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Approximation by modified Szasz–Mirakjan operators on weighted spaces

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Abstract. The theorems on weighted approximation and the order of approximation of continuous functions by modified Szasz–Mirakjan operators on all positive semi-axis are established.

Keywords. Szasz–Mirakjan operator; order of convergence; weighted space.

Introduction

In this paper, an extension to convergence of the modified Szasz–Mirakjan operators to functions of one and two variables is obtained in polynomial weighted spaces of continuous and unbounded function defined on positive semi-axis. In §1 we investigate the theorems on convergence of modified Szasz–Mirakjan operators to functions of one variable. In §2 we give similar theorems for functions of two variables.

As in [8] we define the modified Szasz–Mirakjan operators $S_n(f; a_n, b_n; x) = S_n(f; x)$ for functions of one variable

$$S_n(f;x) := \mathrm{e}^{-a_n x} \sum_{k=0}^{\infty} \frac{(a_n x)^k}{k!} f\left(\frac{k}{b_n}\right), \quad x \in R_0,\ n \in N \tag{1}$$

where $R_0 = [0, \infty)$, $N := \{1, 2, ...\}$ and $\{a_n\}$, $\{b_n\}$ are given increasing and unbounded sequences of positive numbers such that

$$\lim_{n\to\infty} \frac{1}{b_n} = 0, \quad \frac{a_n}{b_n} = 1 + O\left(\frac{1}{b_n}\right). \tag{2}$$

In [8], Walczak introduced modified Szasz–Mirakjan operators in polynomial weighed spaces of functions of one and two variables. He investigated approximation properties of modified Szasz–Mirakjan operators in the weighted space of continuous functions of two variables for which wf is uniformly continuous and bounded on $R_0 \times R_0$ where $w(x, y)$ is a polynomial weighted function.

Similar theorems on weighted approximation of continuous functions, having exponential growth at infinity, are established in [6]. The approximation problem on unbounded sets has also been investigated by many authors. All of these results give the conditions of convergence of special operators (Szasz, Szasz-type or modified Szasz, Bernstein) on any finite interval of real axis or at any fixed point.

In this study, the theorems on convergence of $S_n(f; x)$ defined by (1) to $f(x)$ is obtained in the weighted spaces of continuous, unbounded on positive semi-axis functions which satisfy $\lim_{x\to\infty} f(x)/(1+x^2) = k < \infty$ where k is a constant only depending on f. In this study we define a weighted modulus of continuity and we obtain the rate of convergence of the operators S_n to function on all positive semi-axis.

For our aim we will use the weighted Korovkin-type theorems, proved by Gadzhiev [1,2]. We now give the Gadzhiev's results in weighted spaces. Therefore we need to introduce the notations of [1].

Let $\rho(x) = 1 + x^2$, $-\infty < x < \infty$ and B_ρ be the set of all functions f defined on the real axis satisfying the condition $|f(x)| \le M_f \rho(x)$ where M_f is a constant depending only on f. B_ρ is a normed space with the norm $\|f\|_\rho = \sup_{x\ge 0} f(x)/\rho(x)$, $f \in B_\rho$. C_ρ denotes the subspace of all continuous functions belonging to B_ρ and C_ρ^k denotes the subspace of all functions $f \in C_\rho$ with $\lim_{|x|\to\infty} f(x)/\rho(x) = k$ where k is a constant depending on f.

Theorem A [1,2] *Let $\{T_n\}$ be the sequence of linear positive operators which are mappings from C_ρ into B_ρ satisfying the conditions*

$$\lim_{n\to\infty} \left\|T_n(t^\nu, x) - x^\nu\right\|_\rho = 0, \quad \nu = 0, 1, 2.$$

Then, for any function $f \in C_\rho^k$,

$$\lim_{n\to\infty} \|T_n f - f\|_\rho = 0,$$

and there exists a function $f^* \in C_\rho \setminus C_\rho^k$ such that

$$\lim_{n\to\infty} \left\|T_n f^* - f^*\right\|_\rho \ge 1.$$

1. Approximation of functions of one variable

Obviously, S_n defined by (1) is a linear positive operator from C_ρ into B_ρ having the properties

$$S_n(1; x) = 1 \tag{3}$$

$$S_n(t; x) = \frac{a_n}{b_n} x \tag{4}$$

$$S_n(t^2; x) = \frac{a_n^2}{b_n^2} x^2 + \frac{a_n}{b_n^2} x. \tag{5}$$

Theorem 1. *Let $\{S_n\}$ be the sequence of linear positive operators defined by* (1). *Then for each function $f \in C_\rho^k$,*

$$\lim_{n\to\infty} \|S_n(f; x) - f(x)\|_\rho = 0.$$

Proof. Clearly, $\|S_n(1;x)-1\|_\rho \to 0$ as $n\to\infty$ on $[0,\infty)$. From the equalities (4) and (2) we get

$$\sup_{x\in R_0}\frac{|S_n(t;x)-x|}{1+x^2}=\left(\frac{a_n}{b_n}-1\right)\sup_{x\in R_0}\frac{x}{1+x^2}=O\left(\frac{1}{b_n}\right)\sup_{x\in R_0}\frac{x}{1+x^2}.$$

Hence we obtain

$$\lim_{n\to\infty}\|S_n(t;x)-x\|_\rho=0.$$

Also, by the equalities (5) and (2) we can write

$$\sup_{x\in R_0}\frac{\left|S_n(t^2;x)-x^2\right|}{1+x^2}\le\left(\frac{a_n^2}{b_n^2}-1\right)\sup_{x\in R_0}\frac{x^2}{1+x^2}+\frac{a_n}{b_n^2}\sup_{x\in R_0}\frac{x}{1+x^2}.$$

Thus we have

$$\lim_{n\to\infty}\left\|S_n(t^2;x)-x^2\right\|_\rho=0.$$

Therefore, the desired result follows from Theorem A. □

We now want to find the order of approximation of the functions $f\in C_\rho^k$ by the operators S_n on all the positive semi-axis. It is well-known that the usual first modulus of continuity $\omega(\delta)$ does not tend to zero, as $\delta\to 0$, on infinite interval. Analogously as in [5] we define a weighted modulus of continuity $\Omega(f;\delta)$ which tends to zero, as $\delta\to 0$, on infinite interval. Let

$$\Omega_n(f;\delta)=\sup_{|h|\le\delta,\,x\in R_0}\sup\frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)},$$

for every $f\in C_\rho^k$. We call $\Omega(f;\delta)$ the weighted modulus of continuity of the function $f\in C_\rho^k$.

Some elementary properties of $\Omega(f;\delta)$ are collected in the following

Lemma 1 [5]. *Let* $f\in C_\rho^k$,

(i) $\Omega(f;\delta)$ *is a monotonically increasing function of* δ, $\delta\ge 0$.

(ii) *For every* $f\in C_\rho^k$, $\lim_{\delta\to 0}\Omega(f,\delta)=0$.

(iii) *For each positive value of* λ

$$\Omega_n(f;\lambda\delta)\le 2(1+\lambda)(1+\delta^2)\Omega(f;\delta). \tag{6}$$

From the inequality (6) *and the definition of* $\Omega(f;\delta)$ *we get*

$$|f(t)-f(x)|\le 2\left(1+\frac{|t-x|}{\delta}\right)(1+\delta^2)\Omega(f;\delta)(1+x^2)(1+(t-x)^2) \tag{7}$$

for every $f\in C_\rho^k$ *and* $x,t\in[0,\infty)$.

Theorem 2. *If $f \in C^k_\rho$, then the inequality*

$$\sup_{x\geq 0} \frac{|S_n(f;x)-f(x)|}{(1+x^2)^3} \leq K_1 \Omega(f, b_n^{-1/2})$$

is satisfied for a sufficiently large n, where K_1 is a constant independent of a_n, b_n.

Proof. From (7) we can write

$$\begin{aligned} |S_n(f;x)-f(x)| &\leq 2(1+\delta_n^2)\Omega(f;\delta_n)(1+x^2)\mathrm{e}^{-a_n x}\sum_{k=0}^{\infty}\frac{(a_n x)^k}{k!} \\ &\quad \times \left(1+\frac{|(k/b_n)-x|}{\delta_n}\right)\left(1+\left(\frac{k}{b_n}-x\right)^2\right) \\ &\leq 4\Omega(f;\delta_n)(1+x^2)\left\{1+\frac{1}{\delta_n}\mathrm{e}^{-a_n x}\sum_{k=0}^{\infty}\frac{(a_n x)^k}{k!}\left|\frac{k}{b_n}-x\right|\right. \\ &\quad + \mathrm{e}^{-a_n x}\sum_{k=0}^{\infty}\frac{(a_n x)^k}{k!}\left(\frac{k}{b_n}-x\right)^2 \\ &\quad \left. + \frac{1}{\delta_n}\mathrm{e}^{-a_n x}\sum_{k=0}^{\infty}\frac{(a_n x)^k}{k!}\left|\frac{k}{b_n}-x\right|\left(\frac{k}{b_n}-x\right)^2\right\} \end{aligned}$$

for any $\delta_n > 0$. Applying Cauchy–Schwartz inequality we obtain

$$|S_n(f;x)-f(x)| \leq 4\Omega(f;\delta_n)(1+x^2)\left(1+\frac{2}{\delta_n}\sqrt{A_1}+A_1+\frac{1}{\delta_n}A_2\right) \quad (8)$$

where

$$A_1 = \mathrm{e}^{-a_n x}\sum_{k=0}^{\infty}\frac{(a_n x)^k}{k!}\left(\frac{k}{b_n}-x\right)^2$$

and

$$A_2 = \mathrm{e}^{-a_n x}\sum_{k=0}^{\infty}\frac{(a_n x)^k}{k!}\left(\frac{k}{b_n}-x\right)^4.$$

By simple calculation we get

$$S_n(t^3;x) = \frac{a_n^3}{b_n^3}x^3 + 3\frac{a_n^2}{b_n^3}x^2 + \frac{a_n}{b_n^3}x \quad (9)$$

and

$$S_n(t^4;x) = \frac{a_n^4}{b_n^4}x^4 + 6\frac{a_n^3}{b_n^4}x^3 + 7\frac{a_n^2}{b_n^4}x^2 + \frac{a_n}{b_n^4}x. \quad (10)$$

From (3), (4), (5), (9) and (10) we obtain

$$A_1 = S_n((t-x)^2;x) = \left(\frac{a_n}{b_n}-1\right)^2 x^2 + \frac{a_n}{b_n^2}x \quad (11)$$

and

$$A_2 = S_n((t-x)^4; x) = \left(\frac{a_n^4}{b_n^4} - 4\frac{a_n^3}{b_n^3} + 6\frac{a_n^2}{b_n^2} - 4\frac{a_n}{b_n} + 1\right)x^4 + \left(6\frac{a_n^3}{b_n^4} - 12\frac{a_n^2}{b_n^3} + \frac{a_n}{b_n^2}\right)x^3 + \left(7\frac{a_n^2}{b_n^4} - 4\frac{a_n}{b_n^3}\right)x^2 + \frac{a_n}{b_n^4}x. \tag{12}$$

Using condition (2) in (11) and (12) we can write

$$A_1 = O\left(\frac{1}{b_n}\right)(x^2 + x)$$

and

$$A_2 = O\left(\frac{1}{b_n}\right)(x^4 + x^3 + x^2 + x).$$

Substituting these equalities in (8) we have

$$|S_n(f;x) - f(x)| \le 4\Omega(f;\delta_n)(1+x^2)\left\{1 + \frac{2}{\delta_n}\sqrt{O\left(\frac{1}{b_n}\right)(x^2+x)} + O\left(\frac{1}{b_n}\right)(x^2+x) + \frac{1}{\delta_n}O\left(\frac{1}{b_n}\right)(x^4+x^3+x^2+x)\right\}.$$

Choosing $\delta_n = b_n^{-1/2}$, for sufficiently large n's, we obtain

$$\sup_{x\ge 0}\frac{|S_n(f;x) - f(x)|}{(1+x^2)^3} \le K_1\Omega(f, b_n^{-1/2})$$

where K_1 is a constant independent of a_n, b_n. □

2. Approximation of functions of two variables

Let $C(R_0^2)$ be the set of all real-valued functions of two variables continuous on $R_0^2 := \{(x,y) : x \ge 0,\ y \ge 0\}$. As in §1, we define the weighted function

$$q(x,y) = 1 + x^2 + y^2, \quad (x,y) \in R_0^2$$

and the weighted spaces B_q of all real-valued functions f continuous on R_0^2 satisfying $|f(x,y)| \le M_f\, q(x,y)$ with M_f is a constant depending only on f and the norm is defined by $\|f\|_q = \sup_{(x,y)\in R_0^2} f(x,y)/q(x,y)$. As in §1, C_q denotes the subspace of all continuous functions which belong to B_q and C_q^k denotes the subspace of all functions $f \in C_q$ with $\lim_{(x,y)\to\infty} f(x,y)/q(x,y) = k$ where k is a constant depending on f.

For $(x,y) \in R_0^2,\ m,\ n \in N$ the modified Szasz–Mirakjan operators

$$S_{n,m}(f; a_n, b_n, c_m, d_m; x, y) = S_{n,m}(f; x, y)$$

is defined in [9] as follows:

$$S_{n,m}(f;x,y) := \mathrm{e}^{-a_n x}\mathrm{e}^{-c_m y}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(a_n x)^k}{k!}\frac{(c_m y)^k}{j!} f\left(\frac{k}{b_n},\frac{j}{d_m}\right), \tag{13}$$

where $\{a_n\}$, $\{b_n\}$, $\{c_m\}$, $\{d_m\}$ are given increasing and unbounded sequences of positive numbers such that

$$\lim_{n\to\infty}\frac{1}{b_n}=0,\quad \frac{a_n}{b_n}=1+O\left(\frac{1}{b_n}\right),\quad \lim_{m\to\infty}\frac{1}{d_m}=0,\quad \frac{c_m}{d_m}=1+O\left(\frac{1}{d_m}\right). \tag{14}$$

The approximation theorems for functions of two variables are proved by Volkov [7]. He proved the theorem:

If $\{L_n\}$ is a sequence of linear positive operators satisfying the conditions

$$\lim_{n\to\infty}\|L_n(1;x_1,x_2)-1\|_{C(X)}=0$$
$$\lim_{n\to\infty}\|L_n(t_j;x_1,x_2)-x_j\|_{C(X)}=0,\ j=1,2$$
$$\lim_{n\to\infty}\|L_n(t_1^2+t_2^2;x_1,x_2)-x_1^2+x_2^2\|_{C(X)}=0$$

then for any function $f\in C(X)$, which is bounded in R^2

$$\lim_{n\to\infty}\|L_n(f;x_1,x_2)-f(x_1,x_2)\|_{C(X)}=0$$

where X is a compact set.

Analogously as in Theorem A, the theorems on weighted approximation for functions of several variables are proved by Gadzhiev [3,4].

The following lemma is essential to study the convergence of the sequence $\left\{S_{n,m}\right\}$ to f.

Lemma 2. *For all* $(x,y)\in R_0^2$, *we have*

(i) $S_{n,m}(1;x,y)=1$

(ii) $S_{n,m}(t;x,y)=\frac{a_n}{b_n}x$ *and* $S_{n,m}(s;x,y)=\frac{c_m}{d_m}y$

(iii) $S_{n,m}(t^2+s^2;x,y)=\frac{a_n^2}{b_n^2}x^2+\frac{c_m^2}{d_m^2}y^2+\frac{a_n}{b_n^2}x+\frac{c_m}{d_m^2}y.$

Proof is clearly by definition of the operators $S_{n,m}(f;x,y)$.

Theorem 3. *Let* $\{S_{n,m}\}$ *be a sequence of positive linear operators* $S_{n,m}:C_q\to B_q$ *satisfying the conditions*

$$\lim_{n,m\to\infty}\|S_{n,m}(1;x,y)-1\|_q=0 \tag{15}$$

$$\lim_{n,m\to\infty}\|S_{n,m}(t;x,y)-x\|_q=0 \tag{16}$$

$$\lim_{n,m\to\infty}\|S_{n,m}(s;x,y)-y\|_q=0 \tag{17}$$

and

$$\lim_{n,m\to\infty} \|S_{n,m}(t^2+s^2;x,y) - x^2 + y^2\|_q = 0. \tag{18}$$

Then

$$\lim_{n,m\to\infty} \|S_{n,m}(f;x,y) - f(x,y)\|_q = 0$$

for each $f \in C_q^k$.

Proof. From Lemma 2 and condition (14) we have the conditions (15), (16), (17) and (18). Similar to Theorem A we obtain the desired results. □

Similarly, as in §1 we define the weighted modulus of continuous

$$\Omega(f,\delta_1,\delta_2) = \sup_{|u|<\delta_1,\,|v|\le\delta_2,\,(x,y)\in R_0^2} \frac{|f(x+u,y+v)-f(x,y)|}{q(u,v)\,q(x,y)}$$

for every $f \in C_q^k$. As in §1, note that $\Omega(f,\delta_1,\delta_2) \to 0$ for $\delta_1 \to 0, \delta_2 \to 0$ and for $\lambda_1 > 0,\ \lambda_2 > 0$

$$\Omega(f,\lambda_1\delta_1,\lambda_2\delta_2) \le 4(1+\lambda_1)\,(1+\lambda_2)\,\Omega(f,\delta_1,\delta_2).$$

Theorem 4. *For every* $f \in C_q^k$ *the inequality*

$$\sup_{(x,y)\in R_0^2} \frac{\left|S_{n,m}(f;(x,y)) - f(x,y)\right|}{(1+x^2+y^2)^3} \le K_2\Omega(f, b_n^{-1/4}, d_m^{-1/4})$$

is valid if n, m *are sufficiently large, where* K_2 *is a constant independent of* a_n, b_n, c_m, d_m.

Proof. Since

$$|f(t,s)-f(x,y)| \le 8(1+x^2+y^2)\Omega(f,\delta_n,\delta_m)\left(1+\frac{|t-x|}{\delta_n}\right) \times \left(1+\frac{|s-y|}{\delta_m}\right)\left(1+(t-x)^2\right)\left(1+(s-y)^2\right)$$

we have

$$\begin{aligned}\left|S_{n,m}(f(t,s);x,y)-f(x,y)\right| &\le S_{n,m}\left(|f(t,s)-f(x,y)|\,;x,y\right)\\ &\le 8(1+x^2+y^2)\Omega(f,\delta_n,\delta_m)\mathrm{e}^{-nx}\mathrm{e}^{-my}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(nx)^k}{k!}\frac{(my)^j}{j!}A_1(x)A_2(y)\end{aligned} \tag{19}$$

where

$$A_1 = \left(1+\frac{|(k/b_n)-x|}{\delta_n}\right)\left(1+\left(\frac{k}{b_n}-1\right)^2\right)$$

and

$$A_2 = \left(1+\frac{|(j/d_m)-y|}{\delta_m}\right)\left(1+\left(\frac{j}{d_m}-1\right)^2\right).$$

Since

$$A_1(x) \leq \begin{cases} 2(1+\delta_n^2), & |(k/b_n)-x| \leq \delta_n \\ 2(1+\delta_n^2), & |(k/b_n)-x| \geq \delta_n \end{cases}$$

we obtain for all $x,\ k/b_n \in [0,\infty)$

$$A_1(x) \leq 2(1+\delta_n^2)\left\{1+\frac{((k/b_n)-x)^4}{\delta_n^4}\right\}. \tag{20}$$

Similarly we get

$$A_2(x) \leq 2(1+\delta_m^2)\left\{1+\frac{((j/d_m)-y)^4}{\delta_m^4}\right\}. \tag{21}$$

Using the inequalities (20) and (21) in (19) we have

$$\begin{aligned} \left|S_{n,m}(f(t,s);x,y)-f(x,y)\right| &\leq 4(1+x^2+y^2)\Omega(f,\delta_n,\delta_m) \\ &\times\left(\frac{1}{\delta_n^4}S_n((t-x)^4,x)\,\frac{1}{\delta_m^4}S_m((s-y)^4,y)\right). \end{aligned}$$

From the equality (12)

$$\begin{aligned} \left|S_{n,m}(f(t,s);x,y)-f(x,y)\right| &\leq 4(1+x^2+y^2)\Omega(f,\delta_n,\delta_m)\frac{1}{\delta_n\,\delta_m} \\ &\times\left(O\left(\frac{1}{b_n}\right)O\left(\frac{1}{d_m}\right)(x^4+x^3+x^2+x)(y^4+y^3+y^2+y)\right). \end{aligned}$$

Choosing $\delta_n = b_n^{-1/4},\ \delta_m = d_m^{-1/4}$ we obtain the desired result.

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