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On integral means of star-like functions

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Abstract. We study univalent holomorphic functions in the unit disk $U = \{z : |z| < 1\}$ of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ that satisfy the condition Re $zf'(z)/f(z) > \alpha$ with $\alpha \in [0, 1)$ in U. Some integral means of such funcions are estimated.

Keywords. Star-like function; integral mean; univalent function.

1. Introduction

Let U be the unit disk $U = \{z : |z| < 1\}$ and S denote the usual class of holomorphic univalent functions $f : U \to \mathbb{C}$ normalized by $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$. Let us consider a subclass S_α^* of S given by the condition Re $zf'(z)/f(z) > \alpha$ with $\alpha \in [0, 1)$ in U. Robertson [9] introduced this class of functions in 1936. Apart from the case of the whole class S^* of star-like functions ($\alpha = 0$), the case $\alpha = 1/2$ turns out to be interesting because any function that maps U onto a convex domain is from $S_{1/2}^*$. The class of convex functions we denoted by C, $C \subset S_{1/2}^*$. Let us introduce the functional

$$M_f(r) = M_f(r, \lambda_1, \lambda_2) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(r\mathrm{e}^{i\theta})|^{\lambda_1} |f'(r\mathrm{e}^{i\theta})|^{\lambda_2} \mathrm{d}\theta,$$
$$r \in (0, 1), \quad \lambda_1, \lambda_2 \in \mathbb{R}, \tag{1}$$

which is called the integral mean. This functional attracts much attention (see e.g. [1, 2, 4, 7, 11]). The classical problem is to study the asymptotic behavior of $M_f(r, \lambda_1, \lambda_2)$ as $r \to 1^-$. The case of $\lambda_1 = 0$, $\lambda_2 = p$ is especially known. Let $\beta_f(p)$ be the smallest number such that

$$M_f(r, 0, p) = O((1-r)^{-\beta_f(p)-\varepsilon}) \quad \text{as } r \to 1^-$$

for every $\varepsilon > 0$. An interesting open problem is the Brennan conjecture $\beta_f(-2) \le 1$. We refer the reader to Chapter 8 of [11] for more information about this kind of estimates.

The Koebe function plays a crucial role in the theory of univalent functions

$$k(z) = \frac{z}{(1 - \mathrm{e}^{i\gamma} z)^2},$$

that is extremal in a lot of problems for the classes S and S^*. An analog of the koebe function in the class S^*_α is the function

$$k_\alpha(z) = \frac{z}{(1 - e^{i\gamma} z)^{2(1-\alpha)}}.$$

The main results of our paper are concerned with estimates of the type $M_f(r, \lambda_1, \lambda_2) \le M_{k_\alpha}(r, \lambda_1, \lambda_2)$ for certain values of r, λ_1, λ_2 and $f \in S^*_\alpha$. Let us remind here some remarkable results concerning estimates by the integral mean of the Koebe function. An outstanding achievement by Baernstein [1] about some general integral mean inequalities implies, in particular, that $M_f(r, \lambda, 0) \le M_k(r, \lambda, 0)$ for any $f \in S$ and $r \in [0, 1), \lambda \in \mathbb{R}$ (see also [3]). However, as Baernstein remarked, the analogous inequality for the derivative of a univalent function $M_f(r, 0, \lambda) \le M_k(r, 0, \lambda)$ can be false at least for $\lambda \in (0, 1/3)$. In a positive direction, MacGregor [8] proved that this inequality holds even for higher derivatives for close-to-convex function and $\lambda \ge 1$. Later on, Leung [6] extended MacGregor's result for the first derivative onto a certain subclass of Bazilevič functions.

2. Preliminaries

Before we come to the main results, let us give some simple absolute estimates for integral mean and auxiliary propositions.

For a function f from S we define the function $F(\zeta) \equiv 1/f(1/\zeta) = \zeta + \sum_{n=0}^{\infty} b_n \zeta^{-n}$, where $\zeta = 1/\zeta$ lies in the exterior part U^* of the unit disk U. The function $f \in S^*_\alpha$ if and only if the function F satisfies the condition $\mathrm{Re}\, \zeta F'(\zeta)/F(\zeta) > \alpha$ in U^* and $F(U^*)$ does not contain the origin. We denote such class of functions F by Σ^*_α. If $F \in \Sigma^*_\alpha$, then the sharp coefficient estimate is true [10]:

$$|b_n| \le \frac{2(1-\alpha)}{n+1}. \tag{2}$$

Moreover, the following sharp inequality holds [10]:

$$\sum_{n=0}^{\infty} |b_n|^2 (n + \alpha) \le 1 - \alpha. \tag{3}$$

Let us mention here a result by Barnard and Pearce [2] who proved that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |1/f'(e^{i\theta})| d\theta \le \frac{4}{\pi} \frac{\arctan \sqrt{2\alpha - 1}}{\sqrt{2\alpha - 1}} \frac{1}{\min_\theta |f(e^{i\theta})|}$$

in the class S^*_α. Another result by Pommerenke ([11], Proposition 8.1) states that

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f'(re^{i\theta})|^p d\theta \le p \int_0^r M(\rho)^p \frac{d\rho}{\rho},$$

where $p > 0$, $f \in S$, and $M(r) = \max_{|z|=r} |f(re^{i\theta})|$.

In particular, we are interested in two integral means

$$I_1 \equiv M_f(r, -2, 0) = \int_{-\pi}^{\pi} \frac{d\theta}{|f(re^{i\theta})|^2},$$

$$I_2 \equiv M_f(r, 0, -2) = \int_{-\pi}^{\pi} \frac{d\theta}{|f'(re^{i\theta})|^2}.$$

In turns out that the absolute estimates of I_1 and I_2 play a special role in some problems of fluid mechanics, in particular, in the Hele–Shaw dynamics of the free interface evolution (see [12]). Using (2) and (3) we come to such an estimate in the class S^*_α:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{|f(r\mathrm{e}^{i\theta})|^2} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|F\left(\frac{1}{r}\mathrm{e}^{i\theta}\right)\right|^2\mathrm{d}\theta \\ &= \frac{1}{r^2}+(1-\alpha)|b_0|^2+\alpha|b_0|^2+\sum_{n=1}^{\infty}|b_n|^2r^{2n} \\ &\le \frac{1}{r^2}+4(1-\alpha)^3+(1-\alpha).\end{aligned}$$

In particular, $(1/2\pi)\int_{-\pi}^{\pi}|1/f(\mathrm{e}^{i\theta})|^2\mathrm{d}\theta \le 1+4(1-\alpha)^3+(1-\alpha)$. This inequality is sharp at least for convex functions (for $\alpha=0$ the estimate is 4). For convex functions we can give a sharp estimate of I_2. If $f\in C$, then the function $g(z)=zf'(z)$ is from S^* and $G(z)=1/g(1/z)=z+c_0+\sum_{n=1}^{\infty}c_n/z^n$ is from Σ^*. Therefore, we have

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{|f'(r\mathrm{e}^{i\theta})|^2} &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1}{r}\left|G\left(\frac{1}{r}\mathrm{e}^{i\theta}\right)\right|^2\mathrm{d}\theta \\ &= \frac{1}{r}\left(\frac{1}{r^2}+|c_0|^2+\sum_{n=1}^{\infty}|c_n|^2r^{2n}\right) \\ &\le \frac{1+5r^2}{r^3}.\end{aligned}$$

In particular, if $f\in C$, then $(1/2\pi)\int_{-\pi}^{\pi}|1/f'(\mathrm{e}^{i\theta})|^2\mathrm{d}\theta \le 6$. In the class S^*_α a trivial estimate $(1/2\pi)\int_{-\pi}^{\pi}|1/f'\mathrm{e}^{i\theta})|^2\mathrm{d}\theta < (1+4(1-\alpha)^3+(1-\alpha))/\alpha^2$ holds. However a sharp estimate is still unknown. A specific feature of the above estimates is that they are finite as $r\to 1^-$. Let us remark here that it suffices to suppose that f has angular derivatives on the unit circle to give sense to corresponding limits $r\to 1^-$.

Now let us give some preliminary propositions. Functions from S^*_α admit the following known integral representation

$$f(z)\in S^*_\alpha \Leftrightarrow f(z)=z\exp\left\{-2(1-\alpha)\int_{-\pi}^{\pi}\log(1-\mathrm{e}^{i\theta}z)\mathrm{d}\mu(\theta)\right\},$$

where $\mu(\theta)$ is a non-decreasing function in $\theta\in[-\pi,\pi]$ and $\int_{-\pi}^{\pi}\mathrm{d}\mu(\theta)=1$.

If $\mu(\theta)$ is a piecewise constant function, then we have a set of holomorphic functions $f_n(z)$ that admit the following representation

$$f_n(z)=\frac{z}{\prod_{k=1}^{n}(1-e^{i\theta_k}z)^{2(1-\alpha)\beta_k}}\in S^*_\alpha,\quad \theta_k\in[-\pi,\pi],\quad \beta_k\ge 0,$$
$$\sum_{k=1}^{n}\beta_k=1. \tag{4}$$

Using the known properties of Stieltjes, integral and Vitali's theorem [4] it is easy to show that the set of function (4) is dense in S^*_α, i.e. for every function $f(z)\in S^*_\alpha$ there exists a sequence $\{f_n(z)\}$ satisfying (4) that locally uniformaly converges to $f(z)$ in U. Therefore, we need to prove our results considereing $M_f(r)$ for $f(z)=f_n(z)$.

PROPOSITION 1 . [5]
Let φ, ψ be real-valued even 2π-periodic positive functions in $\mathbb{R}$. If φ, ψ are non-increasing functions in $(0, \pi)$, then

$$\int_{-\pi}^{\pi} \varphi(t)\psi(t+\theta)\mathrm{d}t \leq \int_{-\pi}^{\pi} \varphi(t)\psi(t)\mathrm{d}t.$$

The inequality is sharp for $\theta = 0$.

This proposition is an immediate consequence of a more general result on 'rearrangements' of functions [5].

PROPOSITION 2
Let φ, ψ be non-negative even 2π-periodic functions in $\mathbb{R}$. If φ, ψ do not increase in $(0, \pi)$ or do not decrease in $(0, \pi)$, and if $s > 0$, $\theta_j \in \mathbb{R}$, $\beta_j > 0$, $\sum_{j=1}^{n} \beta_j = 1$, then

$$\int_{-\pi}^{\pi} \prod_{k=1}^{n} \varphi^{s\beta_k}(\theta+\theta_k) \left(\sum_{j=1}^{n} \beta_j \psi(\theta+\theta_j) \right)^t \mathrm{d}\theta \leq \int_{-\pi}^{\pi} \varphi^s(\theta)\psi^t \mathrm{d}\theta. \tag{5}$$

Proposition 2 follows at once from Proposition 1 by making use of the Hölder and Minkowski inequalities ($t \geq 1$).

PROPOSITION 3
Let φ, ψ be non-negative even 2π-periodic functions in $\mathbb{R}$. If φ, ψ do not increase in $(0, \pi)$ and if $s > 0$, $0 < t < 1$, $\theta_j \in \mathbb{R}$, $\beta_j > 0$, $\sum_{j=1}^{n} \beta_j = 1$, then (5) *holds whenever there exists γ, $0 < \gamma < t$, such that $\varphi^s \psi^{-\gamma t/(t-\gamma)}$ does not increase in $(0, \pi)$.*

Proof. We have

$$\int_{-\pi}^{\pi} \prod_{k=1}^{n} \varphi^{s\beta_k}(\theta+\theta_k) \left(\sum_{j=1}^{n} \beta_j \psi(\theta+\theta_j) \right)^t \mathrm{d}\theta$$
$$\leq \prod_{k=1}^{n} \left(\int_{-\pi}^{\pi} \varphi^s(\theta+\theta_k) \left(\sum_{j=1}^{n} \beta_j \psi(\theta+\theta_j) \right)^t \mathrm{d}\theta \right)^{\beta_k},$$

and include

$$\int_{-\pi}^{\pi} \varphi^s(\theta)\psi^{\gamma}(\theta)\psi^{-\gamma}(\theta) \left(\sum_{j=1}^{n} \beta_j \psi(\theta+\theta_j) \right)^t \mathrm{d}\theta = \int_{-\pi}^{\pi} (\varphi^s(\theta)\psi^t(\theta))^{\gamma/t}$$
$$\times \left(\varphi^s(\theta)\psi^{-\gamma t/(t-\gamma)}(\theta) \left(\sum_{j=1}^{n} \beta_j \psi(\theta+\theta_j) \right)^{t^2/(t-\gamma)} \right)^{(t-\gamma)/t} \mathrm{d}\theta.$$

Applying the Hölder inequality we conclude that there exists γ, $0 < \gamma < t$, such that $t^2/(t-\gamma) \geq 1$. The Minkowski inequality implies the statement of the proposition. □

3. Main theorems

Theorem 1. *Let* $f(z) \in S^*_\alpha$, $0 < \alpha 1$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 + \lambda_2 \geq 0$, $\lambda_2 \geq 1$. *Then*

$$M_f(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda_1}|f'(re^{i\theta})^{\lambda_2}d\theta \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |k_\alpha(z)|^{\lambda_1}|k'_\alpha(z)|^{\lambda_2}d\theta,$$

$$0 < r < 1, \quad z = re^{i\theta}.$$

This estimate is sharp with the extremal function $k_\alpha(z)$.

Proof. As we have noted before, we need to prove the theorem for the functions $f(z) = f_n(z)$. We have

$$\begin{aligned} M_{f_n(r)} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_n(z)|^{\lambda_1}|f'_n(z)|^{\lambda_2}d\theta \\ &= \frac{1}{2\pi r^{\lambda_2}}\int_{-\pi}^{\pi} |f_n(z)|^{\lambda_1+\lambda_2}\left|\frac{zf'_n(z)}{f(z)}\right|^{\lambda_2} d\theta \\ &= \frac{1}{2\pi r^{\lambda_2}}\int_{-\pi}^{\pi} \prod_{k=1}^{n}\left(\left|\frac{z}{(1-e^{i\theta_k}z)^{2(1-\alpha)}}\right|^{\lambda_1+\lambda_2}\left|\frac{zf'_n(z)}{f_n(z)}\right|^{\lambda_2}\right)^{\beta_k} d\theta. \end{aligned}$$

Let us calculate

$$\frac{zf'_n(z)}{f_n(z)} - \alpha = (1-\alpha)\sum_{k=1}^{n}\beta_k\frac{1+e^{i\theta_k}z}{1-e^{i\theta_k}z}, \quad 0 \leq \beta_k < 1, \ \sum_{k=1}^{n}\beta_k = 1.$$

Then we obtain

$$\varphi(\theta+\theta_k) = \left|\frac{z}{(1-e^{i\theta_k}z)^{2(1-\alpha)}}\right|, \quad \psi(\theta+\theta_j) = \left|\frac{1+e^{i\theta_j}z(1-2\alpha)}{(1-e^{i\theta_j}z)}\right|.$$

The function $\varphi_1(\theta) = |1 - re^{i\theta}|$ strictly increases in $(0, \pi)$ for every $0 < r \leq 1$, so does the function

$$\psi_1(\theta) = \left|\frac{1-re^{i\theta}}{1+re^{i\theta}(1-2\alpha)}\right|.$$

Applying Propositions 1 and 2 we have

$$\begin{aligned} M_{f_n}(r) &\leq \frac{r^{\lambda_1}}{2\pi}\int_{-\pi}^{\pi}\left|\frac{1}{(1-z)^{2(1-\alpha)}}\right|^{\lambda_1+\lambda_2}\left|\frac{1+z(1-2\alpha)}{1-z}\right|^{\lambda_2} dt \\ &= \int_{-\pi}^{\pi} |k_\alpha(z)|^{\lambda_1}|k'_\alpha(z)|^{\lambda_2}dt. \end{aligned}$$

We come to the conclusion of Theorem 1 as $n \to \infty$. □

Theorem 2. *If* $f(z) \in S^*_\alpha$, $\sqrt{2}-1 \le \alpha < 1$, $\lambda_1, \lambda_2 \in \mathbb{R}, \lambda_1+\lambda_2 \ge 0, 0 \le \lambda_2 < 1$, *and* $(1-\alpha)\lambda_1 + (2-\alpha)\lambda_2 \ge 1$, *then*

$$M_f(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda_1}|f'(re^{i\theta})|^{\lambda_2} d\theta$$

$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |k_\alpha(z)|^{\lambda_1}|k'_\alpha(z)|^{\lambda_2} d\theta, \quad 0 < r < 1,\ z = re^{i\theta}.$$

The estimate is sharp with the extremal function $k_\alpha(z)$*.*

Proof. Again we prove the theorem for $f(z) = f_n(z)$ (see (4)). Applying Proposition 3, we have

$$t = \lambda_2, s = \lambda_1+\lambda_2, \gamma = \frac{\lambda_2(1-\alpha)(\lambda_1+\lambda_2)}{(1-\alpha)\lambda_1+(2-\alpha)\lambda_2}$$

and

$$P(\theta) = \varphi^s(\theta)\psi^{-\gamma t/(t-\gamma)}(\theta) = \varphi^{\lambda_1+\lambda_2}(\theta)\psi^{-(1-\alpha)(\lambda_1+\lambda_2)}(\theta)$$

$$= \left|\frac{1}{(1-z)^{2(1-\alpha)}}\right|^{\lambda_1+\lambda_2}\left|\frac{1+z(1-2\alpha)}{1-z}\right|^{-(1-\alpha)(\lambda_1+\lambda_2)}$$

$$= \left|\frac{1}{(1-z)(1+z(1-\alpha))}\right|^{(1-\alpha)(\lambda_1+\lambda_2)}.$$

In the case $1/2 \le \alpha < 1$, we observe that the function $P_1(\theta) = |1-re^{i\theta}||1+re^{i\theta(1-2\alpha)}|$ strictly increases in $(0,\pi)$ for every $0 < r \le 1$. In the case $\sqrt{2}-1 \le \alpha < 1/2$,

$$P_1(\theta) = |1-re^{i\theta}|^2|1+r(1-2\alpha)e^{i\theta}|^2$$

is a quadratic function of the variable $s = -\cos\theta$. Easy calculation shows that $P_1(\theta)$ strictly increases in $(0,\pi)$ for every $0 < r \le 1$. This completes the proof of Theorem 2. □

Theorem 3. *If* $f(z) \in S^*_\alpha$, $0 < \alpha < \sqrt{2}-1, \lambda_1,\lambda_2 \in \Re, \lambda_1+\lambda_2 \ge 0, 0 \le \lambda_2 \le 1$ *and* $\lambda_1(1-\alpha)+(2-\alpha)\lambda_2 \ge 1$, *then*

$$M_f(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda_1}|f'(re^{i\theta})|^{\lambda_2}\, d\theta$$

$$\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |k_\alpha(z)|^{\lambda_1}|k'_\alpha(z)|^{\lambda_2}\, d\theta,$$

$$0 < r < \frac{-\sqrt{1-2\alpha}+1-\alpha}{\alpha\sqrt{1-2\alpha}}, \quad z = re^{i\theta}.$$

The inequality is sharp with the extremal function $k_\alpha(z)$*.*

Proof. Again we study the behavior of the function $P(\theta)$ for $0 < \alpha < \sqrt{2} - 1$. Let $P_2(\theta) = |1 - z||1 + z(1 - 2\alpha)|$. The function $P_2(\theta)$ increases in $\theta \in (0, \pi)$ for every $r \in (0; r_0)$ with

$$r_0 = \frac{-\sqrt{1-2\alpha} + 1 - \alpha}{\alpha\sqrt{1-2\alpha}} < 1$$

and $0 < \alpha < \sqrt{2} - 1$. Now we apply the argumentation similar to that of Theorem 2 and complete the proof.

COROLLARY 1

If $f(z) \in S^*_\alpha$ and one of the following conditions is satisfied,

(i) $0 \le \alpha < 1, \lambda_1 + \lambda_2 \ge 0, \lambda_2 \ge 1$,
(ii) $\sqrt{2} - 1 \le \alpha < 1, \lambda_1 + \lambda_2 \ge 0, 0 \le \lambda_2 \le 1, (1-\alpha)\lambda_1 + (2-\alpha)\lambda_2 \ge 1$,

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(re^{i\theta})|^{\lambda_1} |f'(re^{i\theta})|^{\lambda_2} d\theta = O\left((1-r)^{1-2\lambda_1(1-\alpha)-\lambda_2(3-2\alpha)-\varepsilon}\right), \quad \text{as } r \to 1^-$$

for every $\varepsilon > 0$.

Proof. The condition $p > 1$ implies that

$$\int_{-\pi}^{\pi} |1 - re^{i\theta}|^{-p} = O\left((1-r)^{1-p}\right) \quad \text{as } r \to 1^-.$$

The conditions (i) or (ii) yields the inequality $p = 2\lambda_1(1-\alpha) + (3-2\alpha)\lambda_2 \ge 1$. Finally, we use the results of Theorems 1 and 2. □

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