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Certain fractional derivative formulae involving the product of a general class of polynomials and the multivariable *H*-function

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Abstract. In the present paper, we obtain three unified fractional derivative formulae (FDF). The first involves the product of a general class of polynomials and the multivariable H-function. The second involves the product of a general class of polynomials and two multivariable H-functions and has been obtained with the help of the generalized Leibniz rule for fractional derivatives. The last FDF also involves the product of a general class of polynomials and the multivariable H-function but it is obtained by the application of the first FDF twice and it involves two independent variables instead of one. The polynomials and the functions involved in all our fractional derivative formulae as well as their arguments which are of the type $x^\rho \prod_{i=1}^{s} \left(x^{t_i} + \alpha_i\right)^{\sigma_i}$ are quite general in nature. These formulae, besides being of very general character have been put in a compact form avoiding the occurrence of infinite series and thus making them useful in applications. Our findings provide interesting unifications and extensions of a number of (new and known) results. For the sake of illustration, we give here exact references to the results (in essence) of five research papers [2, 3, 10, 12, 13] that follow as particular cases of our findings. In the end, we record a new fractional derivative formula involving the product of the Hermite polynomials, the Laguerre polynomials and the product of r different Whittaker functions as a simple special case of our first formula.

Keywords. Riemann–Liouville and Erdélyi–Kober fractional operators; fractional derivative formulae; general class of polynomials; multivariable H-function; generalized Leibniz rule.

1. Introduction

We shall define the fractional integrals and derivatives of a function $f(x)$ ([10], pp. 528–529) (see also [6–8]) as follows:

Let α, β and γ be complex numbers. The fractional integral $(\mathrm{Re}(\alpha) > 0)$ and derivative $(\mathrm{Re}(\alpha) < 0)$ of a function $f(x)$ defined on $(0, \infty)$ is given by

$$I_{0,x}^{\alpha,\beta,\gamma} f(x) = \begin{cases} \dfrac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \displaystyle\int_0^x (x-t)^{\alpha-1} F\left(\alpha+\beta, -\gamma; \alpha; 1-\dfrac{t}{x}\right) f(t)\,\mathrm{d}t, \\ \hfill (\mathrm{Re}(\alpha) > 0), \\ \dfrac{\mathrm{d}^q}{\mathrm{d}x^q} I_{0,x}^{\alpha+q,\beta-q,\gamma-q} f(x),\ (\mathrm{Re}(\alpha) \le 0,\ 0 < \mathrm{Re}(\alpha)+q \le 1, \\ \hfill q = 1, 2, 3, \ldots), \end{cases} \tag{1}$$

where F is the Gauss hypergeometric function.

The operator I includes both the Riemann–Liouville and the Erdélyi–Kober fractional operators as follows:

The Riemann–Liouville operator

$$R_{0,x}^{\alpha} f(x) = \begin{cases} I_{0,x}^{\alpha,-\alpha,\gamma} f(x) = \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x (x-t)^{\alpha-1} f(t)\,\mathrm{d}t, \\ \qquad\qquad (\mathrm{Re}(\alpha) > 0), \\ \dfrac{\mathrm{d}^q}{\mathrm{d}x^q} R_{0,x}^{\alpha+q} f(x), \ (\mathrm{Re}(\alpha) \le 0,\ 0 < \mathrm{Re}(\alpha) + q \le 1, \\ \qquad\qquad q = 1, 2, 3, \ldots). \end{cases} \tag{2}$$

The Erdélyi–Kober operators

$$E_{0,x}^{\alpha,\gamma} f(x) = I_{0,x}^{\alpha,o,\gamma} f(x) = \frac{x^{-\alpha-\gamma}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^{\gamma} f(t)\,\mathrm{d}t, \ (\mathrm{Re}(\alpha) > 0). \tag{3}$$

Also, $S_n^m[x]$ occurring in the sequel denotes the general class of polynomials introduced by Srivastava ([11], p. 1, eq. (1))

$$S_n^m[x] = \sum_{k=0}^{[n/m]} \frac{(-n)_{mk}}{k!} A_{n,k} x^k, \quad n = 0, 1, 2, \ldots, \tag{4}$$

where m is an arbitrary positive integer and the coefficients $A_{n,k}(n, k \ge 0)$ are arbitrary constants, real or complex. On suitably specializing the coefficients $A_{n,k}$, $S_n^m[x]$ yields a number of known polynomials as its special cases. These include, among others, the Hermite polynomials, the Jacobi polynomials, the Laguerre polynomials, the Bessel polynomials, the Gould–Hopper polynomials, the Brafman polynomials and several others ([16], pp. 158–161).

The H-function of r complex variables $z_1, \ldots, z_r$ was introduced by Srivastava and Panda [15]. We shall define and represent it in the following form ([14], p. 251, eq. (C. 1)]:

$$\begin{aligned} H[z_1, \ldots, z_r] &= H_{P,Q:P',Q';\ldots;P^{(r)},Q^{(r)}}^{0,N:M',N';\ldots;M^{(r)},N^{(r)}} \\ &\left[\begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \left| \begin{matrix} \left(a_j; \alpha'_j, \ldots, \alpha_j^{(r)}\right)_{1,P} : \left(c'_j, \gamma'_j\right)_{1,P'}; \ldots; \left(c_j^{(r)}, \gamma_j^{(r)}\right)_{1,P^{(r)}} \\ \left(b_j; \beta'_j, \ldots, \beta_j^{(r)}\right)_{1,Q} : \left(d'_j, \delta'_j\right)_{1,Q'}; \ldots; \left(d_j^{(r)}, \delta_j^{(r)}\right)_{1,Q^{(r)}} \end{matrix} \right. \right] \\ &= \frac{1}{(2\pi\omega)^r} \int_{L_1} \cdots \int_{L_r} \phi_1(\xi_1) \ldots \phi_r(\xi_r) \psi(\xi_1, \ldots, \xi_r) z_1^{\xi_1} \ldots z_r^{\xi_r} \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_r \end{aligned} \tag{5}$$

where $w = \sqrt{-1}$,

$$\phi_i(\xi_i) = \frac{\prod_{j=1}^{M^{(i)}} \Gamma\left(d_j^{(i)} - \delta_j^{(i)} \xi_i\right) \prod_{j=1}^{N^{(i)}} \Gamma\left(1 - c_j^{(i)} + \gamma_j^{(i)} \xi_i\right)}{\prod_{j=M^{(i)}+1}^{Q^{(i)}} \Gamma\left(1 - d_j^{(i)} + \delta_j^{(i)} \xi_i\right) \prod_{j=N^{(i)}+1}^{P^{(i)}} \Gamma\left(c_j^{(i)} - \gamma_j^{(i)} \xi_i\right)} \quad \forall i \in \{1, \ldots, r\} \tag{6}$$

$$\psi(\xi_1, \ldots, \xi_r) = \frac{\prod_{j=1}^{N} \Gamma\left(1 - a_j + \sum_{i=1}^{r} \alpha_j^{(i)} \xi_i\right)}{\prod_{j=N+1}^{P} \Gamma\left(a_j - \sum_{i=1}^{r} \alpha_j^{(i)} \xi_i\right) \prod_{j=1}^{Q} \Gamma\left(1 - b_j + \sum_{i=1}^{r} \beta_j^{(i)} \xi_i\right)}. \quad (7)$$

The nature of contours $L_1, \ldots, L_r$ in (5), the various special cases and other details of the above function can be found in the book referred to above.

It may be remarked here that all the Greek letters occurring in the left-hand side of (5) are assumed to be positive real numbers for standardization purposes. The definition of this function will, however, be meaningful even if some of these quantities are zero.

Again, it is assumed that the various multivariable H-functions occurring in the paper always satisfy their appropriate conditions of convergence ([14], pp. 252–253, eqs (C.5 and C.6)).

2. Main results

2.1 *Fractional derivative formula 1*

$$\begin{aligned}
& I_{0,x}^{\alpha,\beta,\gamma} \left\{ x^{\rho} \prod_{i=1}^{s} \left(x^{t_i} + \alpha_i\right)^{\sigma_i} \prod_{j=1}^{t} S_{n_j}^{m_j} \left[e_j x^{\lambda_j} \prod_{i=1}^{s} \left(x^{t_i} + \alpha_i\right)^{\eta_i^{(j)}} \right] \right. \\
& \quad \left. H \left[z_1 x^{u_1} \prod_{i=1}^{s} \left(x^{t_i} + \alpha_i\right)^{-v_i'}, \ldots, z_r x^{u_r} \prod_{i=1}^{s} \left(x^{t_i} + \alpha_i\right)^{-v_i^{(r)}} \right] \right\} \\
& = \alpha_1^{\sigma_1} \ldots \alpha_s^{\sigma_s} x^{\rho-\beta} \sum_{k_1=0}^{[n_1/m_1]} \cdots \\
& \quad \sum_{k_t=0}^{[n_t/m_t]} \frac{(-n_1)_{m_1 k_1} \ldots (-n_t)_{m_t k_t}}{k_1! \ldots k_t!} A'_{n_1,k_1}, \ldots, A^{(t)}_{n_t,k_t} \\
& \quad e_1^{k_1} \ldots e_t^{k_t} \alpha_1^{\eta_1' k_1 + \cdots + \eta_1^{(t)} k_t} \ldots \alpha_s^{\eta_s' k_1 + \cdots + \eta_s^{(t)} k_t} x^{\lambda_1 k_1 + \cdots + \lambda_t k_t} \\
& \quad H^{0,N+s+2:M',N';\ldots;M^{(r)},N^{(r)};1,0;\ldots;1,0}_{P+s+2,Q+s+2:P',Q';\ldots;P^{(r)},Q^{(r)};\underbrace{0,1;\ldots;0,1}_{s}} \left[\begin{array}{l} z_1 \alpha_1^{-v_1'} \ldots \alpha_s^{-v_s'} x^{u_1} \\ \vdots \\ z_r \alpha_1^{-v_1^{(r)}} \ldots \alpha_s^{-v_s^{(r)}} x^{u_r} \\ \alpha_1^{-1} x^{t_1} \\ \vdots \\ \alpha_s^{-1} x^{t_s} \end{array} \right| \\
& (-\rho - \lambda_1 k_1 - \cdots - \lambda_t k_t; u_1, \ldots, u_r, t_1, \ldots, t_s), (\beta - \gamma - \rho - \lambda_1 k_1 \\
& \quad - \cdots - \lambda_t k_t; u_1, \ldots, u_r, t_1, \ldots, t_s), \\
& (\beta - \rho - \lambda_1 k_1 - \cdots - \lambda_t k_t; u_1, \ldots, u_r, t_1, \ldots, t_s), (-\alpha - \gamma - \rho - \lambda_1 k_1 \\
& \quad - \cdots - \lambda_t k_t; u_1, \ldots, u_r, t_1, \ldots, t_s),
\end{aligned}$$

$$\left(1+\sigma_1+\eta_1'k_1+\cdots+\eta_1^{(t)}k_t;\, v_1',\ldots,v_1^{(r)},1,\frac{0,\ldots,0}{s-1}\right),\ldots,$$

$$\left(1+\sigma_s+\eta_s'k_1+\cdots+\eta_s^{(t)}k_t;\, v_s',\ldots,v_s^{(r)},\frac{0,\ldots,0}{s-1},1\right),$$

$$\left(1+\sigma_1+\eta_1'k_1+\cdots+\eta_1^{(t)}k_t;\, v_1',\ldots,v_1^{(r)},\frac{0,\ldots,0}{s}\right),\ldots,$$

$$\left(1+\sigma_s+\eta_s'k_1+\cdots+\eta_s^{(t)}k_t;\, v_s',\ldots,v_s^{(r)},\frac{0,\ldots,0}{s}\right),$$

$$\left.\begin{matrix}\left(a_j;\alpha_j',\ldots,\alpha_j^{(r)},\frac{0,\ldots,0}{s}\right)_{1,P}:\left(c_j',\gamma_j'\right)_{1,P'};\ldots;\left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,P^{(r)}}; \\ \_\_\_\_;\ldots;\_\_\_\_ \\ \left(b_j;\beta_j',\ldots,\beta_j^{(r)},\frac{0,\ldots,0}{s}\right)_{1,Q}:\left(d_j',\delta_j'\right)_{1,Q'};\ldots;\left(d_j^{(r)},\delta_j^{(r)}\right)_{1,Q^{(r)}}; \\ \frac{(0,1);\ldots;(0,1)}{s}\end{matrix}\right] \tag{8}$$

provided that

(i) $\mathrm{Re}(\alpha)>0$; the quantities $t_1,\ldots,t_s,\lambda_1,\eta_1',\ldots,\eta_s',\ldots,\lambda_t,\eta_1^{(t)},\ldots,\eta_s^{(t)},u_1,v_1',\ldots,v_s',u_r,v_1^{(r)},\ldots,v_s^{(r)}$ are all positive (some of them may however decrease to zero provided that the resulting integral has a meaning),

(ii) $\mathrm{Re}(\rho)+\sum_{i=1}^{r}u_i \min\limits_{1\le j\le M^{(i)}}\left[\mathrm{Re}\left(d_j^{(i)}/\delta_j^{(i)}\right)\right]+1>0$

Also the number occurring below the line at any place on the right-hand side of (8) and throughout the paper indicates the total number of zeros/ones/pairs covered by it. Thus $\frac{0,\ldots,0}{r}$, $\frac{1,\ldots,1}{r}$, $\frac{0,1;\ldots;0,1}{r}$ would mean r zeros/r ones/r pairs, and so on.

2.2 *Fractional derivative formula 2*

$$I_{0,x}^{\alpha,\beta,\gamma}\left\{x^\rho\prod_{i=1}^{s}\left(x^{t_i}+\alpha_i\right)^{\sigma_i}\prod_{j=1}^{t}S_{n_j}^{m_j}\left[e_jx^{\lambda_j}\prod_{i=1}^{s}\left(x^{t_i}+\alpha_i\right)^{\eta_i^{(j)}}\right]\right.$$

$$H\left[z_1x^{u_1}\prod_{i=1}^{s}\left(x^{t_i}+\alpha_i\right)^{-v_i'},\ldots,z_rx^{u_r}\prod_{i=1}^{s}\left(x^{t_i}+\alpha_i\right)^{-v_i^{(r)}}\right]$$

$$\left.H^*\left[z_{r+1}x^{u_{r+1}}\prod_{i=1}^{s-1}\left(x^{t_i}+\alpha_i\right)^{-v_i^{(r+1)}},\ldots,z_{r+\tau}x^{u_{r+\tau}}\prod_{i=1}^{s-1}\left(x^{t_i}+\alpha_i\right)^{-v_i^{(r+\tau)}}\right]\right\}$$

$$=\alpha_1^{\sigma_1}\ldots\alpha_s^{\sigma_s}x^{\rho-\beta}\sum_{l=0}^{\infty}\sum_{k_1=0}^{[n_1/m_1]}\cdots\sum_{k_t=0}^{[n_t/m_t]}\left[\begin{matrix}-\beta\\ l\end{matrix}\right]\frac{(-n_1)_{m_1k_1}\ldots(-n_t)_{m_tk_t}}{k_1!\ldots k_t!}$$

$$A'_{n_1,k_1}, \ldots, A^{(t)}_{n_t k_t} e_1^{k_1} \ldots e_t^{k_t} \alpha_1^{\eta'_1 k_1+\cdots+\eta_1^{(t)} k_t} \ldots \alpha_s^{\eta'_s k_1+\cdots+\eta_s^{(t)} k_t} x^{\lambda_1 k_1+\cdots+\lambda_t k_t}$$

$$H^{0,N+N_1+2s+3:M',N';\ldots;M^{(r)},N^{(r)};1,0;\ldots;1,0;M^{(r+1)},N^{(r+1)};\ldots;M^{(r+\tau)},N^{(r+\tau)};1,0;\ldots;1,0}_{P+P_1+2s+3,Q+Q_1+2s+3:P',Q';\ldots;P^{(r)},Q^{(r)};\frac{0,1;\ldots;0,1}{s};P^{(r+1)},Q^{(r+1)};\ldots;P^{(r+\tau)},Q^{(r+\tau)};\frac{0,1;\ldots;0,1}{s-1}}$$

$$\left[\begin{array}{l} z_1\alpha_1^{-v'_1}\ldots\alpha_s^{-v'_s}x^{u_1} \\ \vdots \\ z_r\alpha_1^{-v_1^{(r)}}\ldots\alpha_s^{-v_s^{(r)}}x^{u_r} \\ \alpha_1^{-1}x^{t_1} \\ \vdots \\ \alpha_s^{-1}x^{t_s} \\ z_{r+1}\alpha_1^{-v_1^{(r+1)}}\ldots\alpha_{s-1}^{-v_{s-1}^{(r+1)}}x^{u_{r+1}} \\ \vdots \\ z_{r+\tau}\alpha_1^{-v_1^{(r+\tau)}}\ldots\alpha_{s-1}^{-v_{s-1}^{(r+\tau)}}x^{u_{r+\tau}} \\ \alpha_1^{-1}x^{t_1} \\ \vdots \\ \alpha_{s-1}^{-1}x^{t_{s-1}} \end{array}\right|$$

$$\left(-\lambda_1 k_1 - \cdots - \lambda_t k_t; u_1, \ldots, u_r, t_1, \ldots, t_s, \frac{0, \ldots, 0}{\tau+s-1}\right),$$
$$\left(l-\gamma-\lambda_1 k_1 - \cdots - \lambda_t k_t; u_1, \ldots, u_r, t_1, \ldots, t_s, \frac{0, \ldots, 0}{\tau+s-1}\right),$$
$$\left(l-\lambda_1 k_1 - \cdots - \lambda_t k_t; u_1, \ldots, u_r, t_1, \ldots, t_s, \frac{0, \ldots, 0}{\tau+s-1}\right),$$
$$\left(-\alpha-\gamma-\lambda_1 k_1 - \cdots - \lambda_t k_t; u_1, \ldots, u_r, t_1, \ldots, t_s, \frac{0, \ldots, 0}{\tau+s-1}\right),$$

$$\left(1+\eta'_1 k_1 + \cdots + \eta_1^{(t)} k_t; v'_1, \ldots, v_1^{(r)}, 1, \frac{0, \ldots, 0}{\tau+2s-2}\right), \ldots,$$
$$\left(1+\sigma_s+\eta'_s k_1 + \cdots + \eta_s^{(t)} k_t; v'_s, \ldots, v_s^{(r)}, \frac{0, \ldots, 0}{s-1}, 1, \frac{0, \ldots, 0}{\tau+s-1}\right),$$
$$\left(1+\eta'_1 k_1 + \cdots + \eta_1^{(t)} k_t; v'_1, \ldots, v_1^{(r)}, \frac{0, \ldots 0}{\tau+2s-1}\right), \ldots,$$
$$\left(1+\sigma_s+\eta'_s k_1 + \cdots + \eta_s^{(t)} k_t; v'_s, \ldots, v_s^{(r)}, \frac{0, \ldots, 0}{\tau+2s-1}\right),$$

$$\left(a_j; \alpha'_j, \ldots, \alpha_j^{(r)}, \frac{0, \ldots, 0}{\tau+2s-1}\right)_{1,N},$$
$$\left(-\rho; \frac{0, \ldots, 0}{r+s}, u_{r+1}, \ldots, u_{r+\tau}, t_1, \ldots, t_{s-1}\right),$$

$$\left(b_j;\beta'_j,\ldots,\beta^{(r)}_j,\frac{0,\ldots,0}{\tau+2s-1}\right)_{1,Q},$$

$$\left(\beta-l-\rho;\frac{0,\ldots,0}{r+s},u_{r+1},\ldots,u_{r+\tau},t_1,\ldots,t_{s-1}\right),$$

$$\left(\beta-l-\gamma;\frac{0,\ldots,0}{r+s},u_{r+1},\ldots,u_{r+\tau},t_1,\ldots,t_{s-1}\right),$$

$$\left(1+\sigma_1;\frac{0,\ldots,0}{r+s},v^{(r+1)}_1,\ldots,v^{(r+\tau)}_1,1,\frac{0,\ldots,0}{s-2}\right),\ldots,$$

$$\left(-\alpha-\gamma-\rho;\frac{0,\ldots,0}{r+s},u_{r+1},\ldots,u_{r+\tau},t_1,\ldots t_{s-1}\right),$$

$$\left(1+\sigma_1;\frac{0,\ldots,0}{r+s},v^{(r+1)}_1,\ldots,v^{(r+\tau)}_1,\frac{0,\ldots,0}{s-1}\right),\ldots,$$

$$\left(1+\sigma_{s-1};\frac{0,\ldots,0}{r+s},v^{(r+1)}_{s-1},\ldots,v^{(r+\tau)}_{s-1},\frac{0,\ldots,0}{s-2},1\right),$$

$$\left(a'_j;\frac{0,\ldots,0}{r+s},\alpha^{(r+1)}_j,\ldots,\alpha^{(r+\tau)}_j,\frac{0,\ldots,0}{s-1}\right)_{1,P_1},$$

$$\left(1+\sigma_{s-1};\frac{0,\ldots,0}{r+s},v^{(r+1)}_{s-1},\ldots,v^{(r+\tau)}_{s-1},\frac{0,\ldots,0}{s-1}\right),$$

$$\left(b'_j;\frac{0,\ldots,0}{r+s},\beta^{(r+1)}_j,\ldots,\beta^{(r+\tau)}_j,\frac{0,\ldots,0}{s-1}\right)_{1,Q_1}$$

$$\left(a_j;\alpha'_j,\ldots,\alpha^{(r)}_j,\frac{0,\ldots,0}{\tau+2s-1}\right)_{N+1,P}:\left(c'_j,\gamma'_j\right)_{1,P'};\ldots;$$

$$\left(c^{(r)}_j,\gamma^{(r)}_j\right)_{1,P^{(r)}};\underline{\quad};\ldots;\underline{\quad};$$

$$:\left(d'_j,\delta'_j\right)_{1,Q'};\ldots;\left(d^{(r)}_j,\delta^{(r)}_j\right)_{1,Q^{(r)}};\frac{(0,1);\ldots;(0,1)}{s};$$

$$\left.\begin{matrix}\left(c^{(r+1)}_j,\gamma^{(r+1)}_j\right)_{1,P^{(r+1)}};\ldots;\left(c^{(r+\tau)}_j,\gamma^{(r+\tau)}_j\right)_{1,P^{(r+\tau)}};\underline{\quad};\ldots;\underline{\quad}\\ \left(d^{(r+1)}_j,\delta^{(r+1)}_j\right)_{1,Q^{(r+1)}};\ldots;\left(d^{(r+\tau)}_j,\delta^{(r+\tau)}_j\right)_{1,Q^{(r+\tau)}};\frac{(0,1);\ldots;(0,1)}{s-1}\end{matrix}\right],\tag{9}$$

here $H^*[z_{r+1},\ldots,z_{r+\tau}]$ stands for the following multivariable H-function of τ complex variables $z_{r+1},\ldots,z_{r+\tau}$ ([14], p. 251, eq. (C.1)):

$$H^*[z_{r+1},\ldots,z_{r+\tau}] = H^{0,N_1:M^{(r+1)},N^{(r+1)};\ldots;M^{(r+\tau)},N^{(r+\tau)}}_{P_1,Q_1:P^{(r+1)},Q^{(r+1)};\ldots;P^{(r+\tau)},Q^{(r+\tau)}}$$

$$\left[\begin{array}{c|l} z_{r+1} & \left(a'_j;\alpha_j^{(r+1)},\ldots,\alpha_j^{(r+\tau)}\right)_{1,P_1} : \left(c_j^{(r+1)},\gamma_j^{(r+1)}\right)_{1,P^{(r+1)}};\ldots; \\ & \qquad\qquad \left(c_j^{(r+\tau)},\gamma_j^{(r+\tau)}\right)_{1,P^{(r+\tau)}} \\ \vdots & \\ z_{r+\tau} & \left(b'_j;\beta_j^{(r+1)},\ldots,\beta_j^{(r+\tau)}\right)_{1,Q_1} : \left(d_j^{(r+1)},\delta_j^{(r+1)}\right)_{1,Q^{(r+1)}};\ldots; \\ & \qquad\qquad \left(d_j^{(r+\tau)},\delta_j^{(r+\tau)}\right)_{1,Q^{(r+\tau)}} \end{array}\right]. \tag{10}$$

The function occurring on the right-hand side of (9) is the H-function of $r+2s+\tau-1$ variables provided that

(i) $\mathrm{Re}(\alpha)>0$; the quantities $t_1,\ldots,t_s,\lambda_1,\eta'_1,\ldots,\eta'_s,\ldots,\lambda_t,\eta_1^{(t)},\ldots,\eta_s^{(t)},u_1,v'_1,\ldots,v'_s,u_r,v_1^{(r)},\ldots,v_s^{(r)},u_{r+1},v_1^{(r+1)},\ldots,v_{s-1}^{(r+1)},u_{r+\tau},v_1^{(r+\tau)},\ldots,v_{s-1}^{(r+\tau)}$ are all positive (some of them may however decrease to zero provided that the resulting integral has a meaning),

(ii) $\mathrm{Re}(\rho)+\sum_{i=1}^{r+\tau} u_i \min\limits_{1\le j\le M^{(i)}}\left[\mathrm{Re}\left(d_j^{(i)}/\delta_j^{(i)}\right)\right]+1>0.$

2.3 *Fractional derivative formula 3*

$$I_{0,x}^{\alpha,\beta,\gamma} I_{0,y}^{\alpha',\beta',\gamma'}\left\{x^{\rho}y^{\rho'}\prod_{i=1}^{s}\left(x^{t_i}+\alpha_i\right)^{\sigma_i}\left(y^{t'_i}+\beta_i\right)^{\sigma'_i}\prod_{j=1}^{t}S_{n_j}^{m_j}\right.$$
$$\left[e_j x^{\lambda_j}y^{\zeta_j}\prod_{i=1}^{s}\left(x^{t_i}+\alpha_i\right)^{\eta_i^{(j)}}\left(y^{t'_i}+\beta_i\right)^{\tau_i^{(j)}}\right]$$
$$H\left[z_1x^{u_1}y^{u'_1}\prod_{i=1}^{s}\left(x^{t_i}+\alpha_i\right)^{-v'_i}\left(y^{t'_i}+\beta_i\right)^{-w'_i},\ldots,z_rx^{u_r}y^{u'_r}\right.$$
$$\left.\left.\prod_{i=1}^{s}\left(x^{t_i}+\alpha_i\right)^{-v_i^{(r)}}\left(y^{t'_i}+\beta_i\right)^{-w_i^{(r)}}\right]\right\}$$
$$=\alpha_1^{\sigma_1}\ldots\alpha_s^{\sigma_s}\beta_1^{\sigma'_1}\ldots\beta_s^{\sigma'_s}x^{\rho-\beta}y^{\rho'-\beta'}\sum_{k_1=0}^{[n_1/m_1]}\cdots$$
$$\sum_{k_t=0}^{[n_t/m_t]}\frac{(-n_1)_{m_1k_1}\ldots(-n_t)_{m_tk_t}}{k_1!\ldots k_t!}A'_{n_1,k_1},\ldots,A^{(t)}_{n_tk_t}$$
$$e_1^{k_1}\ldots e_t^{k_t}\alpha_1^{\eta'_1k_1+\cdots+\eta_1^{(t)}k_t}\ldots\alpha_s^{\eta'_sk_1+\cdots+\eta_s^{(t)}k_t}\beta_1^{\tau'_1k_1+\cdots+\tau_1^{(t)}k_t}\ldots\beta_s^{\tau'_sk_1+\cdots+\tau_s^{(t)}k_t}$$
$$x^{\lambda_1k_1+\cdots+\lambda_tk_t}y^{\zeta_1k_1+\cdots+\zeta_tk_t}$$

$$H^{0,N+2s+4:M',N';\dots;M^{(r)},N^{(r)};1,0;\dots;1,0}_{P+2s+4,Q+2s+4:P',Q';\dots;P^{(r)},Q^{(r)};\frac{0,1;\dots;0,1}{2s}}\left[\begin{array}{l} z_1\alpha_1^{-v_1'}\dots\alpha_s^{-v_s'}\beta_1^{-w_1'}\dots\beta_s^{-w_s'}x^{u_1}y^{u_1'} \\ \vdots \\ z_r\alpha_1^{-v_1^{(r)}}\dots\alpha_s^{-v_s^{(r)}}\beta_1^{-w_1^{(r)}}\dots\beta_s^{-w_s^{(r)}}x^{u_r}y^{u_r'} \\ \alpha_1^{-1}x^{t_1} \\ \vdots \\ \alpha_s^{-1}x^{t_s} \\ \beta_1^{-1}y^{t_1'} \\ \vdots \\ \beta_s^{-1}y^{t_s'} \end{array}\right|$$

$$\left(-\rho-\lambda_1k_1-\cdots-\lambda_tk_t;u_1,\dots,u_r,\frac{0,\dots,0}{s},t_1,\dots,t_s\right),$$

$$\left(\beta-\gamma-\rho-\lambda_1k_1-\cdots-\lambda_tk_t;u_1,\dots,u_r,\frac{0,\dots,0}{s},t_1,\dots,t_s\right),$$

$$\left(\beta-\rho-\lambda_1k_1-\cdots-\lambda_tk_t;u_1,\dots,u_r,\frac{0,\dots,0}{s},t_1,\dots,t_s\right),$$

$$\left(-\alpha-\gamma-\rho-\lambda_1k_1-\dots\lambda_tk_t;u_1,\dots,u_r,\frac{0,\dots,0}{s},t_1,\dots,t_s\right),$$

$$\left(1+\sigma_1+\eta_1'k_1+\cdots+\eta_1^{(t)}k_t;v_1',\dots,v_1^{(r)},\frac{0,\dots,0}{s},1,\frac{0,\dots,0}{s-1}\right),\dots,$$

$$\left(1+\sigma_s+\eta_s'k_1+\cdots+\eta_s^{(t)}k_t;v_s',\dots,v_s^{(r)},\frac{0,\dots,0}{2s-1},1\right),$$

$$\left(1+\sigma_1+\eta_1'k_1+\cdots+\eta_1^{(t)}k_t;v_1',\dots,v_1^{(r)},\frac{0,\dots,0}{2s}\right),\dots,$$

$$\left(1+\sigma_s+\eta_s'k_1+\cdots+\eta_s^{(t)}k_t;v_s',\dots,v_s^{(r)},\frac{0,\dots,0}{2s}\right),$$

$$\left(-\rho'-\zeta_1k_1-\cdots-\zeta_tk_t;u_1',\dots,u_r',t_1',\dots,t_s',\frac{0,\dots,0}{s}\right),$$

$$\left(\beta'-\gamma'-\rho'-\zeta_1k_1-\cdots-\zeta_tk_t;u_1',\dots,u_r',t_1',\dots,t_s',\frac{0,\dots,0}{s}\right),$$

$$\left(\beta'-\rho'-\zeta_1k_1-\cdots-\zeta_tk_t;u_1',\dots,u_r',t_1',\dots,t_s',\frac{0,\dots,0}{s}\right),$$

$$\left(-\alpha'-\gamma'-\rho'-\zeta_1k_1-\cdots-\zeta_tk_t;u_1',\dots,u_r',t_1',\dots,t_s',\frac{0,\dots,0}{s}\right),$$

$$\left(1+\sigma_1'+\tau_1'k_1+\cdots+\tau_1^{(t)}k_t;w_1',\ldots,w_1^{(r)},1,\frac{0,\ldots,0}{2s-1}\right),\ldots,$$
$$\left(1+\sigma_s'+\tau_s'k_1+\cdots+\tau_s'k_t;w_s',\ldots,w_s^{(r)},\frac{0,\ldots,0}{s-1},1,\frac{0,\ldots,0}{s}\right),$$
$$\left(1+\sigma_1'+\tau_1'k_1+\cdots+\tau_1^{(t)}k_t;w_1',\ldots,w_1^{(r)},\frac{0,\ldots,0}{2s}\right),\ldots,$$
$$\left(1+\sigma_s'+\tau_s'k_1+\cdots+\tau_s'k_t;w_s',\ldots,w_s^{(r)},\frac{0,\ldots,0}{2s}\right),$$
$$\left.\begin{matrix}\left(a_j;\alpha_j',\ldots,\alpha_j^{(r)},\frac{0,\ldots,0}{2s}\right)_{1,P}:\left(c_j',\gamma_j'\right)_{1,P'};\ldots;\\ \left(c_j^{(r)},\gamma_j^{(r)}\right)_{1,P^{(r)}};\underline{\quad};\ldots;\underline{\quad}\\ \left(b_j;\beta_j',\ldots,\beta_j^{(r)},\frac{0,\ldots,0}{2s}\right)_{1,Q}:\left(d_j',\delta_j'\right)_{1,Q'};\ldots;\\ \left(d_j^{(r)},\delta_j^{(r)}\right)_{1,Q^{(r)}};\frac{(0,1);\ldots;(0,1)}{2s}\end{matrix}\right] \tag{11}$$

provided that

(i) $\mathrm{Re}(\alpha)>0$; $\mathrm{Re}(\alpha')>0$; the quantities $t_1,t_1',\ldots,t_s,t_s',\lambda_1,\eta_1',\ldots,\eta_s',\ldots,\lambda_t,$ $\eta_1^{(t)},\ldots,\eta_s^{(t)},\zeta_1,\tau_1',\ldots,\tau_s',\ldots,\zeta_t,\tau_1^{(t)},\ldots,\tau_s^{(t)},u_1,v_1',\ldots,v_s',u_1',w_1',\ldots,w_s',$ $\ldots,u_r,v_1^{(r)},\ldots,v_s^{(r)},u_r',w_1^{(r)},\ldots,w_s^{(r)}$ are all positive (some of them may however decrease to zero provided that the resulting integral has a meaning),

(ii) $\mathrm{Re}(\rho)+\sum_{i=1}^r u_i\min\limits_{1\le j\le M^{(i)}}\left[\mathrm{Re}\left(d_j^{(i)}/\delta_j^{(i)}\right)\right]+1>0$ and $\mathrm{Re}(\rho')+\sum_{i=1}^r u_i'\min\limits_{1\le j\le M^{(i)}}$ $\left[\mathrm{Re}\left(d_j^{(i)}/\delta_j^{(i)}\right)\right]+1>0.$

Proof of (8). To prove the fractional derivative formula (FDF) 1, we first express the product of a general class of polynomials occurring on its left-hand side in the series form given by (4), replace the multivariable H-function occurring therein by its well-known Mellin–Barnes contour integral given by (5), interchange the order of summations, $(\xi_1,\ldots,\xi_r)$-integrals and taking the fractional derivative operator inside (which is permissible under the conditions stated with (8)) and make a little simplification. Next, we express the terms $\left(x^{t_1}+\alpha_1\right)^{\sigma_1+\eta_1'k_1+\cdots+\eta_1^{(t)}k_t-v_1'\xi_1-\cdots-v_1^{(r)}\xi_r},\ldots,\left(x^{t_s}+\alpha_s\right)^{\sigma_s+\eta_s'k_1+\cdots+\eta_s^{(t)}k_t-v_s'\xi_1-\cdots-v_s^{(r)}\xi_r}$ so obtained in terms of Mellin–Barnes contour integral ([14], p. 18, eq. (2.6.4); p. 10, eq. (2.1.1)). Now, interchanging the orders of $(\xi_{r+1},\ldots,\xi_{r+s})$ and $(\xi_1,\ldots,\xi_r)$-integrals (which is also permissible under the conditions stated with (8)), and evaluating the x-integral thus obtained by using the known formula ([9], p. 16, Lemma 1)

$$I_{0,x}^{\alpha,\beta,\gamma}\left\{x^\lambda\right\}=\frac{\Gamma(1+\lambda)\Gamma(1-\beta+\gamma+\lambda)}{\Gamma(1-\beta+\lambda)\Gamma(1+\alpha+\gamma+\lambda)}x^{\lambda-\beta},$$
$$\mathrm{Re}(\lambda)>\max[0,\mathrm{Re}(\beta-\gamma)]-1 \tag{12}$$

and reinterpreting the multivariable Mellin–Barnes contour integral so obtained in terms of the H-function of $r+s$ variables, we easily arrive at the desired formula (8) after a little simplification.

Proof of (9). To prove FDF 2, we take

$$f(x) = x^{\rho} \prod_{i=1}^{s-1} \left(x^{t_i} + \alpha_i\right)^{\sigma_i}$$
$$H^* \left[z_{r+1} x^{u_{r+1}} \prod_{i=1}^{s-1} \left(x^{t_i} + \alpha_i\right)^{-v_i^{(r+1)}}, \ldots, z_{r+\tau} x^{u_{r+\tau}} \prod_{i=1}^{s-1} \left(x^{t_i} + \alpha_i\right)^{-v_i^{(r+\tau)}} \right]$$

and

$$g(x) = \left(x^{t_s} + \alpha_s\right)^{\sigma_s} \prod_{j=1}^{t} S_{n_j}^{m_j} \left[e_j x^{\lambda_j} \prod_{i=1}^{s} \left(x^{t_i} + \alpha_i\right)^{\eta_i^{(j)}} \right]$$
$$H \left[z_1 x^{u_1} \prod_{i=1}^{s} \left(x^{t_i} + \alpha_i\right)^{-v_i'}, \ldots, z_r x^{u_r} \prod_{i=1}^{s} \left(x^{t_i} + \alpha_i\right)^{-v_i^{(r)}} \right]$$

in the left-hand side of (9); and apply the following generalized Leibniz rule for the fractional integrals

$$I_{0,x}^{\alpha,\beta,\gamma} \{f(x)g(x)\} = \sum_{l=0}^{\infty} \begin{Bmatrix} -\beta \\ l \end{Bmatrix} I_{0,x}^{\alpha,\beta-l,\gamma} \{f(x)\} \, I_{0,x}^{\alpha,l,\gamma} \{g(x)\} \tag{13}$$

we easily obtain FDF 2 after a little simplification on making use of FDF 1 and a known result ([4], p. 91, eq. (6)).

Proof of (11). To prove FDF 3, we use the formula FDF 1 twice, first with respect to the variable y, and then with respect to the variable x_i; here x and y are independent variables.

3. Special cases and applications

The fractional derivative formulae 1, 2 and 3 established here are unified in nature and act as key formulae. Thus the general class of polynomials involved in FDF 1, 2 and 3 reduce to a large spectrum of polynomials listed by Srivastava and Singh ([16], pp. 158–161), and so from formulae 1, 2 and 3 we can further obtain various fractional derivative formulae involving a number of simpler polynomials. Again, the multivariable H-function occurring in these formulae can be suitably specialized to a remarkably wide variety of useful functions (or product of several such functions) which are expressible in terms of E, F, G and H-functions of one, two or more variables. For example, if $N = P = Q = 0$ (or $N_1 = P_1 = Q_1 = 0$), the multivariable H-function occurring in the left-hand side of these formulae would reduce immediately to the product of r (or τ) different H-functions of Fox [1]. Thus the table listing various special cases of the H-function ([5], pp. 145–159) can be used to derive from these fractional derivative formulae a number of other FDF involving any of these simpler special functions.

On reducing the operator defined by (1) to the Riemann–Liouville operator given by (2) we arrive at three fractional derivative formulae involving these operators but we do not record them here explicitly. Again, our FDF 1, 2 and 3 will also give rise in essence to a number of other FDF lying scattered in the literature (see [12], pp. 563–564, eqs (2.1)–(2.3), [13], pp. 644–645, eqs (2.1)–(2.3), [3], pp. 71–72, eq. (2.1) and [2], p. 171, eq. (3.1)) on making suitable substitutions.

Also, if we take $\sigma_i = 0 = \nu_i' = \ldots = \nu_i^{(r)}, i = 1, \ldots, s$ and $n_j = 0, j = 1, \ldots, t$ in (8) (the polynomials $S_0^{m_1}, \ldots, S_0^{m_t}$ will reduce to $A'_{0,0}, \ldots, A_{0,0}^{(t)}$ respectively which can be taken to be unity without loss of generality), we arrive at the formula given by ([10], p. 532, eq. (4.1)).

If in FDF 1, we take $t = 2$ and reduce the polynomial $S_{n_1}^{m_1}$ to the Hermite polynomial ([16], p. 158, eq. (1.4)), the polynomial $S_{n_2}^{m_2}$ to the Laguerre polynomial ([16], p. 159, eq. (1.8)), the multivariable H-function to the product of r different Whittaker functions ([14], p. 18, eq. (2.6.7)), we arrive at the following new and interesting special case of the FDF 1 after a little simplification

$$I_{0,x}^{\alpha,\beta,\gamma}\left\{x^{\rho+\sum_{l=1}^{r} b_l+\frac{n_1}{2}} \prod_{i=1}^{s}\left(x^{t_i}+\alpha_i\right)^{\sigma_i} H_{n_1}\left\{\frac{1}{2\sqrt{x}}\right\} L_{n_2}^{(\theta)}(x) \prod_{l=1}^{r}(\exp)^{-\frac{z_l x}{2}} W_{\mu_l \nu_l}(z_l x)\right\}$$
$$= \frac{\prod_{l=1}^{r}(z_l)^{-b_l}\, \alpha_1^{\sigma_1}\ldots\alpha_s^{\sigma_s} x^{\rho-\beta}}{\Gamma(-\sigma_1)\ldots\Gamma(-\sigma_s)} \sum_{k_1=0}^{[n_1/2]}\sum_{k_2=0}^{[n_2]} \frac{(-n_1)_{2k_1}(-n_2)_{k_2}}{k_1!k_2!}$$
$$(-1)^{k_1}\binom{n_2+\theta}{n_2}\frac{1}{(\theta+1)_{k_2}} x^{k_1+k_2} H^{0,2:2,0;\ldots;2,0;1,1;\ldots;1,1}_{2,2:\underbrace{1,2;\ldots;1,2}_{r};\underbrace{1,1;\ldots;1,1}_{s}}\left[\begin{matrix} z_1 x \\ \vdots \\ z_r x \\ \alpha_1^{-1}x^{t_1} \\ \vdots \\ \alpha_s^{-1}x^{t_s}\end{matrix}\right|$$
$$(-\rho-k_1-k_2; 1,\ldots,1,t_1,\ldots,t_s), (\beta-\gamma-\rho-k_1-k_2; 1,\ldots,1, t_1,\ldots,t_s):$$
$$\left(\beta-\rho-k_1-k_2; \frac{1,\ldots,1,}{r}, t_1,\ldots,t_s\right), \left(-\alpha, -\gamma-\rho-k_1-k_2; \frac{1,\ldots,1}{r}, t_1,\ldots,t_s\right):$$
$$\left.\begin{matrix}(b_1-\mu_1+1,1);\ldots;(b_r-\mu_r+1,1);(1+\sigma_1,1);\ldots;(1+\sigma_s,1) \\ \left(b_1\pm\nu_1+\frac{1}{2},1\right);\ldots;\left(b_r\pm\nu_r+\frac{1}{2},1\right); \underbrace{(0,1);\ldots;(0,1)}_{s}\end{matrix}\right]. \tag{14}$$

The conditions of validity of (14) can be easily obtained from those of (8).

Several other interesting and useful special cases of our main fractional derivative formulae 1, 2 and 3 involving the product of a large variety of polynomials (which are special cases of $S_{n_1}^{m_1}, \ldots, S_{n_t}^{m_t}$) and numerous simple special functions involving one or more variables (which are particular cases of the multivariable H-function) can also be obtained but we do not record them here for lack of space.

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