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Slow motion of a sphere away from a wall: Effect of surface roughness on the viscous force

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Abstract. An asymptotic analysis is given for the effect of roughness exhibited through the slip parameter β on the motion of the sphere, moving away from a plane surface with velocity V. The method replaces the no-slip condition at the rough surface by slip condition and employs the method of inner and outer regions on the sphere surface. For $\beta > 0$, we have the classical slip boundary condition and the results of the paper are then of interest in the microprocessor industry.

Keywords. Viscous force; slip flow; meridional plane.

1. Introduction

The solution of the problem of motion of a sphere in a semi-infinite expanse of an incompressible viscous fluid bounded by a plane surface has been of considerable interest for many years. O'Neill and Stewartson [9] in 1967 employed the method of matched asymptotic expansions for obtaining the slow flow solution of a sphere moving parallel to a nearby plane. Cooley and O'Neill [2] considered the problem of a sphere approaching a plane and it was afterwards extended by O'Neill and Ranger [8] in 1983 by taking also into account the deformation of the interface. In 1990, a mathematical model was developed by O'Brien and Van Den Brule [7] for the cleansing of the silicon substrates in which a dirt particle taken as a sphere, was made to move away from the plane substrate through the action of surface tension. All solid surfaces considered in the existing literature taken in this paper have been taken as smooth but all solid surfaces are rough on a microscopic scale and therefore they are likely to be affected by roughness [4–6].

Following the work of Cooley and O'Neill for a smooth surface, we have obtained the effect of roughness on the motion of a sphere moving away from a plane surface. The method divides the whole region into two parts: an inner region consisting of the fluid in the neighborhood of O (figure 1), the nearest point of the plane from the sphere and outer region consisting of the rest of the flow. The no-slip condition used, was here replaced by an effective slip condition obtained by Miksis and Davis [6] within the limit of small amplitude of arbitrary shaped roughness. The slip condition [6] for two-dimensional flow bounded by the rough plane $y = 0$ with u as the velocity component along its direction is given by

$$u(x, o, t) + \langle h \rangle \frac{\partial u}{\partial y}(x, o, t) = 0, \tag{1.1}$$

where $\langle h \rangle$ is the average amplitude of the surface roughness.

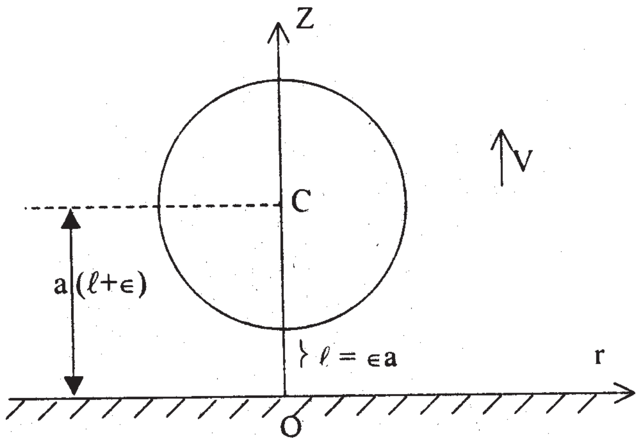

Figure 1.

Using the same method as in [6], we have derived the roughness condition at $z = 0$ for flow in the meridional (r,z) plane as

$$u(r, o, t) - \beta' \frac{\partial u}{\partial z}(r, o, t) = 0, \tag{1.2}$$

where $\langle h \rangle$ has been replaced by the more convenient symbol $-\beta'$. This conforms to locating the boundary $z = 0$ at the crests as in Hocking's paper [5] and also to the classical Basset's [1] slip condition. The slip can also arise when there is a surface coating [5,6] of a different fluid. Such reporting has earlier been made by Scheneel [10] also. Slip is also observed [11] for rarefied gas flow when molecular mean free path is comparable to the microscopic length scale of the asperities. But in all the cases, it may be noted that the theory advanced is tenable only for small values of the slip parameter $|\beta|$.

2. Formulation of the problem

The physical situation may be modelled by a spherical particle of radius a moving with the speed V away from the rough substrate taken as the place $z = 0$ (figure 1). In the cylindrical polar coordinate system (r,θ,z), with space coordinate non-dimensionalised by a, and with the pole at O, the point of the plane nearest to the sphere, further corresponding velocity components are (u,o,w), non-dimensionalised by V.

The motion may be considered quasi-static [7] and instantaneous position of its centre at $(o, o(l+\varepsilon)a)$, where $\varepsilon = \ell/a$, ℓ being the minimum clearance between the sphere and the substrate. Further, suppose that the fluid is incompressible, has a constant density ρ, viscosity μ and that the Reynolds number $Va\rho/\mu$ is sufficiently small to neglect the inertia terms in the Navier–Stokes equations.

Using the method outlined by Cooley and O'Neill [2], it is found that the motion is governed by the Stokes stream function Ψ satisfying the differential equation

$$\Lambda^4\psi \equiv \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)^2 \psi = 0, \tag{2.1}$$

the velocity being given by

$$u = \frac{1}{r}\frac{\partial \psi}{\partial z}, \quad w = -\frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{2.2}$$

In terms of ψ, the relevant boundary conditions (slip condition (1.2)) and vanishing of normal velocity w may be put as

$$\frac{\partial \psi}{\partial z} = \beta \frac{\partial^2 \psi}{\partial z^2} \tag{2.3}$$

where $\beta = \beta'/a$, non-diamensional slip parameter, and

$$\frac{\partial \psi}{\partial r} = 0, \text{ on the substrate surface } z = 0. \tag{2.4}$$

Also $w = 1$ and $u = 0$ providing

$$\psi = -\frac{1}{2}r^2, \quad \frac{\partial \psi}{\partial z} = 0 \quad \text{on } z = \delta. \tag{2.5}$$

Here $a\delta$ is the clearance between the sphere and the plane, and when expressed in terms of r we have

$$\delta = 1 + \varepsilon - (1 - r^2)^{1/2}. \tag{2.6}$$

Since δ is of $o(\varepsilon)$ in the inner region, r is of $o(\varepsilon^{1/2})$.

To investigate the motion in the inner region, we introduce new variables R, Z defined by

$$r = \varepsilon^{1/2} R, \quad z = \varepsilon Z. \tag{2.7}$$

Using the above variables (2.6) now give

$$\frac{\delta}{\varepsilon} = H + \frac{1}{8}R^4 \varepsilon + O(\varepsilon^2), \tag{2.8}$$

where $H = 1 + \frac{1}{2}R^2$.

First of the conditions (2.5) suggests that we choose

$$\psi = \varepsilon \psi_0(R, Z) + \varepsilon^2 \psi_1(R, Z) + \cdots . \tag{2.9}$$

Substituting ψ from the above in eq. (2.1), expressed in terms of R, Z and then equating to zero the coefficients of different power of ε, we get

$$\frac{\partial^4 \psi_0}{\partial Z^4} = 0, \tag{2.10}$$

$$\frac{\partial^4 \psi_i}{\partial Z^4} + 2\frac{\partial^2}{\partial Z^2}\left\{\frac{\partial^2 \psi_{i-1}}{\partial R^2} - \frac{1}{R}\frac{\partial \Psi_{i-1}}{\partial R}\right\} = 0, \tag{2.11}$$

where $i = 1, 2 \ldots$

Substitution of (2.9) into (2.3) and (2.4) yields

$$\psi_{iZ}(R,0) = \beta\psi_{iZZ}(R,0), \tag{2.12}$$

$$\psi_{iR}(R,0) = 0 \quad \text{for } i = 0, 1, 2, \dots , \tag{2.13}$$

the subscript Z denotes differentiation with respect to Z. Further, (2.5) provides up to $O(\varepsilon)$, the condition on the sphere as

$$\psi_0(R,H) = -\frac{1}{2}R^2, \quad \psi_{0Z}(R,H) = 0, \tag{2.14}$$

$$\psi_1(R,H) = -\frac{1}{8}R^4\psi_{0Z}(R,H), \tag{2.15}$$

$$\psi_{1Z}(R,H) = -\frac{1}{8}R^4\psi_{0ZZ}(R,H). \tag{2.16}$$

The solution of (2.10), using (2.12)–(2.14) gives

$$\psi_0(R,Z) = \frac{ZR^2}{2H^3(H+4\beta)}\{2(H+\beta)Z^2 - 3H^2Z - 6H^2\beta\}. \tag{2.17}$$

Using the above value of ψ_0 in (2.11) for $i = 1$ provides the differential equation for ψ_1 as

$$\frac{\partial^4\psi_1}{\partial Z^4} = 12ZM + 4N, \tag{2.18}$$

where

$$\begin{aligned} M = \Bigg[& \frac{4R^2}{H^3(H+4\beta)} - \frac{8R^2\beta}{(H+4\beta)} + \frac{8R^2\beta}{H^4(H+4\beta)} + \frac{8R^2}{H^2(H+4\beta)^2} \\ & + \frac{16R^2\beta^2}{H^4(H+4\beta)} - \frac{2R^4}{H^4(H+4\beta)} - \frac{6R^4\beta}{H^5(H+4\beta)} - \frac{2R^4}{H^3(H+4\beta)^2} \\ & - \frac{16R^4\beta^2}{H^5(H+4\beta)^2} - \frac{14R^4\beta}{H^4(H+4\beta)^2} - \frac{8R^4}{H^2(H+4\beta)^3} - \frac{40R^4\beta}{H^3(H+4\beta)^3} \\ & - \frac{6R^4\beta^2}{H^4(H+4\beta)^3} - \frac{32R^4\beta^3}{H^5(H+4\beta)^3} + \frac{24R^2\beta}{H^3(H+4\beta)^2} \Bigg], \end{aligned} \tag{2.19}$$

$$N = -\frac{12R^2(H+2\beta)}{H^2(H+4\beta)^2} - \frac{3R^4}{H^2(H+4\beta)^2} + \frac{12R^4(H+2\beta)^2}{H^3(H+4\beta)^3}. \tag{2.20}$$

Solution (2.18) which satisfies (2.15) and (2.16) is

$$\psi_1 = \frac{1}{10}MZ^5 + \frac{N}{6}Z^4 + \frac{A}{6}Z^3 + \frac{B}{2}Z^2 + CZ + D, \tag{2.21}$$

where

$$\begin{aligned} A = & -\frac{3MH^2}{5}\left(3 - \frac{4\beta}{(H+4\beta)}\right) - 2NH\left(1 - \frac{\beta}{H+4\beta}\right) \\ & - \frac{9}{4}\frac{R^6}{H^4}\left(1 - \frac{2\beta}{(H+4\beta)}\right)^2, \end{aligned} \tag{2.22}$$

$$B = \frac{3}{4}\frac{R^6}{H^2}\frac{(H+2\beta)}{(H+4\beta)^2} + \frac{2}{5}\frac{MH^4}{(H+4\beta)} + \frac{NH^3}{3(H+4\beta)}, \tag{2.23}$$

$$C = \beta B \quad \text{and} \quad D = 0 \tag{2.24}$$

Expressions (2.17) and (2.21) provide us with the first two terms of the expansion (2.9) for ψ valid in the inner region.

3. The force acting on the sphere

The method uses the well-known drag formula [3] for a sphere. The cylindrical components of the force are $(o, o, 6\pi\mu V a f)$, where

$$6f = \int_\gamma r^3 \frac{\partial}{\partial n}\left(\frac{\Lambda^2\psi}{r^2}\right) \mathrm{d}s. \tag{3.1}$$

The integral is taken around the meridian section γ of the sphere making a positive right angle with the direction n, where n is the normal drawn outwards from the sphere.

3.1 *Contribution of inner region*

Writing $\Lambda^2\psi$ in terms of the inner variable R, Z and the inner solution expansion (2.9) for ψ

$$\Lambda^2\psi = \varepsilon^{-1}\frac{\partial^2\psi_0}{\partial Z^2} + \frac{\partial^2\psi_1}{\partial Z^2} + \frac{\partial^2\psi_0}{\partial R^2} - \frac{1}{R}\frac{\partial\psi_0}{\partial R} + O(\varepsilon). \tag{3.2}$$

Now in the inner region on the surface of the sphere, line element ds is

$$\mathrm{d}s \equiv \varepsilon^{1/2}\left\{1 + \frac{1}{2}\varepsilon R^2 + O(\varepsilon^2)\right\}\mathrm{d}R, \tag{3.3}$$

where

$$\frac{\partial}{\partial n} \equiv -\varepsilon^{-1}\frac{\partial}{\partial Z} + \frac{R\partial}{\partial R} + \frac{1}{2}R^2\frac{\partial}{\partial Z} + O(\varepsilon). \tag{3.4}$$

The inner region may be taken to extend some value $R = R_0$ [1] and then for this part eq. (3.1) yields

$$-6f^i = \frac{1}{\varepsilon}\int_0^{Ro}\left(R\frac{\partial^3\psi_0}{\partial Z^3}\right)_{Z=H}\mathrm{d}R + \int_0^{Ro}\left[R\frac{\partial^3\psi_1}{\partial Z^3} - R^4\frac{\partial}{\partial R}\left(\frac{1}{R^2}\frac{\partial^2\psi_0}{\partial Z^2}\right) + R\frac{\partial^3\psi_0}{\partial Z\partial R^2} - \frac{\partial^2\psi_0}{\partial Z\partial R}\right]_{Z=H}\mathrm{d}R. \tag{3.5}$$

$$-6f^i = I_1 + I_2 + I_3 + I_4 + I_5 \tag{3.6}$$

where

$$I_1 = \frac{1}{\varepsilon}\int_0^{Ro}\left(R\frac{\partial^3\psi_0}{\partial Z^3}\right)_{Z=H}\mathrm{d}R = \frac{6}{\varepsilon}\int_0^{Ro}\frac{R^3(H+\beta)\mathrm{d}R}{H^3(H+4\beta)}$$

$$= \frac{12}{\varepsilon}\left[-\frac{1}{H_0} + \frac{1}{2H_0^2}(3\beta+1) - \frac{\beta}{H_0^3}(4\beta+1) + \frac{3\beta^2}{H_0^4} + \frac{1}{2}(1-\beta+2\beta^2) + o(\beta^3)\right].$$

Here

$$H_0 = 1 + \frac{1}{2}R_0^2$$

$$I_2 = \int_0^{Ro} \left(R\frac{\partial^3 \psi}{\partial Z^3} \right)_{Z=H} \mathrm{d}R$$

$$= \int_0^{Ro} \left[\frac{3}{5}\left\{7+\frac{4\beta}{H+4\beta}\right\}\left\{\frac{4R^2}{H(H+4\beta)}+\frac{8R^2\beta}{H^2(H+4\beta)}+\frac{8R^2}{(H+4\beta)^2} + \frac{16R^2\beta^2}{H^2(H+4\beta)^2} - \frac{2R^4}{H^2(H+4\beta)} - \frac{6R^4\beta}{H^3(H+4\beta)} - \frac{2R^4}{H(H+4\beta)^2} - \frac{16R^4\beta^2}{H^3(H+4\beta)^2} - \frac{14R^4\beta}{H^2(H+4\beta)^2} - \frac{8R^4}{(H+4\beta)^3} - \frac{40R^4\beta}{H(H+4\beta)^3} - \frac{6R^4\beta^2}{H^2(H+4\beta)^3} - \frac{32R^4\beta^3}{H^3(H+4\beta)^3} + \frac{24R^2\beta}{H(H+4\beta)^2}\right\} + 2\left(1+\frac{\beta}{H+4\beta}\right)\left\{-\frac{24R^2\beta}{(H+4\beta)^2} - \frac{12R^2}{(H+4\beta)^2} - \frac{3R^4}{H(H+4\beta)^2} + \frac{12R^4}{(H+4\beta)^3} + \frac{48R^4\beta^2}{H^2(H+4\beta)^3} + \frac{48R^4\beta}{H(H+4\beta)^3}\right\} - \frac{3}{2}\frac{R^6}{H^2(H+4\beta)^2} - \frac{9}{2}\frac{R^6\beta}{H^3(H+4\beta)^2} - \frac{3R^6\beta^2}{H^4(H+4\beta)^2} - \frac{3}{4}\frac{R^6}{H^4} + \frac{3}{2}\frac{R^6\beta}{H^4(H+4\beta)} \right] R\mathrm{d}R,$$

$$I_2 = \left[-\frac{474}{5}\log H_0 - \frac{1}{H_0}\left(\frac{1302}{5}+\frac{1416}{5}\beta\right) + \frac{1}{H_0^2}\left(\frac{459}{5}+444\beta+456\beta^2\right) - \frac{1}{H_0^3}\left(6+\frac{1008}{5}\beta+\frac{3664}{5}\beta^2\right) + \frac{1}{H_0^4}(18\beta+252\beta^2) - \frac{336}{5}\frac{\beta^2}{H_0^5} + \frac{873}{5} + \frac{114}{5}\beta + 92\beta^2 + O(\beta^3) \right],$$

$$I_3 = -\int_0^{Ro} R^4 \frac{\partial}{\partial R}\left(\frac{1}{R^2}\frac{\partial^2 \psi_0}{\partial Z^2}\right)_{Z=H} \mathrm{d}R$$

$$= \int_0^{Ro} \left[\frac{9R^5}{H^3} - \frac{18R^5\beta}{H^3(H+4\beta)} + \frac{3R^5}{H^2(H+4\beta)} - \frac{6R^5\beta}{H^2(H+4\beta)^2} \right] \mathrm{d}R$$

$$= \left[48\log H_0 + \frac{1}{H_0}(96+144\beta) - \frac{6\beta}{H_0^2}(24+144\beta+336\beta^2) - \frac{1}{H_0^3}(48\beta+448\beta^2) - \frac{468\beta^2}{H_0^4} - 72 - 48\beta + 356\beta^2 + O(\beta^3) \right],$$

$$I_4 = -\int_0^{Ro}\left(R\frac{\partial^3\psi_0}{\partial Z\partial R^2}\right)_{Z=H}$$
$$= \int_0^{Ro}\left[-\frac{15R^3}{H^2}+\frac{30R^3\beta}{H^2(H+4\beta)}+\frac{12R^5}{H^3}-\frac{24R^5\beta}{H^3(H+4\beta)}+\frac{12R^5}{H^3}\right.$$
$$\left.+\frac{48R^5\beta^2}{H^3(H+4\beta)^2}-\frac{48R^5\beta}{H^3(H+4\beta)}-\frac{6R^5\beta}{H^3(H+4\beta)}+\frac{6R^5}{H^2(H+4\beta)^2}\right]\mathrm{d}R$$
$$= \left[42\log H_0+\frac{1}{H_0}(114+156\beta)-\frac{1}{H_0^2}(36+186\beta)-78-42\beta+O(\beta^3)\right]$$

and

$$I_5 = -\int_0^{Ro}\left.\frac{\partial^2\psi_0}{\partial Z\partial R}\right|_{Z=H}\mathrm{d}R = \int_0^{Ro}\left[\frac{3R^3}{H^2}-\frac{6R^3\beta}{H^2(H+4\hat{a})}\right]\mathrm{d}R$$
$$= \left[6\log H_0+\frac{6}{H_0}(1+2\beta)-\frac{6\beta}{H_0^2}(1+4\beta)\right.$$
$$\left.+\frac{16\beta^2}{H_0^3}-6-6\beta+8\beta^2+O(\beta^3)\right].$$

The integrals $I_1, I_2, \ldots, I_5$ may be evaluated in closed forms. But since we are interested only in small values of β, we have expanded the integrals and obtained convenient expressions in powers of β.

Substituting the above values in eq. (3.6), we get

$$f^i = \left[-\frac{1}{5}\log H_0+\frac{1}{H_0}\left(\frac{37}{5}-\frac{24\beta}{5}+\frac{2}{\varepsilon}\right)\right.$$
$$-\frac{1}{H_0^2}\left\{\frac{(1+3\beta)}{\varepsilon}+\frac{53}{10}+18\beta-68\beta^2+\frac{1}{\varepsilon}+\frac{3\beta}{\varepsilon}\right\}$$
$$+\frac{1}{H_0^3}\left\{1+\frac{2\beta}{\varepsilon}(1+4\beta)+\frac{68\beta}{5}-\frac{968}{15}\beta^2-\frac{2\beta}{\varepsilon}-\frac{8\beta^2}{\varepsilon}\right\}$$
$$-\frac{\beta}{H_0^4}\left(3-88\beta+\frac{6\beta}{\varepsilon}\right)+\frac{1}{H_0^5}\left(\frac{56}{5}\beta^2\right)-\frac{1}{\varepsilon}(1-\beta+2\beta^2)$$
$$\left.-\frac{31}{10}+\frac{61}{5}\beta-\frac{260}{3}\beta^2+O(\beta^3)\right]. \tag{3.7}$$

Neglecting higher orders of β^2 and also for large R_0 taking$H_0 \approx \frac{1}{2}R_0^2$, we get

$$f^i = -\frac{1}{\varepsilon}(1-\beta)+4\varepsilon^{-1}R_0^{-2}+O(\varepsilon^{-1}R_0^{-4})$$
$$-\frac{1}{5}\log\left(\frac{1}{2}R_0^2\right)-\frac{31}{10}+\frac{61}{5}\beta+P_1(R_0), \tag{3.8}$$

where $P_1(R_0) \to 0$ as $R_0 \to \infty$.

3.2 *Contribution of outer region*

As the slip parameter β is small, it is expected to affect only the region close to the substrate surface $Z = 0$. Thus, the outer region which is far away from the surface remains almost unaffected by the surface roughness. Therefore, the contribution of outer solution to total drag may be taken the same, as for the smooth surface ($\beta = 0$).

The above conjecture is supported by the fact that the slip boundary condition ((1.2) leading to $(\partial\psi/\partial\eta) = [-\beta\eta^2\mu/(2+5\beta\eta\mu)][(\partial^2\psi/\partial\eta^2) - (\partial^2\psi/\partial\xi^2)]$ for small β) for the outer region at $\xi = 0$ [1], although not satisfied in general, for all values of η, will be satisfied in the outer region, i.e., for small values of η, leading to the same outer solution as obtained by Cooley and O'Neill [2].

Equation (5.11) by Cooley and O'Neill [2] gives the expression for the outer solution (f^0)

$$-f^0 = \eta_0^2 + \frac{2}{5}\log\eta_0 - \frac{2}{5} + P_1^*(\eta_0) + \frac{1}{6}\int_0^\infty \left\{ sA - 12s^{-3} - \frac{12s^{-1}e^{-25}}{5} \right\} \mathrm{d}s, \tag{3.9}$$

where $A = 2(1 + s + s^{-1}e^{-s}\sin hs)/\sin h^2 s - s^2$ and $P_1^*(\eta_0) \to 0$ as $\eta \to \infty$ and η_0 is such that the outer solution is valid for $0 \le \eta \le \eta_0$ and is related to R_0 [1] as

$$\eta_0^2 = 4\varepsilon^{-1}R_0^{-2} - 2 + O(\varepsilon R_0^{-2}). \tag{3.10}$$

Here we are using f^0 with negative sign because the direction of the motion of the sphere considered in this paper is opposite to that taken by Cooley and O'Neill [2]. The total viscous force (f) acting on the sphere may now be expressed as the sum of inner (f^i) and outer solution (f^0) given by (3.8) and (3.9)

$$f = f^i + f^0. \tag{3.11}$$

Putting η_0^2 from (3.10) into (3.11), we get

$$f \cong -\left[\frac{1}{\varepsilon}(1-\beta) + \frac{1}{5}\log\left(\frac{2}{\varepsilon}\right) + \frac{7}{10} - \frac{61}{5}\beta + P_1(R_0) + P_1^*(\eta_0) + \frac{1}{3}\int_0^\infty \left\{ \frac{s^2 + s + e^{-s}\sin hs}{\sin h^2 s - s^2} - \frac{6}{s^3} - \frac{6e^{-2s}}{5s} \right\} \mathrm{d}s. \tag{3.12}$$

Integrating the above result, we get

$$f = -\left[\frac{1}{\varepsilon}(1-\beta) - \frac{1}{5}\log\varepsilon - \frac{61}{5}\beta + 0.971280 \right]. \tag{3.13}$$

The derived result is seen to agree with the result (5.13) of Cooley and O'Neill [2] when the reversal of the sphere velocity V is taken into consideration and $\beta \to 0$.

4. Conclusion

The result (3.13) shows that the effect of the slip parameter β is to decrease the viscous force, thereby aids the removal of dust particle and hence the cleansing of silicon substrates [7], a useful result in the microprocessor industry. It may be pointed out here that the slip may be attained by chemically treating the surface of the substrate [10].

Appendix

A. Slip boundary condition

A.1 *Formulation*

Let the soild/liquid interface be denoted by $z = h(r)$, and we assume that the fluid is in the region $z > h(r)$ with $o < r < \infty$ (see figure 2).

Let Ω denote the region containing fluid of velocity μ and density ρ and L denotes the characteristic length on which we measure the flow field. Then in terms of non-dimensional variables, the equations of motion for the fluid in the region $z > h(r)$ are given by

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0, \tag{A.1.1}$$

$$\hat{\rho}\ \mathrm{Re}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z}\right) + \frac{\partial p}{\partial r} = \hat{\mu}\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2}\right], \tag{A.1.2}$$

and

$$\hat{\rho}\ \mathrm{Re}\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right) + \frac{\partial p}{\partial z} = \hat{\mu}\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2}\right], \tag{A.1.3}$$

where u represents the r-component of velocity, w represents the z-component of velocity and p, the pressure.

We define the Reynolds number $\mathrm{Re} = \rho L U_\infty/\mu$, where U_∞ is the velocity far from the soild surface, the density $(\hat{\rho})$ and viscosity $\hat{\mu}$ are taken to be unity for single phase flow over a rough surface.

The no-slip condition on the solid surface $z = h$ is given as

$$u = 0, \quad w = 0. \tag{A.1.4}$$

We assume

$$\varepsilon_1 = d/L \ll 1, \tag{A.1.5}$$

where d is a macroscopic length scale of the surface roughness.

Now, we derive an effective boundary condition [6] in the limit of ε_1 tending to zero. The rough surface is given as

$$z = h(r) = \varepsilon_1 \hat{h}(\hat{r}, r), \tag{A.1.6}$$

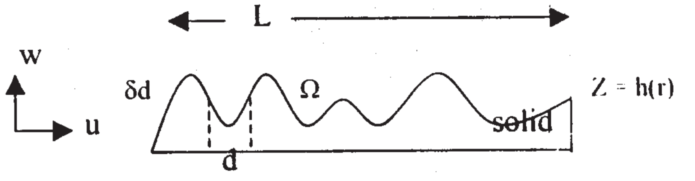

Figure 2.

where the fast space variable is defined by

$$\hat{r} = \frac{r}{\varepsilon_1}. \tag{A.1.7}$$

The arbitrary shaped roughness, h is of small amplitude. Here, the roughness varies on two scales: the slow scale r and the fast scale $\hat{r}$. Thus, it requires that r derivative in (A.1.1)–(A.1.4) should be replaced by

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial r} + \frac{1}{\varepsilon_1}\frac{\partial}{\partial \hat{r}}. \tag{A.1.8}$$

Hence, continuity eq. (A.1.1) reduces to

$$\frac{1}{r}\frac{\partial(ru)}{\partial r} + \frac{1}{\varepsilon_1}\frac{\partial u}{\partial \hat{r}} + \frac{\partial w}{\partial z} = 0, \tag{A.1.9}$$

and Navier–Stokes equations (A.1.2) and (A.1.3) may be written as

$$\hat{\rho}\ \mathrm{Re}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{u}{\varepsilon_1}\frac{\partial u}{\partial \hat{r}} + w\frac{\partial u}{\partial z}\right) + \frac{\partial p}{\partial r} + \frac{1}{\varepsilon_1}\frac{\partial p}{\partial \hat{r}}$$
$$= \hat{\mu}\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{\varepsilon_1^2}\frac{\partial^2 u}{\partial \hat{r}^2} + \frac{2}{\varepsilon_1}\frac{\partial^2 u}{\partial r \partial \hat{r}} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r\varepsilon_1}\frac{\partial u}{\partial \hat{r}} + \frac{\partial^2 u}{\partial z^2} - \frac{u}{r^2}\right], \tag{A.1.10}$$

$$\hat{\rho}\ \mathrm{Re}\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{u}{\varepsilon_1}\frac{\partial w}{\partial \hat{r}} + w\frac{\partial w}{\partial z}\right) + \frac{\partial p}{\partial z}$$
$$= \hat{\mu}\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{\varepsilon_1^2}\frac{\partial^2 w}{\partial \hat{r}^2} + \frac{2}{\varepsilon_1}\frac{\partial^2 w}{\partial r \partial \hat{r}} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r\varepsilon_1}\frac{\partial w}{\partial \hat{r}} + \frac{\partial^2 w}{\partial z^2}\right]. \tag{A.1.11}$$

The average $\langle f\rangle$ of a variable $f = f(\hat{r}, r, z, t)$ is defined as

$$\langle f\rangle(r, z, t) = \lim_{\chi\to\infty}\frac{1}{\chi}\int_0^{\chi} f(\hat{r}, r, z, t)\mathrm{d}\hat{r}. \tag{A.1.12}$$

Now, there are two regions to consider; the outer region, where z is of the order one and inner region, where z is of the order ε_1. But, since we are interested in finding the roughness boundary condition, let us start with the inner region.

A.2 *Inner region*

We introduce the inner scaling

$$u = \varepsilon_1\hat{u}, \quad w = \varepsilon_1\hat{w}, \quad h = \varepsilon_1\hat{h}, \quad z = \varepsilon_1\hat{z}, \quad p = \hat{p}, \tag{A.2.1}$$

using (A.2.1)in (A.1.9), (A.1.10) and (A.1.11), we get

$$\varepsilon_1\frac{\partial \hat{u}}{\partial r} + \frac{\partial \hat{u}}{\partial \hat{r}} + \frac{\partial \hat{w}}{\partial \hat{z}} + \frac{\varepsilon_1\hat{u}}{r} = 0, \tag{A.2.2}$$

$$\varepsilon_1^2 \hat{\rho} \mathrm{Re} \left(\frac{\partial \hat{u}}{\partial t} + \hat{u} \frac{\partial \hat{u}}{\partial \hat{r}} + \varepsilon_1 \hat{u} \frac{\partial \hat{u}}{\partial r} + \hat{w} \frac{\partial \hat{u}}{\partial \hat{z}} \right) + \frac{\partial \hat{p}}{\partial \hat{r}} + \varepsilon_1 \frac{\partial \hat{p}}{\partial r}$$
$$= \hat{\mu} \left[\frac{\partial^2 \hat{u}}{\partial \hat{r}^2} + \frac{\partial^2 \hat{u}}{\partial \hat{z}^2} + 2\varepsilon_1 \frac{\partial^2 \hat{u}}{\partial \hat{r} \partial r} + \varepsilon_1^2 \frac{\partial^2 \hat{u}}{\partial \hat{r}^2} + \frac{\varepsilon_1^2}{r} \frac{\partial \hat{u}}{\partial \hat{r}} + \frac{\varepsilon_1}{r} \frac{\partial \hat{u}}{\partial \hat{r}} - \frac{\varepsilon_1^2 u}{r^2} \right], \tag{A.2.3}$$

$$\varepsilon_1^2 \hat{\rho} \mathrm{Re} \left(\frac{\partial \hat{w}}{\partial t} + \hat{u} \frac{\partial \hat{w}}{\partial \hat{r}} + \varepsilon_1 \hat{u} \frac{\partial \hat{w}}{\partial r} + \hat{w} \frac{\partial \hat{w}}{\partial \hat{z}} \right) + \frac{\partial \hat{p}}{\partial \hat{r}}$$
$$= \hat{\mu} \left[\frac{\partial^2 \hat{w}}{\partial \hat{r}^2} + \frac{\partial^2 \hat{w}}{\partial \hat{z}^2} + 2\varepsilon_1 \frac{\partial^2 \hat{w}}{\partial r \partial \hat{r}} + \varepsilon_1^2 \frac{\partial^2 \hat{w}}{\partial r^2} + \frac{\varepsilon_1^2}{r} \frac{\partial \hat{w}}{\partial r} + \frac{\varepsilon_1}{r} \frac{\partial \hat{w}}{\partial \hat{r}} \right]. \tag{A.2.4}$$

Also from the boundary conditions (A.1.4), we have

$$\hat{u} = \hat{w} = 0 \quad \text{in } \hat{z} = \hat{h}. \tag{A.2.5}$$

Solutions of system (A.2.2)– (A.2.5) are found as a regular perturbation series in ε_1, e.g.,

$$\hat{u} = \hat{u}_0 + \varepsilon_1 \hat{u}_1 + \varepsilon_1^2 \hat{u}_2 + \cdots . \tag{A.2.6}$$

Substituting eq. (A.2.6) into (A.2.2)–(A.2.4) and equating to zero the coefficients of like power of ε_1. At leading order, we find that $\hat{u}_0$, $\hat{w}_0$ and $\hat{p}_0$ satisfy the Stokes equations, i.e.,

$$\frac{\partial \hat{u}_0}{\partial \hat{r}} + \frac{\partial \hat{w}_0}{\partial \hat{z}} = 0. \tag{A.2.7}$$

$$\frac{\partial \hat{p}_0}{\partial \hat{r}} = \hat{\mu} \left[\frac{\partial^2 \hat{u}_0}{\partial \hat{r}^2} + \frac{\partial^2 \hat{u}_0}{\partial \hat{z}^2} \right], \tag{A.2.8}$$

$$\frac{\partial \hat{p}_0}{\partial \hat{z}} = \hat{\mu} \left[\frac{\partial^2 \hat{w}_0}{\partial \hat{r}^2} + \frac{\partial^2 \hat{w}_0}{\partial \hat{z}^2} \right]. \tag{A.2.9}$$

Suppose we average (A.2.7)–(A.2.9) and interchanging the $\hat{z}$ derivative with the average, we find

$$\frac{\partial \langle \hat{w}_0 \rangle}{\partial \hat{z}} = 0, \tag{A.2.10}$$

$$\frac{\partial^2 \langle \hat{u}_0 \rangle}{\partial \hat{z}^2} = 0, \tag{A.2.11}$$

$$\frac{\partial \langle \hat{p}_0 \rangle}{\partial \hat{z}} = 0, \tag{A.2.12}$$

whence (A.2.10) and (A.2.12) imply that $\langle \hat{w}_0 \rangle$ and $\langle \hat{p}_0 \rangle$ are independent of $\hat{z}$ and (A.2.11) shows that $\langle \hat{u}_0 \rangle$ is linear function of $\hat{z}$, i.e.,

$$\langle \hat{u}_0 \rangle (r, \hat{z}, t) = a(r, t) + b(r, t) \hat{z} \tag{A.2.12a}$$

where a and b are two functions which are to be determined by matching the inner solution with the outer.

The no-slip condition for the leading term of the outer solution [6] can be written as

$$u_0(r, 0, t) = w_0(r, 0, t) = 0. \tag{A.2.13}$$

Continuing with the matching, we find that $b(r,t) = \partial u_0(r,0,t)/\partial z$ and we rewrite (A.2.11) as

$$\langle \hat{u}_0 \rangle (r, \hat{z}, t) = [\hat{z} - \hat{c}(r,t)] \frac{\partial u_0}{\partial z}(r, 0, t), \tag{A.2.14}$$

where $\hat{c}(r,t)$, defined by $a(r,t) = -\hat{c}(r,t)(\partial u_0/\partial z)(r,0,t)$.

Applying the next order matching condition for the outer solution and using expression (A.2.14), we obtain

$$u_1(r, 0, t) = -\hat{c}(r,t) \frac{\partial u_0}{\partial z}(r, 0, t). \tag{A.2.14a}$$

Multiplying (A.2.14a) by ε_1 and then adding (A.2.14) at $z = 0$, we obtain the effective boundary condition up to the order ε as

$$u(r, 0, t) + c(r,t) \frac{\partial u_0}{\partial z}(r, 0, t) = 0, \tag{A.2.14b}$$

where $c(r,t) = \varepsilon_1 \hat{c}(r,t)$.

In order to solve eqs (A.2.7)–(A.2.9), we assume the amplitude of the roughness to be small in the inner region. Therefore, introducing δ as a small parameter, we have

$$\hat{h} = \delta \bar{h}(\hat{r}, r), \tag{A.2.15}$$

where $\bar{h}$ is of the order one. Hence, the final result will be valid to order $\varepsilon_1 \delta$. Here, we assume δ as

$$\varepsilon_1 \ll \delta \ll 1. \tag{A.2.16}$$

Now, applying regular perturbation series in δ

$$\begin{aligned} \hat{u}_0 &= \hat{U}_0(\hat{r}, r, \hat{z}, t) + \delta \hat{U}_1(\hat{r}, r, \hat{z}, t) + \delta^2 \hat{U}_2(\hat{r}, r, \hat{z}, t) + \cdots \\ \hat{w}_0 &= \hat{W}_0(\hat{r}, r, \hat{z}, t) + \delta \hat{W}_1(\hat{r}, r, \hat{z}, t) + \delta^2 \hat{W}_2(\hat{r}, r, \hat{z}, t) + \cdots \\ \hat{p}_0 &= \hat{P}_0(\hat{r}, r, \hat{z}, t) + \delta \hat{P}_1(\hat{r}, r, \hat{z}, t) + \delta^2 \hat{P}_2(\hat{r}, r, \hat{z}, t) + \cdots . \end{aligned} \tag{A.2.17}$$

Rescaling the variables for the inner solution as

$$\hat{u}_0 = \delta \bar{u}, \quad \hat{w}_0 = \delta^2 \bar{w}, \quad \hat{p}_0 = \bar{p}, \quad \hat{z} = \delta \bar{z}, \quad \hat{h} = \delta \bar{h}, \tag{A.2.18}$$

we again find a solution as a power series in δ

$$\bar{u} = \bar{u}_0(\hat{r}, r, \bar{z}, t) + \delta \bar{u}_1(\hat{r}, r, \bar{z}, t) + \delta^2 \bar{u}_2(\hat{r}, r, \bar{z}, t) + \cdots . \tag{A.2.19}$$

Substituting (A.2.18) and (A.2.19) into (A.2.7)– (A.2.9), we find that the leading order equations are

$$\frac{\partial \bar{u}_0}{\partial \hat{r}} + \frac{\partial \bar{w}_0}{\partial z} = 0, \tag{A.2.20}$$

$$\frac{\partial^2 \bar{u}_0}{\partial \bar{z}^2} = 0, \tag{A.2.21}$$

$$\frac{\partial \bar{p}_0}{\partial z} = 0. \tag{A.2.22}$$

These three equations [eqs (A.2.20)–(A.2.22)] can be solved with the help of equation (A.2.5) which at leading order in δ are

$$\bar{u}_0 = 0, \quad \bar{w}_0 = 0, \tag{A.2.23}$$

along $\bar{z} = \bar{h}(\hat{r}, r)$. The solutions of (A.2.20)–(A.2.22) are

$$\bar{u}_0 = (\bar{z} - \bar{h})K, \tag{A.2.24}$$

$$\bar{w}_0 = \frac{1}{2}(\bar{h}^2 - \bar{z}^2)\frac{\partial K}{\partial \hat{r}} - (\bar{h} - \bar{z})\frac{\partial (K\bar{h})}{\partial \hat{r}}, \tag{A.2.25}$$

$$\bar{p}_0 = \bar{p}_0(\hat{r}, r, t) \tag{A.2.26}$$

where $K = K(\hat{r}, r, t)$ will be determined by matching. Higher order terms can also be obtained in the same way.

Next, expanding the dependent variables about $\hat{z} = 0$ we find the matching condition along $\hat{z} = 0$ as

$$\hat{U}_0(\hat{r}, r, 0, t) = 0, \quad \hat{U}_1(\hat{r}, r, 0, t) = -\bar{h}K \quad \text{and} \quad \frac{\partial \hat{U}_0}{\partial \hat{z}} = K, \tag{A.2.27}$$

and also for $\hat{W}_0$ we have

$$\hat{W}_0 = \hat{W}_1 \frac{\partial \hat{W}_0}{\partial \hat{z}} = 0, \tag{A.2.28}$$

which gives the leading order solution as a linear shearing flow

$$\hat{U}_0 = \hat{z}\frac{\partial u_0}{\partial z}(r, 0, t), \quad \hat{W}_0 = 0. \tag{A.2.29}$$

The pressure $\hat{p}_0$ must be constant (independent of $\hat{r}$, $\hat{z}$).

Thus, the leading order term for $\hat{P}_0$ will be

$$\hat{P}_0 = \hat{p}_0.$$

From eq. (A.2.27), we find that K is equal to

$$K = \frac{\partial u_0}{\partial z}(r, 0, t) \quad \text{at } z = 0. \tag{A.2.30}$$

Now, the next order matching condition for the velocity U_1 from eq. (A.2.27) is given as

$$\langle \hat{U}_1 \rangle = -\langle \bar{h} \rangle \frac{\partial u_0}{\partial z}(r, 0, t), \quad \langle \hat{W}_1 \rangle = 0. \tag{A.2.31}$$

The above result, along with (A.2.14), (A.2.17) and (A.2.29) imply that the slip coefficient is

$$\langle h \rangle = c(r, t). \tag{A.2.32}$$

Thus, the effective boundary condition to order $\varepsilon\delta$ along $z = 0$ is given by

$$u(r, 0, t) + \langle h \rangle \frac{\partial u}{\partial z}(r, 0, t) = 0. \tag{A.2.33}$$

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