Proc. Indian Acad. Sci. (Math. Sci.) Vol. 113, No. 1, February 2003, pp. 71–76.

Questions concerning matrix algebras and invariance of spectrum

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MS received 2 January 2002; revised 2 April 2002

Abstract. Let A and B be unital Banach algebras with A a subalgebra of B. Denote the algebra of all $n \times n$ matrices with entries from A by $M_n(A)$. In this paper we prove some results concerning the open question: If A is inverse closed in B, then is $M_n(A)$ inverse closed in $M_n(B)$? We also study related questions in the setting where A is a symmetric Banach *-algebra.

Keywords. Banach algebra; inverse closed; symmetric *-algebra; matrix algebra.

1. Introduction

Throughout, A and B are unital Banach algebras, A is a subalgebra of B, and the unit of B is in A. In this paper, we continue studying the relationships among the three properties:

A is inverse closed in B if whenever $a \in A$ and $a^{-1} \in B$, then $a^{-1} \in A$.

A is SRP in B if $r_A(a) = r_B(a)$ for all $a \in A$ (here $r_A(a)$ is the spectral radius of $a \in A$ in A; SRP stands for 'spectral radius preserving').

When A has an involution $*$, *A is *-inverse closed in B* if whenever $a = a^* \in A$ and $a^{-1} \in B$, then $a^{-1} \in A$.

Note that A is inverse closed in B is equivalent to: for all $a \in A$, $\sigma(a; A) = \sigma(a; B)$ (here $\sigma(a; A)$ denotes the spectrum of a relative to the algebra A).

A Banach *-algebra A with unit 1 is *symmetric* if $1 + a^*a$ is invertible for all $a \in A$. This is equivalent to the property that $\sigma(a^*a; A) \subseteq [0, \infty)$ for all $a \in A$. The relationships among the three properties above when A is a symmetric Banach *-algebra is the subject of the author's paper [3]. In this paper we study these properties in algebras of $n \times n$ matrices over a Banach algebra; notation: $M_n(A)$ denotes the algebra of all $n \times n$ matrices with entries from A. Much of our work in this paper centers on the open question:

Question. If A is inverse closed in B, then is $M_n(A)$ inverse closed in $M_n(B)$?

It is known that when A is commutative, then this question has an affirmative answer; this follows from [6, Theorem 1.1]. Many other results related to this question are known. We list two useful results; Fact 1 is proved in [9, Theorem 2.2.14, p. 219], and Fact 2 is proved in [3, Proposition 2].

Fact 1. If A is SRP in B, then $M_n(A)$ is SRP in $M_n(B)$.

Dedicated to Professor Ashoke K. Roy on his retirement.

Fact 2. Assume that A is SRP in B. Let $\overline{A}$ be the closure of A in B. If $a \in A$ and $a^{-1} \in \overline{A}$, then $a^{-1} \in A$.

The matrix algebras $M_n(A)$ and $M_n(B)$ are Banach algebras with respect to natural norms: For $(D, \| \|_D)$ a Banach algebra, define for $T = (t_{jk}) \in M_n(D)$,

$$\|T\| = \sum_{j=1}^{n} \sum_{k=1}^{n} \|t_{jk}\|_D.$$

Then $(M_n(D), \| \|)$ is a Banach algebra. We use the norm defined above, or any equivalent norm, as the standard norm on $M_n(D)$.

Combining Facts 1 and 2, we have an affirmative answer to the Question when A is dense in B. (The proof is easy: Assume that A is inverse closed in B, and A is dense in B. By Fact 1, $M_n(A)$ is SRP in $M_n(B)$. Since A is dense in B, it is easy to see that $M_n(A)$ is dense in $M_n(B)$. It follows from Fact 2 that $M_n(A)$ is inverse closed in $M_n(B)$). We state this result as a proposition.

PROPOSITION 3

If A is inverse closed in B, and A is dense in B, then $M_n(A)$ is inverse closed in $M_n(B)$.

2. Results for a general Banach algebra A

For a Banach algebra D, we let $\mathrm{Inv}_l(D)$, $\mathrm{Inv}_r(D)$, and $\mathrm{Inv}(D)$ denote the set of left invertible elements of D, the set of right invertible elements of D, and the set of invertible elements of D, respectively. Also, we write $GL_n(A) = \mathrm{Inv}\ (M_n(A))$.

We introduce an equivalence relation on $M_n(B)$: For $T, S \in M_n(B)$, we write $T \approx S(GL_n(A))$ if there exist $V, W \in GL_n(A)$ such that $VTW = S$. It is clear that this is an equivalence relation on $M_n(B)$. For convenience we usually write $T \approx S$ with the expression $GL_n(A)$ omitted. Since $GL_n(A)$ is a group, if $T \in M_n(A)$, $S \in GL_n(A)$, and $T \approx S$, then $T \in GL_n(A)$. In particular, it is easy to check that if $T \in M_n(A)$ and S is obtained from T by a finite sequence of interchanges of two rows or two columns, then $T \approx S$. For example,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \approx \begin{pmatrix} c & d \\ a & b \end{pmatrix},$$

since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ a & b \end{pmatrix}.$$

Now we prove a preliminary result which we believe is known (we have been told that this proposition follows from results in [7]).

PROPOSITION 4

Assume that A is inverse closed in B, and A is continuously embedded in B. ('A is continuously embedded in B' means that there exists $J > 0$ such that $J\ \|a\|_A \geq \|a\|_B$ for all $a \in A$.) Let $T = (t_{jk}) \in M_2(A)$ with $T^{-1} \in M_2(B)$. Also assume that some entry in T is contained in $\overline{\mathrm{Inv}_l(A)} \cup \overline{\mathrm{Inv}_r(A)}$. Then $T^{-1} \in M_2(A)$.

Proof. By interchanging rows and columns of T if necessary, we may assume that $t_{11} \in \overline{\mathrm{Inv}_l(A)}$. First assume that $t_{11} \in \mathrm{Inv}_l(A)$. Choose $a \in A$ with $at_{11} = 1$, and choose $\lambda \in \mathbf{C}$ such that $\lambda + t_{21} \in \mathrm{Inv}(A)$. Let

$$R = \begin{pmatrix} \lambda a & 1 \\ 1 & 0 \end{pmatrix} T = \begin{pmatrix} \lambda + t_{21} & r_{12} \\ t_{11} & t_{12} \end{pmatrix}.$$

Note that by construction, $r_{11} = \lambda + t_{21} \in \mathrm{Inv}(A)$. Let

$$S = \begin{pmatrix} 1 & 0 \\ -t_{11}r_{11}^{-1} & 1 \end{pmatrix} R = \begin{pmatrix} \lambda + t_{21} & r_{12} \\ 0 & s_{22} \end{pmatrix}.$$

Note that $V = \begin{pmatrix} \lambda a & 1 \\ 1 & 0 \end{pmatrix}$ and $W = \begin{pmatrix} 1 & 0 \\ -t_{11}r_{11}^{-1} & 1 \end{pmatrix}$ are in $GL_n(A)$. Also, $S = WVT$.

Thus, $S^{-1} \in M_2(B)$, and $\lambda + t_{21} \in \mathrm{Inv}(B)$. It follows from [9, the criterion on p. 78] that $s_{22} \in \mathrm{Inv}(B)$. Since A is inverse closed in B, $\lambda + t_{21},\ s_{22} \in \mathrm{Inv}(A)$. It follows from a straightforward computation that $S \in GL_n(A)$. Also, $T \approx S$, and therefore, $T^{-1} \in M_2(A)$.

Now assume that $t_{11} \in \overline{\mathrm{Inv}_l(A)}$. Choose $\{a_m\} \subseteq \mathrm{Inv}_l(A)$ with $\|t_{11} - a_m\|_A \to 0$, and so $\|t_{11} - a_m\|_B \to 0$. Since $GL_n(B)$ is open, we may assume for all m,

$$T_m = \begin{pmatrix} a_m & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \in GL_n(B).$$

By the previous argument, $T_m \in GL_n(A)$ for all m. Also, $T_m^{-1} \to T^{-1}$ in $M_2(B)$. It follows from Facts 1 and 2 that $T^{-1} \in M_2(A)$. ■

Let $U_n(A)$ be the algebra of upper triangular matrices in $M_n(A)$, that is, $T = (t_{jk}) \in U_n(A)$ if $t_{jk} = 0$ whenever $j > k$. When A is inverse closed in B, it is easy to check that $U_n(A)$ is inverse closed in $U_n(B)$. For example, suppose in the case $n = 2$ that $T = \begin{pmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{pmatrix} \in U_2(A)$ has inverse $S = \begin{pmatrix} s_{11} & s_{12} \\ 0 & s_{22} \end{pmatrix} \in U_2(B)$. Then $ST = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ implies that $s_{11}t_{11} = 1$ and $s_{22}t_{22} = 1$. Also, $TS = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ implies that $t_{11}s_{11} = 1$ and $t_{22}s_{22} = 1$. Thus, t_{11} and t_{22} have inverses s_{11} and s_{22} in B. By hypothesis, $s_{11}, s_{22} \in A$. Also, $t_{11}s_{12} + t_{12}s_{22} = 0$. Therefore $s_{12} = -s_{11}t_{12}s_{22} \in A$. This proves $S \in U_2(A)$.

It is known that $T \in U_n(A)$ can have an inverse $S \in M_n(A)$ with $S \notin U_n(A)$; see for example the author's paper [4].

These remarks lead to the question: If A is inverse closed in B, then is $U_n(A)$ inverse closed in $M_n(B)$? Now we prove that this question has an affirmative answer. In fact, we do the result for a subalgebra larger than $U_n(A)$. We define this algebra now.

Let $I(A)$ denote the largest inessential ideal of A; see [1, Fact 3.1]. To illustrate this concept in the case of operators, let $B(X)$ and $K(X)$ denote the algebra of all bounded linear operators on a Banach space X, and the space of all compact operators on X, respectively. Then $I(B(X))$ is the closed ideal of inessential operators on X as originally defined by Kleinecke [8]. It is always true that $K(X) \subseteq I(B(X))$. The key fact here for our purposes is that the spectral theory of an element $a \in I(A)$ is exactly like that of a compact operator. In particular, for $a \in I(A)$, $\sigma(a; A)$ is a finite set or a sequence converging to 0; see [1, Remark 2.6]. This implies that $a \in \overline{\mathrm{Inv}(A)}$.

DEFINITION 5

The matrix $T = (t_{jk}) \in M_n(A)$ is in $U_n^e(A)$ if $t_{jk} \in I(A)$ whenever $j > k$. The matrix $T = (t_{jk}) \in M_n(A)$ is in $L_n^e(A)$ if $t_{jk} \in I(A)$ whenever $j < k$.

Theorem 6. *Assume that A is inverse closed in B, and A is continuously embedded in B. If $T \in U_n^e(A)$ (or $T \in L_n^e(A)$) and $T^{-1} \in M_n(B)$, then $T^{-1} \in M_n(A)$.*

Proof. First assume that $n = 2$. Assume that $T \in U_2^e(A)$ has $T^{-1} \in M_2(B)$. By assumption, $t_{21} \in I(A)$, so t_{21} is a limit of invertible elements in A. Thus Proposition 4 applies and so $T^{-1} \in M_2(A)$.

The proof proceeds by induction. Assume that the result holds for n. Assume as a first case that: $T \in M_{n+1}(A)$, $T^{-1} \in M_{n+1}(B)$, $t_{jk} \in I(A)$ *provided that* $j > k$ *and* $j \geq 3$, and $t_{11} \in \text{Inv}(A)$. We may assume that $t_{11} = 1$.

Define W and V in $M_{n+1}(A)$ by:

$$\begin{aligned} w_{kk} &= 1, 1 \leq k \leq n+1; \quad w_{1k} = -t_{1k},\ 2 \leq k \leq n+1; \\ w_{jk} &= 0, \quad \text{otherwise}; \\ v_{kk} &= 1, 1 \leq k \leq n+1; \quad v_{j1} = -t_{j1},\ 2 \leq k \leq n+1; \\ v_{jk} &= 0, \quad \text{otherwise}. \end{aligned}$$

It is easy to check that both W and V are invertible in $M_{n+1}(A)$. Set $S = VTW$. By direct computation

$$s_{11} = 1; \quad s_{1k} = 0,\ 2 \leq k \leq n+1; \quad s_{j1} = 0,\ 2 \leq j \leq n+1.$$

Define $\widetilde{S} \in M_n(A)$ by deleting the first row and the first column of S. A direct matrix computation verifies that $\widetilde{S} \in U_n^e(A)$. Since $S^{-1} \in M_{n+1}(B)$, and noting the form of the first row and the first column of S displayed above, we have $\widetilde{S}^{-1} \in M_n(B)$. Then by the induction hypothesis, $\widetilde{S}^{-1} \in M_n(A)$. It follows that $S^{-1} \in M_{n+1}(A)$.

In the remaining case, $t_{11} \notin \text{Inv}(A)$. Interchange rows 1 and 2 of T, so that t_{21} is in the (1, 1) position. Call this new matrix R, and note that $R \approx T$. Since $r_{11} = t_{21} \in I(A)$, we can choose $\lambda_m \to 0$ such that $\lambda_m + r_{11} \in \text{Inv}(A)$ for all m. Let R_m be the matrix with $\lambda_m + r_{11}$ in the (1, 1)-position, and with r_{jk} in the (j, k)-position otherwise. Since R is invertible in $M_{n+1}(B)$ and $R_m \to R$ in norm, we may assume that $R_m^{-1} \in M_{n+1}(B)$ for all m. Now each R_m satisfies the hypotheses of the first case considered above, and so $R_m^{-1} \in M_{n+1}(A)$ for all m. Thus, $R^{-1} \in M_{n+1}(A)$ by applying Facts 1 and 2. ■

Now we look at a concrete situation where Theorem 6 applies. Let (Y, μ) be a σ-finite measure space. Let $K(x, y)$ be a kernel with the property that

$$||| K |||_\infty \equiv ess \sup_{x \in Y} \int_Y |K(x, y)| \, d\mu(y) < \infty.$$

For such a kernel K, define the integral operator $T_K : L^\infty \to L^\infty$ by

$$T_K(f)(x) \equiv \int_Y K(x, y) f(y) \, d\mu(y),\ f \in L^\infty.$$

Then $T_K \in B(L^\infty)$. The set of all such integral operators form a subalgebra of $B(L^\infty)$ which is called the algebra of Hille–Tamarkin operators on L^∞; see [5, 11.5]. We use the notation H_∞ for this algebra. H_∞ is a Banach algebra with respect to the norm $|||K|||_\infty$. This space of integral operators is an important class of operators which contains many interesting examples.

We assume that H_∞ does not contain the identity operator (the usual situation). Let H^1_∞ be the algebra H_∞ with the identity operator on L^∞ adjoined. It follows from results in [5, 11.5] that H^1_∞ is inverse closed in $B(L^\infty)$. Let $J = \{T \in H^1_\infty : T$ is a compact operator on $L^\infty\}$. Certainly, $J \subseteq I(B(L^\infty))$. Theorem 6 applies to any operator $T = (t_{jk}) \in M_n(H^1_\infty)$ with the property that $t_{jk} \in J$ whenever $j > k$.

3. Results when A is a symmetric Banach *-algebra

The following fact, which we use repeatedly, is proved in [10, Theorem 9.8.4 and the preceding remarks, p. 1011].

Fact 7. When A is a symmetric Banach *-algebra, then $M_n(A)$ is symmetric.

The next proposition follows from a result of the author in [2] and Fact 7.

PROPOSITION 8

*When A is a symmetric Banach *-algebra, A is continuously embedded in B, and A is closed in B, then $M_n(A)$ is inverse closed in $M_n(B)$ for all n.*

Proof. By Fact 7, $M_n(A)$ is symmetric. Also, $M_n(A)$ is continuously embedded in $M_n(B)$, and $M_n(A)$ is closed in $M_n(B)$. Then the proposition follows from [2, Theorem]. ■

Theorem 9. *Assume that A is a symmetric Banach *-algebra, that A is *-inverse closed in B, and that A is continuously embedded in B. Then $M_n(A)$ is symmetric and *-inverse closed in $M_n(B)$.*

Proof. $M_n(A)$ is symmetric by Fact 7.

Assume that $n = 2$. Suppose that $T = T^* \in M_2(A)$ and that $T^{-1} \in M_2(B)$. Write $T = (t_{jk})$, and note that $t_{11} = t^*_{11}$. Since A is symmetric, t_{11} is the limit of the sequence of invertible elements $\{\frac{i}{n} + t_{11}\}$. By Proposition 4, it follows that $T^{-1} \in M_2(A)$.

Now note that $M_2(M_{2^n}(A))$ can be naturally identified as a Banach *-algebra with $M_{2^{n+1}}(A)$. Therefore by the case where $n = 2$ proved above, and induction we have:

$$M_{2^n}(A) \text{ is *-inverse closed in } M_{2^n}(B) \text{ for } n \geq 1.$$

Assume that $2^{n-1} < m < 2^n$ for some $n \geq 2$. For $T = (t_{jk}) \in M_m(A)$, define $\widetilde{T} = (\widetilde{t_{jk}}) \in M_{2^n}(A)$ by

$$\widetilde{t_{jk}} = t_{jk},\ 1 \leq j, k \leq m; \quad \widetilde{t_{kk}} = 1,\ m < k \leq 2^n; \quad \widetilde{t_{jk}} = 0, \quad \text{otherwise.}$$

It is straightforward to check that $T \to \widetilde{T}$ is a unital *-algebra monomorphism of $M_m(A)$ into $M_{2^n}(A)$, and that T is invertible in $M_m(A)$ if and only if $\widetilde{T}$ is invertible in $M_{2^n}(A)$. The same definition as given above defines a unital algebra monomorphism of $M_m(B)$ into $M_{2^n}(B)$, and again, T is invertible in $M_m(B)$ if and only if $\widetilde{T}$ is invertible in $M_{2^n}(B)$.

Finally, if $T = T^* \in M_m(A)$ and $T^{-1} \in M_m(B)$, then $\widetilde{T} = \widetilde{T}^* \in M_{2^n}(A)$ and that $\widetilde{T}^{-1} \in M_{2^n}(B)$. As proved previously, this implies $\widetilde{T}^{-1} \in M_{2^n}(A)$, and so $T^{-1} \in M_m(A)$. ■

The following corollary extends [3, Theorem 11] which is the case $n = 1$.

COROLLARY 10

*Assume that A is a symmetric Banach *-algebra, and that A is either *-inverse closed in B and continuously embedded in B, or A is* SRP *in B. Also, assume that for some $M > 0$,*

$$\|a^*\| \le M\|a\| \quad \text{for all } a \in A.$$

Then $M_n(A)$ is inverse closed in $M_n(B)$ for $n \ge 1$.

Proof. Since A is symmetric, $M_n(A)$ is symmetric by Fact 7. If A is *-inverse closed and continuously embedded in B, then by Theorem 9, $M_n(A)$ is *-inverse closed in $M_n(B)$. If A is SRP in B, then by Fact 1, $M_n(A)$ is SRP in $M_n(B)$. Also, it is easy to check that for all $T \in M_n(A)$, $\|T^*\| \le M\|T\|$. In both cases it follows from [3, Theorem 11] that $M_n(A)$ is inverse closed in $M_n(B)$. ■

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