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A general theorem characterizing some absolute summability methods

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Abstract. A general theorem is given which gives the necessary and sufficient conditions satisfied by a sequence (ε_n) in order to have the series $\sum a_n\varepsilon_n$ summable to $|A|$ whenever $\sum a_n$ is summable to $|A|$ for some summability method A.

Keywords. Absolute summability; summability factors; infinite series.

1. Introduction

Let $\sum a_n$ be a given infinite series with sequence of partial sums (s_n). By u_n we denote the nth $(C, 1)$ mean of the sequence (s_n). The series $\sum a_n$ is said to be summable to $|C, 1|_k$, $k \geq 1$, if (see [2])

$$\sum_{n=1}^{\infty} n^{k-1}|u_n - u_{n-1}|^k < \infty.$$

Let (p_n) be a sequence of positive numbers such that

$$P_n = p_0 + p_1 + \cdots + p_n \to \infty \text{ as } n \to \infty \ (P_{-1} = p_{-1} = 0).$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n}\sum_{v=0}^{n} p_v s_v$$

defines the sequence (t_n) of the Riesz means of the sequence (s_n) generated by the sequence of coefficients (p_n). We say that the series $\sum a_n$ is summable $|R, p_n|_k$, $k \geq 1$, if

$$\sum_{n=1}^{\infty} n^{k-1}|t_n - t_{n-1}|^k < \infty.$$

In the special case when $p_n = 1$ for all values of n (resp. $k = 1$), $|R, p_n|_k$ summability is the same as $|C, 1|_k$ (resp. $|R, p_n|$) summability. Recently Bor [1] proved the following:

Theorem 1. *Let $k > 1$. In order for $|R, p_n|_k \Rightarrow |R, q_n|_k$ to hold, $q_n P_n = O(p_n Q_n)$ is necessary. If we suppose that*

$$\sum_{n=v}^{\infty} \frac{n^{k-1}q_n^k}{Q_n^k Q_{n-1}} = O\left\{\frac{(vq_v)^{k-1}}{Q_v^k}\right\}, \tag{1}$$

then $q_n P_n = O(p_n Q_n)$ is also sufficient.

Quite recently Sarigöl and Bor [3] proved the following:

Theorem 2. *Let* $1 \leq k < \infty$. *Then* $\lambda \in \left(\left|\bar{N}, p_n\right|, \left|\bar{N}, q_n\right|_k\right)$ *iff*

(a) $$\lambda_n = O(1),$$

(b) $$\Delta\lambda_n = O\left(\frac{p_n}{P_n}\right),$$

(c) $$\lambda_n = O\left\{\left(\frac{p_n}{P_n}\right)\left(\frac{Q_n}{q_n}\right)^{1/k}\right\},$$

as $n \to \infty$, *where* $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$.

Theorem 3. *Let* $1 < k < \infty$. *Then* $\lambda \in \left(\left|\bar{N}, p_n\right|_k, \left|\bar{N}, q_n\right|\right)$ *iff*

(a) $$\sum_{v=1}^{\infty}\left(\frac{p_v}{P_v}\right)\left|\left(\frac{P_v}{p_v}\right)\Delta\lambda_v + \lambda_{v+1}\right|^{k^*} < \infty,$$

(b) $$\sum_{v=1}^{\infty}\left(\frac{p_v}{P_v}\right)\left\{\frac{q_v P_v}{p_v Q_v}|\lambda_v|\right\}^{k^*} < \infty,$$

where $1/k + 1/k^* = 1$.

We prove the following:

Theorem 1.1. *Suppose that* (f_n), (g_n), (H_n) *and* (T_n) *be sequences of positive numbers such that* $F_n = \sum_{v=1}^{n} f_v \to \infty$, $G_n = \sum_{v=1}^{n} g_v \to \infty$ *as* $n \to \infty$. *Write*

$$Y_n = \frac{1}{H_n F_{n-1}}\sum_{v=1}^{n} F_{v-1} x_v \varepsilon_v$$

$$X_n = \frac{1}{T_n G_{n-1}}\sum_{v=1}^{n} G_{v-1} x_v$$

and suppose that $\Delta\varepsilon_n \geq 0$, $F_n \Delta\varepsilon_n / f_n G_n \to 0$ *as* $n \to \infty$

$$\Delta(G_{n-1}) = O\left(\frac{f_n G_n}{F_n}\right),$$

$$\sum_{n=v+1}^{\infty}\frac{1}{H_n^k F_{n-1}} = O\left(\frac{1}{f_v H_v^k}\right),$$

and the implication

$$\sum_{n=v+1}^{\infty}\left|\frac{F_n \Delta\varepsilon_n}{f_n G_n H_n}\right|^k = O(1)\sum_{n=v}^{\infty}\frac{1}{T_n^k G_{n-1}^k} \Rightarrow \Delta\varepsilon_n = O\left(\frac{f_n H_n}{F_n T_n}\right)$$

holds. Then necessary and sufficient conditions that the implication

$$\sum \left|X_n\right|^k < \infty \Rightarrow \sum \left|Y_n\right|^k < \infty$$

is satisfied is

(i) $$\varepsilon_n = O\left(\frac{H_n}{T_n}\right),$$

(ii) $$\Delta\varepsilon_n = O\left(\frac{f_n H_n}{F_n T_n}\right).$$

2. Lemmas

Lemma 1. [1]. *Let $k \geq 1$ and let $A = (a_{nv})$ be an infinite matrix. In order that $A \in (l^k, l^k)$ it is necessary that*

$$a_{nv} = O(1)(\textit{for all } n, v).$$

Lemma 2. *Let $Q_n \to \infty$ as $n \to \infty$, then*

(a) $$\frac{1}{Q^k_{v-1}} = O(1)\sum_{n=v}^{\infty} \frac{n^{k-1}q^k_n}{Q^k_n Q^k_{n-1}}, \textit{provided } Q_n = O(nq_n),$$

(b) $$\sum_{n=v}^{\infty} \frac{n^{k-1}q^k_n}{Q^k_n Q^k_{n-1}} = O\left(\frac{1}{Q^k_v}\right), \textit{provided } nq_n = O(Q_n).$$

Proof.

(a) $$\frac{1}{Q^r_{v-1}} = \sum_{n=v}^{\infty} \Delta\left(\frac{1}{Q^r_{n-1}}\right) = O(1)\sum_{n=v}^{\infty} \frac{Q^r_n - Q^r_{n-1}}{Q_n Q_{n-1}}$$

$$= O(1)\sum_{n=v}^{\infty} \frac{(Q_n - Q_{n-1})(Q^{r-1}_n + Q^{r-2}_n Q_{n-1} + \cdots + Q^{r-1}_{n-1})}{Q^r_n Q^r_{n-1}}$$

$$= O(1)\sum_{n=v}^{\infty} \frac{q_n Q^{r-1}_n}{Q^r_n Q^r_{n-1}} = O(1)\sum_{n=v}^{\infty} \frac{q_n}{Q_n Q^r_{n-1}} = O(1)\sum_{n=v}^{\infty} \frac{n^{k-1}q^h_n}{Q^k_n Q^r_{n-1}}.$$

If r is not an integer, the result follows by the mean value theorem.

(b) $$\sum_{n=v}^{\infty} \frac{n^{k-1}q^h_n}{Q^k_n Q^k_{n-1}} = O\left(\frac{1}{Q^{k-1}_{v-1}}\right)\sum_{n=v}^{\infty} \frac{q_n}{Q_n Q_{n-1}}$$

$$= O\left(\frac{1}{Q^{k-1}_{v-1}}\right)\sum_{n=v}^{\infty} \left(\frac{1}{Q_{n-1}} - \frac{1}{Q_n}\right)$$

$$= O\left(\frac{1}{Q^k_{v-1}}\right).$$

Lemma 3. *Let $Q_n \to \infty$ as $n \to \infty$, then*

(a) $$\frac{1}{Q^k_{v-1}} = O(1)\sum_{n=v}^{\infty} \frac{q_n}{Q_n Q^k_{n-1}},$$

(b) $$\sum_{n=v}^{\infty} \frac{q_n}{Q_n Q^k_{n-1}} = O(1)\left(\frac{1}{Q^k_{v-1}}\right).$$

Proof. Follows in the same manner as in the proof of Lemma 2.

3. Proof of Theorem 1.1

We have via Abel's transformaiton

$$
\begin{aligned}
Y_n &= \frac{1}{H_n F_{n-1}} \sum_{v=1}^{n} G_{v-1} x_v \frac{F_{v-1}\varepsilon_v}{G_{v-1}} \\
&= \frac{1}{H_n F_{n-1}} \left[\sum_{v=1}^{n-1} \left(\sum_{r=1}^{v} G_{r-1} x_r \right) \Delta \left(\frac{F_{v-1}\varepsilon_v}{G_{v-1}} \right) \right. \\
&\qquad \left. + \left(\sum_{r=1}^{n} G_{r-1} x_r \right) \frac{F_{n-1}\varepsilon_n}{G_{n-1}} \right] \\
&= \frac{1}{H_n F_{n-1}} \sum_{v=1}^{n-1} \left(-f_v T_v X_v \varepsilon_v - \frac{F_v \Delta G_{v-1} T_v X_v \varepsilon_v}{G_v} \right. \\
&\qquad \left. + \frac{F_v G_{v-1} T_v X_v \Delta\varepsilon_v}{G_v} \right) + \frac{T_n X_n \varepsilon_n}{H_n} \\
&= Y_{n,1} + Y_{n,2} + Y_{n,3} + Y_{n,4}, \text{ say.}
\end{aligned} \tag{2}
$$

In order to prove sufficiency, by Minkowski's inequality, it is sufficient to show that

$$
\sum_{n=1}^{\infty} |Y_{n,j}|^k < \infty, \; j = 1, 2, \ldots, 4.
$$

Applying Hölder's inequality,

$$
\begin{aligned}
\sum_{n=2}^{m+1} |Y_{n,1}|^k &= \sum_{n=2}^{m+1} \frac{1}{H_n^k F_{n-1}^k} \left| \sum_{v=1}^{n-1} f_v T_v X_v \varepsilon_v \right|^k \\
&\le \sum_{n=2}^{m+1} \frac{1}{H_n^k F_{n-1}} \sum_{v=1}^{n-1} f_v T_v^k |X_v|^k |\varepsilon_v|^k \left\{ \sum_{v=1}^{n-1} \frac{f_v}{F_{n-1}} \right\}^{k-1} \\
&= O(1) \sum_{v=1}^{m} T_v^k |X_v|^k |\varepsilon_v|^k \sum_{n=v+1}^{m+1} \frac{f_v}{H_v^k F_{n-1}} \\
&= O(1) \sum_{v=1}^{m} \frac{T_v^k |X_v|^k |\varepsilon_v|^k}{H_v^k},
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=2}^{m+1} \left| Y_{n,2} \right|^k &= \sum_{n=2}^{m+1} \frac{1}{H_n^k F_{n-1}^k} \left| \sum_{v=1}^{n-1} \frac{F_v \Delta G_{v-1} T_v X_v \varepsilon_v}{G_v} \right|^k \\
&\le \sum_{n=2}^{m+1} \frac{1}{H_n^k F_{n-1}} \sum_{v=1}^{n-1} \frac{f_v F_v^k |\Delta G_{v-1}|^k T_v^k |X_v|^k |\varepsilon_v|^k}{f_v^k G_v^k} \\
&\quad \times \left\{ \sum_{v=1}^{n-1} \frac{f_v}{F_{n-1}} \right\}^{k-1} \\
&\le O(1) \sum_{v=1}^{m} \frac{F_v^k |\Delta G_{v-1}|^k T_v^k |X_v|^k |\varepsilon_v|^k}{f_v^k G_v^k} \sum_{n=v+1}^{m+1} \frac{f_v}{H_n^k F_{n-1}}
\end{aligned}
$$

$$= O(1)\sum_{v=1}^{m}\frac{T_v^k|X_v|^k|\varepsilon_v|^k}{H_v^k},$$

$$\sum_{n=2}^{m+1}\left|Y_{n,3}\right|^k = \sum_{n=2}^{m+1}\frac{1}{H_n^k F_{n-1}^k}\left|\sum_{v=1}^{n-1}\frac{F_v G_{v-1} T_v X_v \Delta\varepsilon_v}{G_v}\right|^k$$

$$\leq \sum_{n=2}^{m+1}\frac{1}{H_n^k F_{n-1}}\sum_{v=1}^{n-1}\frac{f_v F_v^k G_{v-1}^k T_v^k |X_v|^k |\Delta\varepsilon_v|^k}{f_v^k G_v^k}$$

$$\times\left\{\sum_{v=1}^{n-1}\frac{f_v}{F_{n-1}}\right\}^{k-1}$$

$$\leq O(1)\sum_{v=1}^{m}\frac{F_v^k G_{v-1}^k T_v^k |X_v|^k |\Delta\varepsilon_v|^k}{f_v^k G_v^k}\sum_{n=v+1}^{m+1}\frac{f_v}{H_n^k F_{n-1}}$$

$$= O(1)\sum_{v=1}^{m}\frac{F_v^k G_{v-1}^k T_v^k |X_v|^k |\Delta\varepsilon_v|^k}{f_v^k G_v^k H_v^k},$$

$$\sum_{n=1}^{m}\left|Y_{n,4}\right|^k = O(1)\sum_{n=1}^{m}\frac{T_n^k|X_n|^k|\varepsilon_n|^k}{H_n^k}.$$

Necessity of (i). Using the result of Bor in [1], the transformation from (X_n) into (Y_n) maps $l^k$to l^k and hence the diagonal elements of this transformation are bounded (by Lemma 1) and so (i) is necessary.

Necessity of (ii). By (2), we have

$$Y_{n,3} = Y_n - Y_{n,1} - Y_{n,2} - T_{n,4}.$$

By Minkowski's inequality, the hypothesis, and the necessity of (i), we have

$$\sum_{n=1}^{\infty}|Y_{n,3}|^k = O(1)\sum_{n=1}^{\infty}|X_n|^k \tag{3}$$

Now, taking $v \geq 1$ and choosing

$$x_v = 1, \quad x_j = 0\ (j \neq v),$$

we have

$$X_n = \begin{cases} 0, & n < v \\ \dfrac{G_{v-1}}{T_n G_{n-1}}, & n \geq v \end{cases}.$$

Equation (3) gives

$$\sum_{n=v}^{\infty}\frac{1}{H_n^k F_{n-1}^k}\left|\sum_{r=1}^{n-1}\frac{F_r G_{r-1} T_r X_r \Delta\varepsilon_r}{G_r}\right|^k = O(1)\sum_{n=v}^{\infty}\frac{G_{v-1}^k}{T_n^k G_{n-1}^k},$$

$$\sum_{n=v}^{\infty}\frac{1}{H_n^k F_{n-1}^k}\left|\sum_{r=1}^{n-1}\frac{F_r \Delta\varepsilon_r}{G_r}\right|^k = O(1)\sum_{n=v}^{\infty}\frac{1}{T_n^k G_{n-1}^k}.$$

Now, as $F_n \Delta\varepsilon_n / f_n G_n \to 0$,

$$\frac{F_n \Delta\varepsilon_n F_{n-1}}{G_n f_n} = \frac{F_n \Delta\varepsilon_n}{G_n f_n} \sum_{r=1}^{n-1} f_r = O(1) \sum_{r=1}^{n-1} \frac{F_r \Delta\varepsilon_r}{G_r},$$

then

$$\sum_{n=v}^{\infty} \left| \frac{F_n \Delta\varepsilon_n}{G_n f_n H_n} \right|^k = O(1) \sum_{n=v}^{\infty} \frac{1}{T_n^k G_{n-1}^k}$$

which, by the hypothesis implies

$$\Delta\varepsilon_n = O\left(\frac{f_n H_n}{F_n T_n} \right).$$

This completes the proof of the theorem.

Remark. We may assume that $P_n, Q_n \to \infty$ as $n \to \infty$.

(1) If we put $x_n = a_n, f_n = q_n, H_n = (Q_n/q_n)^{1/k}$ in the formula defining Y_n, then the condition $\sum |Y_n|^k < \infty$ is equivalent to $|\bar{N}, q_n|_k$ summability of $\sum a_n \varepsilon_n$.

(2) If we put $x_n = a_n, f_n = q_n, H_n = Q_n/n^{1-(1/k)} q_n$ in the formula defining Y_n, then the condition $\sum |Y_n|^k < \infty$ is equivalent to $|R, q_n|_k$ summability of $\sum a_n \varepsilon_n$.

(3) If we put $x_n = a_n, g_n = p_n, T_n = (P_n/p_n)^{1/k}$ in the formula defining X_n, then the condition $\sum |X_n|^k < \infty$ is equivalent to $|\bar{N}, p_n|_k$ summability of $\sum a_n$.

(4) If we put $x_n = a_n, g_n = p_n, T_n = P_n/n^{1-(1/k)} p_n$ in the formula defining X_n, then the condition $\sum |X_n|^k < \infty$ is equivalent to $|R, p_n|_k$ summability of $\sum a_n$.

4. Applications

In what follows we assume that $P_n, Q_n \to \infty$ as $n \to \infty$ and $\Delta\varepsilon_n \geq 0$.

Theorem 3.1. *Suppose $P_n = O(P_{n-1})$, $p_n Q_n = O(P_n q_n)$, $P_n q_n = O(p_n Q_n)$ and $Q_n \Delta\varepsilon_n / q_n P_n \to 0$ as $n \to \infty$. Then necessary and sufficient conditions that $\sum a_n \varepsilon_n$ be summable to $|\bar{N}, q_n|_k$ whenever $\sum a_n$ is summable to $\left|\bar{N}, p_n\right|_k$, $k \geq 1$, is*

$$\varepsilon_n = O\left(\frac{p_n Q_n}{P_n q_n} \right)^{1/k},$$

$$\Delta\varepsilon_n = O\left(\frac{q_n}{Q_n} \right).$$

Proof. It is sufficient to show that

$$\sum_{n=v}^{\infty} \frac{Q_n^k |\Delta\varepsilon_n|^k q_n}{q_n^k P_n^k Q_n} = O(1) \sum_{n=v}^{\infty} \frac{p_n}{P_n P_{n-1}^k}$$

implies that $\Delta\varepsilon_n = O(q_n/Q_n)$. By Lemma 3,

$$\frac{Q_v^k Q_{v-1}^{k_1}|\Delta\varepsilon_v|^k}{q_v^k P_v^k}\sum_{n=v}^{\infty}\frac{q_n}{Q_n Q_{n-1}^{k_1}} = O\left(\frac{1}{P_{v-1}^k}\right),$$

for some $k_1 > 0$ such that $Q_n^k Q_{n-1}^{k_1}|\Delta\varepsilon_n|^k \big/ q_n^k P_n^k \to \infty$ as $n \to \infty$. Again by Lemma 3,

$$\frac{Q_v^k\left|\Delta\varepsilon_v\right|^k}{q_v^k P_v^k} = O\left(\frac{1}{P_{v-1}^k}\right).$$

Therefore

$$\Delta\varepsilon_v = O\left(\frac{q_v}{Q_v}\frac{P_v}{P_{v-1}}\right) = O\left(\frac{q_v}{Q_v}\right).$$

Theorem 3.2. *Suppose $P_n = O(P_{n-1})$, $p_n Q_n = O(P_n q_n)$, $P_n q_n = O(p_n Q_n)$ and $Q_n\Delta\varepsilon_n/q_n P_n \to 0$ as $n \to \infty$, $Q_n = O(nq_n)$, $np_n = O(P_n)$, and*

$$\sum_{n=v+1}^{\infty}\frac{n^{k-1}q_n^k}{Q_n^k Q_{n-1}} = O\left\{\frac{(vq_v)^{k-1}}{Q_v^k}\right\}.$$

Then necessary and sufficient conditions that $\sum a_n\varepsilon_n$ be summable to $|R, q_n|_k$ whenever $\sum a_n$ is summable to $|R, p_n|_k$, $k \geq 1$, is

(i) $$\varepsilon_n = O\left(\frac{p_n Q_n}{P_n q_n}\right),$$

(ii) $$\Delta\varepsilon_n = O\left(\frac{p_n}{P_n}\right).$$

Proof. It is sufficient to show that condition

$$\sum_{n=v}^{\infty}\frac{n^{k-1}|\Delta\varepsilon_n|^k}{P_n^k} = O(1)\sum_{n=v}^{\infty}\frac{n^{k-1}p_n^k}{P_n^k P_{n-1}^k}$$

implies that $\Delta\varepsilon_n = O(p_n/P_n)$. We have

$$\frac{Q_v^k Q_{v-1}^{k_1}|\Delta\varepsilon_v|^k}{q_v^k P_v^k}\sum_{n=v}^{\infty}\frac{n^{k-1}q_n^k}{Q_n^k Q_{n-1}^{k_1}} = O(1)\sum_{n=v}^{\infty}\frac{n^{k-1}p_n^k}{P_n^k P_{n-1}^k}$$

for some $k_1 > 0$ such that $Q_n^k Q_{n-1}^{k_1}|\Delta\varepsilon_n|^k/q_n^k P_n^k \to \infty$ as $n \to \infty$. By Lemma 2,

$$\frac{Q_v^k|\Delta\varepsilon_v|^k}{q_v^k P_v^k} = O\left(\frac{1}{P_{v-1}^k}\right).$$

Therefore

$$\Delta\varepsilon_v = O\left(\frac{q_v}{Q_v}\right) = O\left(\frac{p_v}{P_v}\right).$$

Note. It may be mentioned that we can find the necessary and sufficient conditions satisfied by (ε_n) in order to have $\sum a_n \varepsilon_n$ summable to $\left|\bar{N}, q_n\right|_k$ whenever $\sum a_n$ is summable to $\left|R, p_n\right|_k$ and $\sum a_n \varepsilon_n$ be summable to $\left|R, q_n\right|_k$ whenever $\sum a_n$ is summable to $\left|\bar{N}, p_n\right|_k$, by an application of Theorem 1.1.

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