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A complete analogue of Hardy's theorem on $SL_2(\mathbb{R})$ and characterization of the heat kernel

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Abstract. A theorem of Hardy characterizes the Gauss kernel (heat kernel of the Laplacian) on $\mathbb{R}$ from estimates on the function and its Fourier transform. In this article we establish a *full group* version of the theorem for $SL_2(\mathbb{R})$ which can accommodate functions with arbitrary K-types. We also consider the 'heat equation' of the Casimir operator, which plays the role of the Laplacian for the group. We show that despite the structural difference of the Casimir with the Laplacian on $\mathbb{R}^n$ or the Laplace–Beltrami operator on the Riemannian symmetric spaces, it is possible to have a heat kernel. This heat kernel for the full group can also be characterized by Hardy-like estimates.

Keywords. Hardy's theorem; uncertainty principle; heat kernel, Casimir.

1. Introduction

A modified version of the Hardy's theorem states that if for a measurable function f on $\mathbb{R}$, $|f(x)| \leq C_1 \mathrm{e}^{-\alpha|x|^2}$ and $|\widehat{f}(x)| \leq C_2 \mathrm{e}^{-\beta|x|^2}$, $x \in \mathbb{R}$ for positive α, β, then (i) $\alpha \cdot \beta > 1/4$ implies $f = 0$ but (ii) $\alpha \cdot \beta = 1/4$ implies that f is a constant multiple of the Gauss kernel $\mathrm{e}^{-\alpha x^2}$. Through a number of articles in recent past, the first assertion of the theorem is established as a fairly general phenomenon of harmonic analysis on semi-simple, nilpotent and some other Lie groups. However, due to intrinsic difficulties, research remains incomplete in most of these cases as to the second assertion of the theorem, that is, which functions satisfy the sharpest possible decay conditions.

The purpose of this article is to extend the above result of Hardy (both (i) and (ii)) for the *full group* $SL_2(\mathbb{R})$. This article may be considered as a starting point to understand Hardy's result in its totality and in particular its relation with characterization of the heat kernel in the context of a full group. Our setting is sufficient to exhibit the new feature of considering the heat kernel of the Casimir operator on the group and at the same time concrete enough to provide explicit relations between the estimates and nontrivial isotypic components of the function.

Let G be $SL_2(\mathbb{R})$ and let K be its maximal compact subgroup $SO_2(\mathbb{R})$. As we are dealing with the group G our aim is to obtain a version of the theorem which accommodates functions with no restriction on K-types. Our first result characterizes a function with given arbitrary K-types satisfying the Hardy-like estimates. In fact, we get an explicit relationship between the vanishing of a K-isotypic component of a function and the estimates it satisfies. We find it necessary to strengthen the estimates using polynomials to preserve nontrivial isotypic components. We observe that a sharp point of the estimates can be achieved, only in terms of this polynomial say P, keeping $\alpha \cdot \beta = 1/4$. On one side (when deg $P = \upsilon < 0$ in Theorem 3.2) the function is identically zero and on the other side ($\upsilon = 0$), it is good enough to generate an L^1-module which is dense in the set of L^p-functions of its isotypic class for $p \in [1, 2)$ (Theorems 3.5 and 3.6).

Since the Casimir operator plays the role of the Laplacian for the group, we proceed to explore possible relation of Hardy's estimate with the characterization of the heat kernel of the Casimir operator. It is not difficult to see that the heat kernel of the Laplace–Beltrami operator on G/K which is bi-K-invariant cannot fulfil the requirement of being a heat kernel for the full group. But when the heat equation is considered on the group G, replacing the Laplace–Beltrami operator of G/K by the Casimir operator Ω, the situation changes drastically. Because, unlike the cases of Euclidean spaces and the Riemannian symmetric spaces of non-compact type, the general theory does not provide a well-defined heat kernel with nice L^p properties, due to the non-ellipticity of Ω. We show that the object which emerges naturally as a *heat kernel* of Ω and fits perfectly for the full group is only a virtual one. But the situation is saved *only* if we restrict in an arbitrary but fixed K-finite environment. Then, we can still have an actual heat kernel which retains all the nice properties of the Gauss kernel $[1/(2\sqrt{\pi t})^n]\mathrm{e}^{-\|x\|^2/4t}$ on $\mathbb{R}^n$ and there we are able to relate this heat kernel of Ω with the Hardy's estimates.

We conclude with a brief review and references on Hardy's theorem. Hardy proved this theorem on $\mathbb{R}$ in [10]. A well-known stronger version, which we have generalized, can be found, e.g., in [11] and [6]. For Lie groups the Hardy's theorem was first taken up by Sitaram and Sundari in [17]. This triggered considerable attention in the last half-decade. Different versions of this theorem was proved for semi-simple Lie groups, symmetric spaces, nilpotent groups, motion groups and solvable extensions of H-type groups. Among these many articles [5,7,16,17] have dealt with semi-simple Lie groups. See [8] for a comprehensive survey and references for Hardy's theorem on semi-simple and other groups. All these results are analogues of the first half of the Hardy's theorem where the estimates force the function to be zero. These results are usually viewed as *mathematical uncertainty principle*. For $SL_2(\mathbb{R})$, Theorem 3.2(ii) in this paper accommodates these results and provides the sharpest possible Hardy-type uncertainty (see Remark 3.3). See also [14] which takes up Hardy's theorem on G/K.

2. Notation and preliminaries

We shall mainly use the notation of Barker [3] with a few variations which will be mentioned here.

Unless stated otherwise, G is $SL_2(\mathbb{R})$, $\mathfrak{g}$ is the Lie algebras of G and $\mathfrak{g}_{\mathbb{C}}$ is the complexification of $\mathfrak{g}$ throughout this article. Let

$$X = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \qquad F = \begin{pmatrix} -1 & i \\ i & 1 \end{pmatrix}$$

and

$$H = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \overline{Y} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}.$$

Then $X, Y, \overline{Y} \in \mathfrak{g}$ and $\{X, E, F\}$ is a basis of $\mathfrak{g}_{\mathbb{C}}$. Let K be a fixed maximal compact subgroup of $SL_2(\mathbb{R})$. Then $K = \{k_\theta | \theta \in \mathbb{R}\}$ where

$$k_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \exp(\theta X).$$

Let $A = \{a_t = \exp 2tH | t \in \mathbb{R}\}$ and $A^+ = \{a_t | t > 0\}$. Then A is an Abelian subgroup of G. Let $\mathfrak{k}$ and $\mathfrak{a}$ denote the Lie algebras of K and A respectively. In our set up the multiplicities of the two positive roots γ and 2γ are 0 and 1 respectively, the half sum of the positive roots $\rho = 1$ and the Weyl group $\mathfrak{W}$ is $\mathbb{Z}_2$.

Let $\widehat{K} = \{\chi_n | n \in \mathbb{Z}\}$ be the set of the characters of K, where $\chi_n(k_\theta) = \mathrm{e}^{in\theta}$. Instead of χ_n, by abuse of language, we will call integers n as K-types. A complex valued function f on G is said to be of left (resp. right) K-type n if $f(kx) = \chi_n(k)f(x)$ (resp. $f(xk) = \chi_n(k)f(x)$) for all $k \in K$ and $x \in G$. A function is of type (m, n) if its left K-type is m and right K-type is n. Suppose F is a finite subset of $\widehat{K} = \mathbb{Z}$. Then we call a function left (resp. right) $K(F)$-finite, if all of its left (resp. right) K-types are in F. A function is $K(F)$-finite when it is both right and left $K(F)$-finite. For a suitable function f, the (m, n)-th isotypic component of f is denoted by $f_{m,n}$ and is given by:

$$\int_K \int_K \overline{\chi_m(k_1)}\, \overline{\chi_n(k_2)} f(k_1 x k_2) \mathrm{d}k_1 \mathrm{d}k_2.$$

Let $\mathfrak{a}^*$ be the real dual of $\mathfrak{a}$ and $\mathfrak{a}^*_{\mathbb{C}}$ be the complexification of $\mathfrak{a}^*$. Then $\mathfrak{a}^*_{\mathbb{C}}$ and $\mathfrak{a}^*$ has obvious identifications with $\mathbb{C}$ and $\mathbb{R}$ respectively. Let M be $\{\pm I\}$, where I is the 2×2 identity matrix. Then the unitary dual of M is $\widehat{M} = \{\sigma^+, \sigma^-\}$ of which σ^+ is the trivial representation of M. Let $\mathbb{Z}^{\sigma^+}$ (resp. $\mathbb{Z}^{\sigma^-}$) be the set of even (resp. odd) integers. Also let $-\sigma^+ = \sigma^-$ and $-\sigma^- = \sigma^+$.

For $\sigma \in \widehat{M}$ and $\lambda \in \mathfrak{a}^*_{\mathbb{C}} = \mathbb{C}$, let $(\pi_{\sigma,\lambda}, H_\sigma)$ be the principal series representation where H_σ is a subspace of $L^2(K)$. Define e_n on K by $e_n(k_\theta) = \mathrm{e}^{in\theta}$. Then, for $n \in \mathbb{Z}^\sigma$, $e_n \in H_\sigma$ transforms according to the K-type n and $\{e_n | n \in \mathbb{Z}^\sigma\}$ is an orthonormal basis for H_σ. The representation $\pi_{\sigma,\lambda}$ is so normalized that it is unitary if and only if $\lambda \in i\mathfrak{a}^* = i\mathbb{R}$. For every $k \in \mathbb{Z}$ there is a discrete series representation π_k which occur as a subrepresentation of $\pi_{\sigma,|k|}$ so that $k \in \mathbb{Z}^{-\sigma}$. For $m, n \in \mathbb{Z}^\sigma$ and $k \in \mathbb{Z}^{-\sigma}$, let $\Phi^{m,n}_{\sigma,\lambda}(x) = \langle \pi_{\sigma,\lambda}(x) e_m, e_n \rangle$ and $\Psi^{m,n}_k(x) = \langle \pi_k(x) e^k_m, e^k_n \rangle_k$, be the matrix coefficients of the principal series and discrete series representations respectively, where $\{e^k_n\}$ are the renormalized basis and $\langle\, ,\, \rangle_k$ is the renormalized inner product of π_k (see [3], p. 20). Also, for every positive integer m, G has exactly one irreducible representation of dimension m which are subrepresentations of $\pi_{\sigma,-m}$ with $m \in \mathbb{Z}^{-\sigma}$. For details of the parameterization of the representations $\pi_{\sigma,\lambda}$ and π_k and their realizations on $L^2(K)$ we refer to [3].

For $f \in C^\infty(G)$, $D \in \mathfrak{g}$ and any admissible representation (π, V), define

$$\pi(D)v = \frac{\mathrm{d}}{\mathrm{d}t} \pi(\exp tD)v|_{t=0}, \qquad v \in V,$$

$$f(x; D) = \frac{\mathrm{d}}{\mathrm{d}t} f(x \exp(tD))|_{t=0}, \qquad f(D; x) = \frac{\mathrm{d}}{\mathrm{d}t} f(\exp(tD)x)|_{t=0}.$$

Let us recall that $\{X, E, F\}$ is a basis of $\mathfrak{g}_{\mathbb{C}}$. The following action of these basis elements on $\{e_n\}$, will be useful for us (see [3], 4.4 and 4.6):

$$\pi_{\sigma,\lambda}(X) e_n = ine_n \tag{1}$$

$$\pi_{\sigma,\lambda}(E)e_n = (n+\lambda+1)e_{n+2} \tag{2}$$

$$\pi_{\sigma,\lambda}(F)e_n = (n-\lambda-1)e_{n-2}. \tag{3}$$

Let d be the distance on G/K induced by the Riemannian metric on it. Define $\sigma(x) = \mathrm{d}(xK, o)$ where $o = eK$. Then $\sigma(\exp H) = |H| = \langle H, H\rangle^{1/2}\ \forall H \in \mathfrak{a}$, where $\langle\ ,\ \rangle$ is the Killing form. Then we have

$$\mathrm{e}^{-\sigma(x)} \le \Xi(x) \le \mathrm{e}^{-\sigma(x)}(1+\sigma(x)) \quad \text{for all } x \in G \tag{4}$$

where $\Xi(x) = \Phi^{0,0}_{\sigma^+,0}(x)$ (see [3], 3.2).

Let $\mathrm{d}k$ and $\mathrm{d}a$ respectively be the Haar measures on K and A and $\int_K \mathrm{d}k = 1$. Then the Haar measure $\mathrm{d}x$ on G is given by $\mathrm{d}x = J(a)\mathrm{d}k_1\mathrm{d}a\mathrm{d}k_2$, where the $J(a)$ is the Jacobian of the $K\bar{A}^+K$ decomposition of G. Also,

$$|J(a)| \le C\mathrm{e}^{2\sigma(a)} \quad \text{for all } a \in A. \tag{5}$$

From the integral expression of $\Phi^{m,n}_{\sigma,\lambda}$ (see [3], 4.8), it follows that $|\Phi^{m,n}_{\sigma,\lambda}(x)| \le C_{m,n}|\Phi^{0,0}_{\lambda}(x)|$, where $C_{m,n}$ is a constant which depends on m, n. Now as $|\Phi^{0,0}_{\lambda}(a)| \le \mathrm{e}^{\lambda^+_{\mathbb{R}}(\log a)}\,\Xi(a)$ (see [9], Proposition 4.6.1), we have

$$|\Phi^{m,n}_{\sigma,\lambda}(a)| \le C\mathrm{e}^{\lambda^+_{\mathbb{R}}(\log a)}\,\Xi(a), \quad a \in A. \tag{6}$$

Here $\lambda^+_{\mathbb{R}}$ is the Weyl translate of $\lambda_{\mathbb{R}}$ which is dominant, i.e., belongs to the positive Weyl chamber. In our case $\lambda^+_{\mathbb{R}}$ is simply $|\lambda_{\mathbb{R}}|$.

For $p \in (0, 2]$ let $\mathcal{S}^p = \{z \in \mathbb{C} \mid |\Re\lambda| \le (2/p) - 1\}$. Then from the estimate (6), it follows that L^q-norm of $\Phi^{m,n}_{\sigma,\lambda}(x)$ is uniformly bounded on $\lambda \in \overset{\circ}{\mathcal{S}^p}$, where $\overset{\circ}{\mathcal{S}^p}$ is the interior of $\mathcal{S}^p$.

For a suitable function f, let $\widehat{f} = (F_H(f), F_B(f))$ where $F_H(f)$ and $F_B(f)$ denote its continuous and discrete Fourier transforms respectively. Precisely

$$F_H(f)(\sigma,\lambda) = \int_G f(x)\pi_{\sigma,\lambda}(x^{-1})\mathrm{d}x$$

and

$$F_B(f)(k) = \int_G f(x)\pi_k(x^{-1})\mathrm{d}x.$$

The (m, n)-th matrix coefficients of $F_H(f)$ and $F_B(f)$ are denoted by $F^{m,n}_H(f)$ and $F^{m,n}_B(f)$. Thus $F^{m,n}_H(f)(\sigma,\lambda) = \int_G f(x)\Phi^{m,n}_{\sigma,\lambda}(x^{-1})\mathrm{d}x$ and $F^{m,n}_B(f)(k) = \int_G f(x)\Psi^{m,n}_k(x^{-1})\mathrm{d}x$. Then clearly, $F^{m,n}_H(f) = F_H(f_{m,n})$. Since m, n determines a unique $\sigma \in \widehat{M}$ by $m, n \in \mathbb{Z}^\sigma$, we may sometimes write $F^{m,n}_H(f)(\lambda)$ for $F^{m,n}_H(f)(\sigma,\lambda)$, omitting the σ as that does not lead to any confusion.

For $p \in (0, 2]$ we denote the L^p-Schwartz spaces of G by $C^p(G)$ and $C^0(G) = \cap_{0\le p\le 2}C^p(G)$ (see [3] for definitions). The space of compactly supported infinitely differentiable functions of G is denoted by $C^\infty_c(G)$. By $C^p(\widehat{G})$, $C^0(\widehat{G})$ and $C^\infty_c(\widehat{G})$ respectively we denote their images under Fourier transform. For any class of function $\mathcal{F}$, $\mathcal{F}_{m,n}$ denotes the corresponding subclass of (m, n)-type functions.

We follow the practice of using C, C' etc. to denote a constant (real or complex) whose value might change from line to line. We use subscripts of C when needed to indicate their dependence on parameters of interest. We may not repeat mentioning these at the particular places.

3. Hardy's theorem

In our parameterizations (see [3], p. 16), principal series representations $\{\pi_{\sigma,\lambda} | (\sigma, \lambda) \in \widehat{M} \times \mathbb{C}\}$ are not irreducible only when $\lambda \in \mathbb{Z}^{-\sigma}$. But when $\pi_{\sigma,\lambda}$ is not irreducible it has irreducible subrepresentations. If $\pi_{\sigma,\lambda}$ has an irreducible subrepresentation which contains e_m but not e_n, for $m, n \in \mathbb{Z}^\sigma$, then $\Phi^{m,n}_{\sigma,\lambda}(x) = \langle \pi_{\sigma,\lambda}(x) e_m, e_n\rangle = 0$ for all $x \in G$ and hence the Fourier transform of any admissible function of (m, n) type is zero at this $(\sigma, \lambda) \in \widehat{M} \times \mathbb{C}$. There are $|m-n|/2$ such zeroes each of order one of the Fourier transform of an (m, n) type function (see [3], Proposition 7.1). Let p_{mn} be the monomial with exactly those zeroes with order one. Then p_{mn} is clearly a polynomial of degree $|m - n|/2$ with integer coefficients. For instance, when $m > n$, then

$$p_{m,n} = (m - 1 - \lambda)(m - 3 - \lambda) \cdots (n + 1 - \lambda).$$

Thus when $m > n$ we see that the coefficients for e_{n-2j} (j positive integer less than $|m - n|/2$) in the right hand side of (3) are factors of $p_{m,n}$. Using that, one can easily show that $\pi_\lambda(F^{|m-n|/2})e_m = p_{m,n}(\lambda)e_n$. Similarly, using (2), we get $\pi_\lambda(E^{|m-n|/2})e_n = p_{m,n}(-\lambda)e_m$. When $m < n$, then in an analogous way we can also prove $\pi_\lambda(E^{|m-n|/2})e_m = p_{m,n}(\lambda)e_n$ and $\pi_\lambda(F^{|m-n|/2})e_n = p_{m,n}(-\lambda)e_m$.

Lemma 3.1. *Let $f \in L^p(G)_{m,n}$ for some $p \in [1, 2)$ and $m, n \in \mathbb{Z}^\sigma$ for some $\sigma \in \mathbb{Z}^\sigma$. Then $F^{m,n}_H(f)(\sigma, \lambda)/p_{m,n}(\lambda)$ is an even analytic function on $\overset{\circ}{\mathcal{S}}{}^p$.*

Proof. Analyticity of $F^{m,n}_H(f)(\sigma, \lambda)/p_{m,n}(\lambda)$ on $\overset{\circ}{\mathcal{S}}{}^p$ is clear from the definition of $p_{m,n}(\lambda)$ and the fact that the Fourier transform of an L^p function is analytically extendable on $\overset{\circ}{\mathcal{S}}{}^p$. We have to prove that it is also even.

Let us first assume $m > n$. Let $g \in C^\infty_c(G)_{m,n}$. Then m, n determines a $\sigma \in \widehat{M}$ by $m, n \in \mathbb{Z}^\sigma$. We write $F^{m,n}_H(g)(\lambda)$ for $F^{m,n}_H(g)(\sigma, \lambda)$. Then, using the relations preceding the lemma, we have

$$\begin{aligned} p_{m,n}(-\lambda)F^{m,n}_H(g)(\lambda) &= p_{m,n}(-\lambda) \int_G g(x)\langle \pi_\lambda(x)e_m, e_n\rangle \mathrm{d}x \\ &= \int_G g(x)\langle \pi_\lambda(x) p_{m,n}(-\lambda)e_m, e_n\rangle \mathrm{d}x \\ &= \int_G g(x)\langle \pi_\lambda(x)\pi_\lambda(E^{|m-n|/2})e_n, e_n\rangle \mathrm{d}x \\ &= \int_G g(x)\Phi^{n,n}_\lambda(x; E^{|m-n|/2})\mathrm{d}x \\ &= \int_G g(x; (-1)^{|m-n|/2}E^{|m-n|/2})\Phi^{n,n}_\lambda(x)\mathrm{d}x \\ &= F^{n,n}_H(h)(\lambda), \end{aligned}$$

where $h = g(x; (-1)^{|m-n|/2}E^{|m-n|/2}) \in C^\infty_c(G)_{n,n}$. Thus $p_{m,n}(-\lambda)F^{m,n}_H(g)(\lambda)$ is in $C^\infty_c(\widehat{G})_{n,n}$. From the characterization of the image of $C^\infty_c(G)$ under the Fourier transform (see [3], Theorem 10.5) it follows that, $p_{m,n}(-\lambda)F^{m,n}_H(g)(\lambda)$ is an even function on $\mathbb{C}$. Therefore, $F^{m,n}_H(g)(\lambda)/p_{m,n}(\lambda)$ is also an even function. As $C^\infty_c(G)_{m,n}$ is dense in $L^p(G)_{m,n}$, for any function $f \in L^p(G)_{m,n}$, there exists $\{f_i\} \in C_c^\infty(G)_{m,n}$ such that $f_i \longrightarrow f$ in L^p. Now as $\sup_{\lambda\in i\mathbb{R}} |F_H^{m,n}(f)(\sigma,\lambda)| \le ||f||_p \sup_{\lambda\in i\mathbb{R}} ||\Phi_{\sigma,\lambda}^{m,n}||_q$ and as the L^q-norm of $\Phi_{\sigma,\lambda}^{m,n}$ is bounded uniformly on $\overset{\circ}{\mathcal{S}}{}^p$, $F_H^{m,n}(f_i) \longrightarrow F_H^{m,n}(f)$ uniformly on $\overset{\circ}{\mathcal{S}}{}^p$. Therefore $F_H^{m,n}(f)(\lambda)/p_{m,n}(\lambda)$ is even on $\overset{\circ}{\mathcal{S}}{}^p$. The case $m < n$ is similar. □

When a function $f \in L^p(G)_{m,n}$ has greater decay so that $F_H^{m,n}(f)(\lambda)$ can be defined on a bigger domain than $\overset{\circ}{\mathcal{S}}{}^p$, then by analyticity, $F_H^{m,n}(f)(\lambda)/p_{m,n}(\lambda)$ will be even on the bigger domain.

Now we are in a position to state the main results:

Theorem 3.2. *Let f be a measurable function on G such that*

$$|f(x)| \le C e^{-\alpha(\sigma(x))^2}\,\Xi(x)(1+\sigma(x))^u \quad \textit{for all } x \in G \tag{7}$$

and

$$||F_H(f)(\sigma,\lambda)|| \le C_\sigma e^{-\beta|\lambda|^2}(1+|\lambda|)^v \quad \textit{for all } (\sigma,\lambda) \in \widehat{M} \times i\mathfrak{a}^* \tag{8}$$

where $||\cdot||$ is the operator norm with respect to the norm of H_σ, $u > 0$ and v are integers and $\alpha, \beta, C, C_\sigma$ are positive constants.

If $\alpha \cdot \beta = 1/4$, then

(i) *for $m, n \in \mathbb{Z}$, if $|m-n|/2 > v$, then $f_{m,n} = 0$,*

(ii) *if $v < 0$, then $f = 0$,*

(iii) *for $\sigma \in \widehat{M}$ and $m, n \in \mathbb{Z}^\sigma$ with $|m-n|/2 \le v$,*

$$F_H^{m,n}(f)(\sigma,\lambda) = P_\beta(m,n,\lambda)e^{\beta\lambda^2} \quad \textit{for all } \lambda \in \mathfrak{a}_{\mathbb{C}}^*$$

and $F_B^{m,n}(f)(k) = P_\beta(m,n,k)e^{\beta k^2}$ for any discrete series π_k such that $\pi_k|_K \supset \chi_n, \chi_m$, where P_β is a polynomial in m, n, λ (resp. in m, n, k) which depends on β. In particular if $|m-n|/2 = v$ then $P_\beta(m,n,\lambda) = C_{m,n}p_{m,n}(\lambda)$.

(iv) *if $v = 0$, then $f_{m,n} = 0$ for $m, n \in \mathbb{Z}$ with $m \ne n$,*
$F_H^{n,n}(f)(\sigma,\lambda) = C_{\beta,n} \cdot e^{\beta\lambda^2}$ for all $(\sigma,\lambda) \in \widehat{M} \times \mathfrak{a}_{\mathbb{C}}^*$ and $n \in \mathbb{Z}^\sigma$ and
$F_B^{n,n}(f)(k) = C_{\beta,n} \cdot e^{\beta\cdot k^2}$, for any discrete series π_k such that $\pi_k|_K \supset \chi_n$.

Proof. Let $\sigma \in \widehat{M}$ and $m, n \in \mathbb{Z}^\sigma$.

Step 1. In this step, we will show that $F_H^{m,n}(f)(\lambda)$ is an entire function and

$$|F_H^{m,n}(f)(\lambda)| \le C e^{\beta|\lambda|^2} \cdot (1+|\lambda|)^{u+2} \tag{9}$$

for all $\lambda \in \mathbb{C}$

$$F_H^{m,n}(f)(\sigma,\lambda) = \int_G f(x)\Phi_{\sigma,\lambda}^{m,n}(x)\mathrm{d}x = \int_{A^+} f(a)\Phi_{\sigma,\lambda}^{m,n}(a)J(a)\mathrm{d}a.$$

Therefore using (7), (6), (5), we have,

$$|F_H^{m,n}(f)(\sigma,\lambda)| \le C \cdot \int_{\bar{A}^+} e^{-\alpha\sigma(a)^2} e^{\lambda_{\mathbb{R}}^+(\log a)}\,\Xi^2(a)\cdot(1+\sigma(a))^u e^{2\sigma(a)}\mathrm{d}a.$$

Now applying (4) we get

$$\begin{aligned}|F_H^{m,n}(f)(\sigma,\lambda)| &\leq C\cdot\int_{\bar{A}^+} \mathrm{e}^{-\alpha\sigma(a)^2}\mathrm{e}^{\lambda_{\mathbb{R}}^+(\log a)}\mathrm{e}^{-2\sigma(a)}\cdot(1+\sigma(a))^{u+2}\mathrm{e}^{2\sigma(a)}\mathrm{d}a\\ &= C\cdot\int_{\bar{A}^+} \mathrm{e}^{-\alpha\sigma(a)^2}\mathrm{e}^{\lambda_{\mathbb{R}}^+(\sigma(a))}\cdot(1+\sigma(a))^{u+2}\mathrm{d}a\\ &\leq C\cdot\int_{\mathfrak{a}} \mathrm{e}^{-\alpha|H|^2}\mathrm{e}^{\lambda_{\mathbb{R}}^+(H)}\cdot(1+H)^{u+2}\mathrm{d}H.\end{aligned}$$

Here $H=\log a$, i.e., $H\in\mathfrak{a}$ such that $\exp H=a$, $\mathrm{d}H$ is the Lebesgue measure on $\mathfrak{a}$. Suppose $H_{\lambda_{\mathbb{R}}}$ corresponds to $\lambda_{\mathbb{R}}^+$ via the isomorphism of $\mathfrak{a}^*$ with $\mathfrak{a}$ through the Killing form (i.e. $\lambda_{\mathbb{R}}^+(H)=\langle H, H_{\lambda_{\mathbb{R}}}\rangle$ for all H) so that $|\lambda_{\mathbb{R}}^+|=|H_{\lambda_{\mathbb{R}}}|$. Then

$$\begin{aligned}|F_H^{m,n}(f)(\sigma,\lambda)| &\leq C\cdot \mathrm{e}^{1/4\alpha|H_{\lambda_{\mathbb{R}}}|^2}\int_{\mathfrak{a}}\mathrm{e}^{-\alpha\langle H-(1/2\alpha)H_{\lambda_{\mathbb{R}}},H-(1/2\alpha)H_{\lambda_{\mathbb{R}}}\rangle}\\ &\qquad\cdot(1+|H|)^{u+2}\mathrm{d}H.\end{aligned}$$

Using translation invariance of Lebesgue measure

$$\begin{aligned}|F_H^{m,n}(f)(\sigma,\lambda)| &\leq C\cdot \mathrm{e}^{1/4\alpha|H_{\lambda_{\mathbb{R}}}|^2}(1+|H_{\lambda_{\mathbb{R}}}|)^{u+2}\\ &\qquad\times\int_{\mathfrak{a}^+}\mathrm{e}^{-\alpha|H|^2}(1+|H|)^{u+2}\mathrm{d}H\\ &= C\cdot \mathrm{e}^{1/4\alpha|\lambda|^2}(1+|\lambda|)^{u+2}\\ &\qquad\times\int_{\mathfrak{a}^+}\mathrm{e}^{-\alpha|H|^2}(1+|H|)^{u+2}\mathrm{d}H \quad \text{as } |\lambda_{\mathbb{R}}^+|=|\lambda_{\mathbb{R}}|\leq|\lambda|\\ &\leq C'\cdot\mathrm{e}^{\beta|\lambda|^2}(1+|\lambda|)^{u+2}\quad \text{as } \beta=\frac{1}{4\alpha}.\end{aligned}$$

Step 2. Let $F(z)=F_H^{m,n}(f)(z)/p_{m,n}(z)$. Then, from Lemma 4.1, F is an even entire function.

Consider $\phi(z)=F(\sqrt{z})$. Then ϕ is also entire and by (9) and (8) respectively, it satisfies the inequalities

$$|\phi(z)|\leq C\mathrm{e}^{\beta|z|}(1+|z|)^s \quad \text{for } z\in\mathbb{C} \tag{10}$$

and

$$|\phi(\lambda)|\leq C_\sigma\mathrm{e}^{-\beta\lambda}(1+|\lambda|)^{s'} \quad \text{for } \lambda\in i\mathbb{R}^+ \tag{11}$$

where $2s=(u+2-|m-n|/2)$ and $2s'=v-|m-n|/2$.

Let us define $\psi(z)=\phi(iz)$. Then from the above two inequalities, we have

$$|\psi(z)|\leq C\mathrm{e}^{\beta|z|}(1+|z|)^s \quad \text{for } z\in\mathbb{C} \tag{12}$$

and

$$|\psi(x)|\leq C'_\sigma\mathrm{e}^{-\beta x}(1+x)^{s'} \quad \text{for } x\in\mathbb{R}^+ \tag{13}$$

where s and s' are as above.

We claim that for $s'' \geq \max\{s, s'\}$,

$$|\psi(z)\mathrm{e}^{\beta z}| \leq C(1+|z|)^{s''} \quad \text{for } z \in \mathbb{C}. \tag{14}$$

Let $F'(z) = w(z,\Theta)\psi(z)/(z+i)^{s''}$ on $D_\Theta = \{r\mathrm{e}^{i\theta} | 0 \leq \theta \leq \Theta\}$ for $\Theta \in (0,\pi)$, where $w(z,\Theta) = w(r,\theta,\Theta) = \exp[\beta iz \cdot \mathrm{e}^{-i\Theta/2}/\sin(\frac{\Theta}{2})]$.

Then

$$|w(r,0,\Theta)| = \mathrm{e}^{\beta r} \tag{15}$$

$$|w(r,\Theta,\Theta)| = \mathrm{e}^{-\beta r} \tag{16}$$

$$w(z,\Theta) \longrightarrow \mathrm{e}^{\beta z} \quad \text{as } \Theta \longrightarrow \pi. \tag{17}$$

From (15) and (13) we have $|F'(x)| = |F'(r)| \leq C[(1+r)^{s'}/(i+r)^{s''}] \leq M$. Again from (16) and (12), we get $|F'(r\mathrm{e}^{i\Theta})| \leq C[(1+r)^{s}/(i+r)^{s''}] \leq M$ for some $M > 0$.

By Phragmén–Lindelöf theorem ([4], II,6.1] we have $|F'(z)| \leq M$ on D_Θ. Now from (17), we conclude that on $\{z \in \mathbb{C} \,|\, \Im z \geq 0\}$, $|\psi(z)\mathrm{e}^{\beta z}/(z+i)^{s''}| \leq M$. Similarly we can prove that on the lower half plane $|\psi(z)\mathrm{e}^{\beta z}/(z-i)^{s''}| \leq M$. Combination of the above two results proves the claim (14).

Step 3. From (14), we get, for $z \in \mathbb{C}$, $\phi(iz) = \psi(z) = C \cdot Q_1(z) \cdot \mathrm{e}^{-\beta z}$ for some polynomial Q_1 such that $\deg Q_1 \leq s''$. Therefore,

$$F(iz) = C_\beta \cdot \mathrm{e}^{-\beta z^2} Q_1(z^2) = C_\beta \cdot \mathrm{e}^{\beta(iz)^2} Q_2((iz)^2). \tag{18}$$

Hence, for some polynomial Q with $0 \leq \deg Q \leq s''$

$$F(z) = C_\beta \cdot \mathrm{e}^{\beta z^2} Q(z^2) \tag{19}$$

and finally

$$F_H^{m,n}(f)(z) = C_\beta \cdot \mathrm{e}^{-\beta z^2} Q(z^2) \cdot p_{m,n}(z) \quad \text{for } z \in \mathbb{C}. \tag{20}$$

Then from (8), $v \geq \deg Q + |m-n|/2$. Therefore $|m-n|/2 > v$ implies that $f_{m,n} = 0$. This proves (i).

If $v < 0$, then $v < |m-n|/2$ for all $m, n \in \mathbb{Z}^\sigma$. Hence $f_{m,n} = 0$ for all m, n and hence $f = 0$. Thus (ii) is proved.

If $v = 0$, then $|m-n|/2 \leq v$ only when $m = n$. Therefore $f_{m,n} = 0$ for all m, n with $m \neq n$. This proves the first part of (iv).

When $m = n$ then $p_{m,n} \equiv 1$ by definition. Therefore from (20)

$$F_H^{n,n}(f)(z) = C_\beta \cdot \mathrm{e}^{-\beta z^2} Q(z^2). \tag{21}$$

But if $v = 0$ then from (8) it is clear that Q can be only constant. This proves the last part of (iv) while (iii) is clear from (20). □

Remark 3.3. From Theorem 3.2(ii) it is easy to see that if the condition (8) is replaced by '$F_H(f)(\sigma,\lambda)$ is $o(\mathrm{e}^{-\beta|\lambda|^2})$', then $f = 0$. With this observation, the above theorem looks like an exact analogue of the original result of Hardy (see [10]). Clearly it takes care of the following Hardy's theorem for $SL_2(\mathbb{R})$ (see [17], Theorem 5.1).

Theorem 3.4. *Let f be a measurable function on G such that*

$$|f(x)| \leq C\mathrm{e}^{-\alpha(\sigma(x))^2} \quad \textit{for all } x \in G \tag{22}$$

and

$$||F_H(f)(\sigma, \lambda)|| \leq C_\sigma \mathrm{e}^{-\beta|\lambda|^2} \quad \textit{for all } (\sigma, \lambda) \in \widehat{M} \times i\mathfrak{a}^* \tag{23}$$

where $||\cdot||$ is the operator norm with respect to the norm of H_σ and α, β, C, C_σ are positive constants with $\alpha \cdot \beta > 1/4$, then $f = 0$.

In contrast to the above theorem it is shown in Theorem 3.2(ii) that even when $\alpha \cdot \beta = \frac{1}{4}$ then also $f = 0$, provided $v < 0$. On the other hand, from Theorem 3.2(iv) we can show that if a nonzero function satisfies the estimates with $v = 0$ then f has nonzero Fourier transform at every representation relevant for f. That is, if the Fourier transform of f vanishes at a (unitary or nonunitary) principal series or a discrete series representation π, then $\pi|_K$ does not contain any of the K-types of f. We notice the sharp contrast between the cases $v = 0$ and $v < 0$.

Our observation related to the case $v = 0$ leads to the following consequence.

Let $p \in [1, 2)$. Suppose for a function $f \in L^p(G)$, $M(f)$ is the L^1-bimodule (under convolution) generated by f in $L^p(G)$. It is well known that if $M(f)$ is dense in $L^p(G)$ then f satisfies these two necessary conditions:

(A) The even and odd parts of f have infinitely many *positive* and *negative* K-types.
(B) The bi-invariant component of f is nonzero.

It can be proved that if a function $f \in L^p(G)$ satisfies these necessary conditions and decays as in Hardy's theorem, then f, in fact, generates a dense L^1-bimodule in $L^p(G)$. Precisely:

Theorem 3.5. *Let f be a measurable function on G which satisfies* (A) *and* (B) *above. Assume also that f and $F_H(f)$ satisfy the inequalities* (7) *and* (8) *respectively with $\alpha \cdot \beta = 1/4$ and $v = 0$. Then f is an L^p function and $\overline{M(f)} = L^p(G)$.*

Proof. Take a discrete series representation π_k. Then $f_{n,n} \neq 0$ for some integer n of parity opposite to k with $n > k$ or $< k$ according as $k > 0$ or < 0. It is possible to find such an $f_{n,n}$, as odd and even parts of f are non-K-finite on both positive and negative sides and as by Theorem 3.2(iv) $f_{m,n} = 0$ almost everywhere for $m \neq n$. This n determines a $\sigma \in \widehat{M}$ by $n \in \mathbb{Z}^\sigma$. Now as f and $F_H^{n,n}(f)(\sigma, \cdot)$ satisfy respectively the inequalities (7) and (8) with $v = 0$ and $\alpha \cdot \beta = 1/4$, by Theorem 3.2(iv) it is clear that $F_{n,n}^H(f)(\sigma, \lambda) \neq 0\ \forall \lambda \in \mathbb{C}$ and hence $F_{n,n}^H(f)(\sigma, \cdot)$ is nonzero on all of the irreducible subrepresentations of $\{\pi_{\sigma,\lambda}|\lambda \in \mathbb{C}\}$ containing the vector e_n. Therefore, $f_{n,n}$ has nonzero Fourier transform on π_k. Because π_k is equivalent to the irreducible subrepresentation of $\pi_{\sigma,|k|}$ which contains e_n, f has nonzero Fourier transform at all principal and discrete series representations. For similar reason f has nonzero Fourier transform also on the limits of discrete series.

If $f_{0,0} \neq 0$, then by the same argument $f_{0,0}$ has nonzero Fourier transform at the trivial representation which is a subrepresentation of $\pi_{\sigma^+,-1}$. Hence $\int_G f(x)\mathrm{d}x = \int_G f(x)\Phi^{0,0}_{\sigma^+,-1}(x)\mathrm{d}x \neq 0$.

If $f_{n,n} \neq 0$, then by Theorem 3.2(iv) clearly,

$$\limsup_{|\lambda|\longrightarrow\infty} |F^H_{n,n}(f)(\sigma,\lambda)\cdot \mathrm{e}^{C' e^{|\lambda|}}| > 0 \quad \text{for all } C' > 0, \lambda \in \mathfrak{a}^*.$$

Now the theorem follows from the Wiener–Tauberian theorem for $SL_2(\mathbb{R})$ (see [15], Theorem 1.2). □

Note that condition (B) above is necessary only when $p = 1$ and can be dropped from the hypothesis of the theorem when $p > 1$. We also have the following simpler version when instead of $L^p(G)$ we restrict on a particular K-type on the right. We know that functions of right K-type n may be considered as sections of certain line bundle on G/K corresponding to n. Let us call that line bundle Γ_n. Using similar argument as above we can show:

Theorem 3.6. *Let s be a measurable L^p-section of the line bundle Γ_n for some $n \in Z$. If s and its continuous Fourier transform satisfy* (7) *and* (8) *respectively with $\alpha\cdot\beta = 1/4$ and $v = 0$, then the left $L^1(\Gamma_n)$-module generated by s is dense in $L^p(\Gamma_n)$.*

4. Heat kernel on $SL_2(\mathbb{R})$

In this section we will explore possible connection of the Hardy's theorem with the characterization of the *heat kernel.*

Let us consider a general noncompact, connected semisimple Lie group for this discussion and call it G. We shall come back again to the particular case of $SL_2(\mathbb{R})$ after that. Let K be a fixed maximal compact subgroup of G. For a function f on $G \times \mathbb{R}^+$, let $\{f_t\}$ denote the parameterized family of function on G defined by $f_t(x) = f(x,t)$.

Nelson (see [13]) constructed the heat kernel $\{k^+_t(x)\}_{t>0}$ on G, which is an analogue of the Gauss kernel $(1/(2\sqrt{\pi t})^n)\mathrm{e}^{-||x||^2/4t}$ on $\mathbb{R}^n$. This $\{k^+_t(x)\}_{t>0}$ constitutes a fundamental solution of the heat equation $(\partial u/\partial t) = \Delta^+ u$ of a Laplace operator Δ^+ of G. Precisely, let $\{X_i\}_{1\le i\le n}$ be a basis of the Lie algebra $\mathfrak{g}$ of G. Consider the left invariant differential operator $\Delta^+ = \sum_{1\le i,j\le n} a_{ij}X_iX_j$, where $\{a_{ij}\}$ is a symmetric positive definite matrix. Then it is known that (see [18], Chapter v, [13], §8), $T_t = \mathrm{e}^{t\Delta^+}$, $t > 0$ defines a semigroup (heat-diffusion semigroup) of operators such that for any $\phi \in C^\infty_c(G)$, $T_t\phi$ is a solution of $\Delta^+ u = (\partial u/\partial t)$, and $T_t\phi \longrightarrow \phi$ a.e. as $t \longrightarrow 0$. Also T_t is an integral operator with kernel k^+_t, i.e., for any $\phi \in C^\infty_c(G)$, $T_t\phi = k^+_t * \phi$. This kernel $k^+_t(x)$ is in $C^\infty(G\times\mathbb{R}^+)$ and is the fundamental solution of $\Delta^+ u = (\partial u/\partial t)$. Moreover, $k^+_t \in L^1(G)\cap L^\infty(G)$. The particular structure of Δ^+ is responsible for the existence of the kernel k^+_t. In fact Magyar [12] has shown that it is enough to consider a real formally negative elliptic right or left invariant differential operator of second order without constant term for this purpose.

Let us now look back at the situation on G/K. Consider the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, where $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$, with respect to the Killing form. Suppose we choose the basis $\{X_i\}_{1\le i\le n}$ of $\mathfrak{g}$ such that $\{X_i\}_{1\le i\le r}$ is a basis of $\mathfrak{k}$ and $\{X_i\}_{r+1\le i\le n}$ is a basis of $\mathfrak{p}$. Then one can find two positive definite symmetric matrices $\{b_{ij}\}$ and $\{c_{kl}\}$, with $1 \le i,j \le r$ and $r+1 \le k,l \le n$ such that $\Omega = \sum c_{kl}X_kX_l - \sum b_{ij}X_iX_j$ is the Casimir operator (see [19]-I, p. 168) of G. But as $\sum b_{ij}X_iX_j$ annihilates any smooth function on G/K (regarded as a right-K-invariant function on G), the action of Ω on G/K is identical with the action of the

Laplace–Beltrami operator, Δ on G/K. It can be shown that for suitable $\{a_{ij}\}$, there exists $\Delta^+ = \sum_{1\le i,j\le n} a_{ij}X_iX_j$ as above, whose restriction to G/K is Δ (see [18], I.7). In fact these $\{a_{ij}\}$ corresponds to a K-invariant positive definite quadratic form on the tangent space of G/K at $o = eK$ and from it we get the G-invariant Riemannian metric on G/K so that Δ becomes the Laplace–Beltrami operator of G/K for that metric. The availability of such a Δ^+ is behind the existence of the fundamental solution of the heat equation of Δ on G/K. Precisely, Δ also gives rise to a semigroup $T_t = \mathrm{e}^{t\Delta}$, $t > 0$ of convolution operators with kernel k_t which is the fundamental solution of the heat equation $\Delta u = \frac{\partial u}{\partial t}$.

Recall that the Casimir operator Ω on the group enjoys the position of the Laplace operator. So, it is natural to consider its 'heat equation'

$$\Omega u = \frac{\partial u}{\partial t}.$$

Note that in the accepted terminology the words 'heat equation' and 'heat kernel' are associated to an evolution equation whose generator is elliptic. The Casimir operator is not elliptic; in fact it is hyperbolic. Therefore, we shall call the evolution equation of the Casimir operator Ω as *pseudo-heat equation* and the kernel of the Casimir semigroup as *pseudo-heat kernel*. This conforms to the terminology of the pseudo-Riemannian manifold. It does not seem to be known if this pseudo-heat equation of Ω on G has a nice fundamental solution.

Nevertheless, we will see below that for this pseudo-heat equation of the Casimir operator on $SL_2(\mathbb{R})$, a formal (pseudo) heat kernel emerges rather naturally and its restriction on every finite subset F of $\widehat{K}$, is a one-parameter family of solutions $\{k_t^F | t \in \mathbb{R}^+\}$ of the pseudo-heat equation of Ω, such that for any left (resp. right) $K(F)$-finite $\phi \in C_c^\infty(SL_2(\mathbb{R}))$, $k_t^F * \phi \longrightarrow \phi$ (resp. $\phi * k_t^F \longrightarrow \phi$) in L^p. We will also see that this k_t^F retains all the nice properties of the heat kernel of G/K or for that matter of the original Gauss kernel on $\mathbb{R}^n$.

From here till the end of this article, G is again $SL_2(\mathbb{R})$. The Casimir operator of $G = SL_2(\mathbb{R})$ is $\Omega = H^2 + H - \bar{Y}Y$ (see [3], 2.6). Then,

Theorem 4.1. *Let $p \in [1,2]$ be fixed and let $\phi(x) \in C_c^\infty(G)$. Then there exists unique $h^\phi(x,t) \in C^\infty(G \times \mathbb{R}^+)$ such that the following holds*:

$$\Omega h_t^\phi - \frac{\partial}{\partial t} h_t^\phi = 0 \quad \textit{for all } t > 0, \tag{24}$$

$$h_t^\phi(x) \longrightarrow \phi(x) \quad \textit{in } L^p \textit{ as } t \longrightarrow 0, \tag{25}$$

$h_t^\phi(x), \Omega h_t^\phi(x), \frac{\partial}{\partial t} h_t^\phi \in L^p(G)$ *for all* $t > 0$ *and* $\lim_{\delta\longrightarrow 0} \left|\left|[h_{t+\delta}^\phi(\cdot) - h_t^\phi(\cdot)]/\delta - \partial h_t^\phi(\cdot)/\partial t\right|\right|_p = 0.$

Proof. For $t > 0$, we define $H_t(\sigma,\lambda)$ by $H_t(\sigma,\lambda)_{m,n} = \mathrm{e}^{(\lambda^2-1)t/4} \cdot F_H^{m,n}(\phi)(\sigma,\lambda)$ for all $(\sigma,\lambda) \in \widehat{M} \times \mathbb{C}$ and $m,n \in \mathbb{Z}^\sigma$. Then it is easy to verify that $H_t(\sigma,\lambda)$ is in $C^0(\widehat{G}) = \cap_p C^p(G)$ (see [3], §19). Therefore, by the isomorphism of $C^0(G)$ with $C^0(\widehat{G})$, there exists $h_t \in C^0(G)$ such that $F_H(h_t)(\sigma,\lambda) = H_t(\sigma,\lambda)$ for all $(\sigma,\lambda) \in \widehat{M} \times \mathbb{C}$.

For proving $h_t(x)$ is a solution of (24), it is enough to show that for every integers m,n of the same parity, $h_{t\,m,n}$ is a solution.

For $m,n \in \mathbb{Z}^\sigma$,

$$F_H^{m,n}(h_t)(\sigma,\lambda) = F_H(h_{t\,m,n})(\sigma,\lambda) = \mathrm{e}^{(\lambda^2-1)t/4} F_H^{m,n}(\phi)(\sigma,\lambda).$$

Therefore,

$$F_H\left(\frac{\partial}{\partial t}h_{t\,m,n}\right)(\sigma,\lambda)=\frac{\partial}{\partial t}F_H^{m,n}(h_t)(\sigma,\lambda)=\frac{\lambda^2-1}{4}F_H^{m,n}(h_t)(\sigma,\lambda).$$

Since Ω is formally self adjoint and $\Omega\Phi^{m,n}_{\sigma,\lambda}(x)=(\lambda^2-1)/4\Phi^{m,n}_{\sigma,\lambda}(x)$ ([3], 4.7) we have from earlier discussion,

$$\begin{aligned}F_H\left(\frac{\partial}{\partial t}h_{t\,m,n}\right)(\sigma,\lambda)&=\frac{\lambda^2-1}{4}\int_G h_{t\,m,n}(x)\Phi^{m,n}_{\sigma,\lambda}(x)\mathrm{d}x\\&=F_H(\Omega h_{t\,m,n}(x))(\sigma,\lambda).\end{aligned}$$

Similarly on any discrete series representation π_k with $\pi_k|_K\supset\chi_n,\chi_m$, $F_B(\frac{\partial}{\partial t}h_{t\,m,n})(k)=F_B(\Omega h_{t\,m,n})(k)$ since $\Omega\Psi^{m,n}_k(x)=(k^2-1)/4\Psi^{m,n}_k(x)$ (see [3], 4.7 and Proposition 7.3). Hence h_t is a $C^0(G)$-solution of (24).

Now as $t\longrightarrow 0$, $H_t\longrightarrow\widehat{\phi}$ in $C^p(\widehat{G})$. Therefore, $h_t\longrightarrow\phi$ in $C^p(G)$ and hence in $L^p(G)$. It is easy to check that h_t satisfies other conditions.

Conversely let us assume that $h^\phi=h\in C^\infty(G\times\mathbb{R}^+)$ is a solution of $\Omega u=(\partial/\partial t)u$ on $G\times\mathbb{R}^+$ such that for some $p\in[1,2]$, $h_t(x)$ satisfies the hypothesis of Theorem 4.1. Note that for any function $f\in L^p(G)$, $F_H^{m,n}(f)(\sigma,\lambda)$ can be defined for all $(\sigma,\lambda)\in\widehat{M}\times\overset{\circ}{\mathcal{S}}{}^p$ and $m,n\in\mathbb{Z}^\sigma$ and $\sup_{\lambda\in i\mathbb{R}}|F_H^{m,n}(f)(\lambda)|\le||f||_p\sup_{\lambda\in i\mathbb{R}}||\Phi^{m,n}_{\lambda_{\,\circ}}||_q$. Suppose $f_i\longrightarrow f$ in L^p. Then, since L^q-norm of $\Phi^{m,n}_\lambda(x)$ is uniformly bounded on $\mathcal{S}^p$, $F_H^{m,n}(f_i)\longrightarrow F_H^{m,n}(f)$ uniformly on $i\mathbb{R}$. Also there are only finitely many discrete series representations π_k such that $\pi_k|_K\supset\chi_n,\chi_m$. Therefore we can say that the (m,n)-th Fourier transform of f_i converges to the Fourier transform of f uniformly on all unitary representations. Taking $f_\delta=(h^\phi_{t+\delta}(\cdot)-h^\phi_t(\cdot))/\delta$ and $f=\partial h^\phi_t(\cdot)/\partial t$, by the last assumption in Theorem 4.1, $(\partial/\partial t)[F_H(h_t)(\lambda)]=F_H[\partial h(\cdot,t)/\partial t](\lambda)$ and $(\partial/\partial t)[F_B(h_t)(k)]=F_B[\partial h(\cdot,t)/\partial t](k)$ for all $\lambda\in i\mathbb{R}$ and $k\in\mathbb{Z}$.

Applying Fourier transform on both sides of (24), we get for each $m,n\in\mathbb{Z}^\sigma$: $F_H^{m,n}(\Omega h_t)(\lambda)=F_H^{m,n}[(\partial/\partial t)h_t](\lambda)=(\partial/\partial t)F_H^{m,n}(h_t)(\lambda)$. Also, $F_H^{m,n}(\Omega h_t)(\lambda)=\int_G\Omega h_t\Phi^{m,n}_\lambda(x)\mathrm{d}x=[(\lambda^2-1)/4]F_H^{m,n}(h_t)(\lambda)$. Therefore $[(\lambda^2-1)/4]F_H^{m,n}(h_t)(\lambda)=(\partial/\partial t)F_H^{m,n}(h_t)(\lambda)$. Solving this we get $F_H^{m,n}(h_t)(\sigma,\lambda)=C_H^{m,n}(\lambda)\mathrm{e}^{(\lambda^2-1)t/4}$ for each $(\sigma,\lambda)\in\widehat{M}\times i\mathbb{R}$.

From similar calculation we also get, $F_B^{m,n}(h_t)(k)=C_B^{m,n}(k)\mathrm{e}^{(k^2-1)t/4}$ for every discrete series π_k, such that $\pi_k|_K\supset\chi_n,\chi_m$. For each pair m, n, $C_H^{m,n}$ and $C_B^{m,n}$ are functions of λ and k respectively.

As L^q-norm of $\Phi^{m,n}_{\sigma,\lambda}$ is uniformly bounded on $\lambda\in\overset{\circ}{\mathcal{S}}{}^p$, as $t\longrightarrow 0$, $F_H^{m,n}(h_t)(\sigma,\lambda)$ converges uniformly to $F_H^{m,n}(\phi)(\sigma,\lambda)$ on $\overset{\circ}{\mathcal{S}}{}^p$ by (25). Therefore, $F_H^{m,n}(h_t)(\sigma,\lambda)=\mathrm{e}^{(\lambda^2-1)t/4}\cdot F_H^{m,n}(\phi)(\sigma,\lambda)$ for each $(\sigma,\lambda)\in\widehat{M}\times i\mathfrak{a}^*$ and $F_B^{m,n}(h_t)(k)=\mathrm{e}^{(k^2-1)t/4}\cdot F_B^{m,n}(\phi)(k)$ for every k such that $\pi_k|_K$ contains χ_m,χ_n. □

Due to the uniqueness of the solution h^ϕ_t in the above theorem, it is natural to define $\{k_t(x)\}_{t>0}$ on G *formally* by the following data: For $\sigma\in\widehat{M}$ and $m,n\in\mathbb{Z}^\sigma$, $F_H^{m,n}(k_t)(\sigma,\lambda)=0$ when $m\ne n$ and $F_H^{n,n}(k_t)(\sigma,\lambda)=\mathrm{e}^{(\lambda^2-1)t/4}$ for all $\lambda\in\mathbb{C}$. Then from the Plancherel theorem (see [19], vol. 2, p. 421) it is clear that there cannot be any L^2 function which matches with this description of k_t. But the behavior of this virtual k_t is similar to that of the heat kernel:

(a) k_{t+s} can be thought of as $k_t * k_s$, because $(k_{t+s})_{n,n} = (k_t)_{n,n} * (k_s)_{n,n}$ and $(k_t)_{n,n}*(k_s)_{m,m} = 0$ for $m \neq n$.
(b) We have seen in the proof of Theorem 4.1 that the solution h_t^ϕ is formally given by $h_t^\phi = k_t * \phi = \phi * k_t$ for any initial function $\phi \in \mathbb{C}_c^\infty(G)$.

However, when restricted to a K-finite environment, k_t is no longer virtual. Let F be a finite subset of $\widehat{K}$. Suppose $k_t^F = \sum_{n\in F}(k_t)_{n,n}$ and let H_F^σ be the finite dimensional subspace of H_σ generated by the vectors which transform according to any of the K-types in $F \cap \mathbb{Z}^\sigma$. Then,

(i) $k_t^F(x)$ is in $C^\infty(G\times\mathbb{R}^+)$ and for every $t \in \mathbb{R}^+$, $k_t^F \in C^0(G)$ and hence in particular $k_t^F \in L^1(G)\cap L^\infty(G)$.
(ii) It is easy to verify that $k_{t+s}^F = k_t^F * k_s^F$ for $t, s \in \mathbb{R}^+$.
(iii) For every $\sigma \in \widehat{M}$, $F_H(k_t^F)(\sigma, \pm 1) = I_F^\sigma$, the identity operator of H_F^σ. This generalizes the well-known property of the heat kernel on the symmetric space G/K: $\int_{G/K} k_t(x)\mathrm{d}x = 1$. On G/K, k_t^F is $(k_t)_{0,0}$ and hence $\int_{G/K} k_t(x)\mathrm{d}x = \int_G k_t(g)\Phi_{-1}^{0,0}(g)\mathrm{d}g = F_H^{0,0}(k_t)(\pm 1)$ as $\Phi_{-1}^{0,0}(x) \equiv 1$.
(iv) For any right (resp. left) $K(F)$-finite $\phi \in C_c^\infty(G)$, the unique solution of the initial value problem (24) and (25) is $h_t^\phi = \phi * k_t^F$ (resp. $h_t^\phi = k_t^F * \phi$).

Thus in the subclass of right or left $K(F)$-finite functions k_t^F plays the role of the heat kernel. We have now all the background to offer the following which relates the Hardy's theorem with the characterization of this pseudo-heat kernel of Ω:

Theorem 4.2. *Let F be a finite subset of $\widehat{K}$. Let f be a measurable function on $G\times\mathbb{R}^+$ such that for each $t \in \mathbb{R}^+$, f_t is $K(F)$-finite,*

$$|f_t(x)| \leq C_t \mathrm{e}^{-1/t(|\log a| + \frac{t}{2}|)^2}(1+|\log a|)^u \quad \textit{for all } k_1, k_2 \in K, a \in A^+, \tag{26}$$

and

$$||F_H(f_t)(\sigma,\lambda)|| \leq C_{t,\sigma}\mathrm{e}^{-(t/4)|\lambda|^2} \quad \textit{for all } (\sigma,\lambda) \in \widehat{M}\times i\mathfrak{a}^* \tag{27}$$

for some positive constants C_t and $C_{t,\sigma}$ and positive integer u. Moreover, if $F_H(f_t)(\sigma,\pm 1) = I_F^\sigma$ for all $\sigma \in \widehat{M}$ then $f_t = k_t^F$.

Remark 4.3. The precise estimate for the heat kernel on $SL_2(\mathbb{R})/SO_2(\mathbb{R})$ is the same as the estimate (26) (see [1], p. 66). Relating partial Fourier transform of a function f with the Jacobi transform of $f|_A$, one can in fact show that k_t^F also satisfies the same estimate. See also the work of Anker *et al* [2] in this direction. Estimate (27) says that the Fourier transform of f_t decays faster than the Fourier transform of the heat kernel. The condition $\widehat{f_{n,n}}(\sigma,\pm 1) = 1$ is a generalization of $\int_{G/K} f(x)\mathrm{d}x = 1$, as noted above.

Proof. From (26) we have

$$|f_t(x)| \leq C_t' \mathrm{e}^{-|\log a|^2/t}\mathrm{e}^{-|\log a|}(1+\log a|)^u.$$

Therefore, using estimate (4),

$$|f_t(x)| \le C_t'' e^{-|\log a|^2/t}\, \Xi(a)(1+\log a|)^{u'} \quad \text{for some } u' \ge u.$$

Now for fixed t, let $\alpha = 1/t$ and $\beta = t/4$. Then from Theorem 3.2(iv) we have, $(f_t)_{m,n} = 0$ for $m \neq n$ and $F_H^{n,n}(f_t)(\sigma,\lambda) = C_{t,n} e^{(t/4)\lambda^2}$ for all $(\sigma,\lambda) \in \widehat{M} \times \mathfrak{a}_{\mathbb{C}}^*$ and $n \in \mathbb{Z}^\sigma \cap F$.

The condition $F_H(f_t)(\sigma,\pm 1) = I_F^\sigma$ implies $F_H^{n,n}(f_t)(\sigma,\pm 1) = 1$, for all $n \in \mathbb{Z}^\sigma \cap F$, for all $\sigma \in \widehat{M}$. From this, we get

$$F_H^{n,n}(f_t)(\sigma,\lambda) = e^{(t/4)(\lambda^2-1)}$$

and

$$F_H^{m,n}(f_t)(\sigma,\lambda) = 0 \quad \text{when } m \neq n,$$

for all $(\sigma,\lambda) \in \widehat{M} \times \mathfrak{a}_{\mathbb{C}}^*$ and $m, n \in \mathbb{Z}^\sigma \cap F$.

Therefore by the definition of k_t^F, $f_t = k_t^F$. □

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