorem of Székelyhidi [17] (see Corollary 8.4 in [12]), we conclude that m is either bounded or multiplicative.

Finally, put $x_1 = x_2 = x$, $y_1 = y_2 = y$, $z_1 = z_2 = z$ and $w_1 = w_2 = w$ in (20) to get

$$\begin{aligned}
&|f(x, y, z, w)^2 - f(x^2 + y^2 + z^2 + w^2, 0, 0, 0)|\\
&\quad \le \min\{K_1(x), K_2(x), L_1(y), L_2(y), M_1(z), M_2(z), N_1(w), N_2(w)\}
\end{aligned}$$

for all $x, y, z, w \in X$.

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