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Representability of GL_E

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Abstract. We prove a necessary and sufficient condition for the automorphisms of a coherent sheaf to be representable by a group scheme.

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The main result of this note is the following theorem.

Theorem 1. (Representability of the functor GL_E**).** *Let S be a noetherian scheme, and E a coherent $\mathcal{O}_S$-module. Let GL_E denote the contrafunctor on S-schemes which associates to any S-scheme $f : T \to S$ the group of all $\mathcal{O}_T$-linear automorphisms of the pullback $E_T = f^*E$ (this functor is a sheaf in the fpqc topology). Then GL_E is representable by a group scheme over S if and only if E is locally free.*

The 'if' part is obvious. The main work is in proving the 'only if' part, for which we need various preliminaries. The following lemma is standard, and is the first step in the construction of a flattening stratification of a noetherian scheme S for a coherent sheaf on $\mathbf{P}^n_S$.

Lemma 2. *If R is a noetherian local ring and E a finite R-module, there exists an ideal $I \subset \mathfrak{m}$ with the following property: the module E/IE is free over R/I, and for any ideal $J \subset R$, the module E/JE is free over R/J if and only if $I \subset J$. By its property, I is unique.*

Proof. Define I to be the ideal generated by the matrix entries of the map $\varphi : R^q \to R^p$ where $R^q \stackrel{\varphi}{\to} R^p \to E \to 0$ is an exact sequence in which p is minimal (equal to the dimension of the vector space $E/\mathfrak{m}E$ over $R/\mathfrak{m}$, where $\mathfrak{m}$ denotes the maximal ideal in R). It can be seen that this I has the desired property. □

Remark 3. As a consequence, if R is a noetherian local ring and E a finite R-module such that $E/\mathfrak{m}^nE$ is a free module over $R/\mathfrak{m}^n$ for each $n \geq 2$, then E is free over R. By the above lemma, $I \subset \mathfrak{m}^n$ for each $n \geq 2$, hence $I = 0$.

Lemma 4. (Srinivas). *Let R be an artin local ring with maximal ideal $\mathfrak{m}$, and let E be a finite R-module, with corresponding ideal I as in Lemma 2. Suppose that the ideal I is a principal ideal and $\mathfrak{m}I = 0$. Then E is isomorphic to a direct sum of the form $R^m \oplus (R/I)^n$, where m, n are non-negative integers.*

Proof. Let $R^q \stackrel{\varphi}{\to} R^p \to E \to 0$ be an exact sequence of R-modules such that $p = \dim_{R/\mathfrak{m}}(E/\mathfrak{m}E)$. The ideal I is generated by the matrix entries of the map $\varphi : R^q \to R^p$.

By assumption, there exists some $a \in \mathfrak{m}$ with $I = (a)$ and $\mathfrak{m}a = 0$. If $a = 0$ then E is free, so now assume $a \neq 0$. Hence every non-zero element of I is of the form ua where $u \in R - \mathfrak{m}$ is some unit of R. Hence the non-zero matrix entries of $\varphi : R^q \to R^p$ (if any) are of the form ua. Hence there is another matrix ψ whose non-zero entries are units of R, with $\phi = a\psi$. Changing the free basis of R^q and R^p gives row and column operations on ψ, which can be used to put it in a block form

$$\begin{pmatrix} 1_{m\times m} & 0_{m\times(q-m)} \\ 0_{(p-m)\times m} & 0_{(p-m)\times(q-m)} \end{pmatrix}$$

The lemma follows. □

Lemma 5. *Let S be a noetherian scheme, and let E be a coherent $\mathcal{O}_S$-module. Let E' be a coherent subsheaf of E, such that the quotient E/E' is locally free. If GL_E is representable, then the subfunctor P of GL_E which consists of automorphisms of E (over base changes) which preserve E' is also representable, and is represented by a closed subgroup scheme of GL_E over S.*

Proof. If $f : F' \to F$ is a homomorphism of coherent sheaves on a scheme T such that F is locally free, then T has a closed subscheme $T_0 \hookrightarrow T$ with the universal property that f vanishes identically under a base-change $T' \to T$ if and only if it factors via $T_0 \hookrightarrow T$. Applying this with $T = GL_E$, $F' = E'_T$, $F = (E/E')_T$, and with $f : E'_T \to (E/E')_T$ the composite $E'_T \to E_T \stackrel{u}{\to} E_T \to (E/E')_T$ where $u : E_T \to E_T$ is the universal family of automorphisms over $T = GL_E$, we get a closed subscheme $P \subset GL_E$ which has the desired properties. □

Lemma 6. *Let X be a scheme, and $I \subset \mathcal{O}_X$ a quasi-coherent ideal sheaf, with $I^n = 0$ for some $n \geq 1$. Suppose that the closed subscheme $Y \subset X$ defined by I is affine. Then X is affine.*

Proof. By induction on n, we can reduce to the case where $I^2 = 0$. As $I^2 = 0$, I becomes an $\mathcal{O}_Y$-module. As I is quasi-coherent over $\mathcal{O}_X$, it is quasi-coherent over $\mathcal{O}_Y$. If F is any quasi-coherent sheaf on X, then we have a short exact sequence $0 \to IF \to F \to F/IF \to 0$. As $I^2 = 0$, both IF and F/IF are $\mathcal{O}_Y$-modules, and these are quasi-coherent. Hence as Y is affine, $H^1(Y, IF) = H^1(Y, F/IF) = 0$. But these are just cohomologies over the space X, as topologically Y is X. Hence by the long exact sequence of $0 \to IF \to F \to F/IF \to 0$, it follows that $H^1(X, F) = 0$. As this holds for every quasi-coherent $\mathcal{O}_X$-module, X is affine by Serre's theorem. □

Lemma 7. *Let A be a ring and $I \subset A$ an ideal with $I^n = 0$ for some $n \geq 1$. Let B be an A-algebra, such that B/IB is finite-type over A (equivalently, over A/I). Let $b_1, \ldots, b_m \in B$ such that $B/I = A[\overline{b}_1, \ldots, \overline{b}_m]$, where $\overline{b}_i \in B/I$ is the residue of b_i. Then B is generated as an A-algebra by $b_1, \ldots, b_m$.*

Proof. By induction on n, we are reduced to the case where $I^2 = 0$. As $B/I = A[\overline{b}_1, \ldots, \overline{b}_m]$, any $x \in B$ can be written as $x = f(b_1, \ldots, b_m) + uy$ where f is a polynomial in m variables over A, $u \in I$, and $y \in B$. Similarly, $y = g(b_1, \ldots, b_m) + vz$ where g is a polynomial in m variables over A, $v \in I$, and $z \in B$. As $I^2 = 0$, we get $x = f(b_1, \ldots, b_m) + ug(b_1, \ldots, b_m)$. Hence $B = A[b_1, \ldots, b_m]$. □

Lemma 8. *Let R be an artin local ring with maximal ideal $\mathfrak{m}$, and let $0 \neq I \subset \mathfrak{m}$ be a non-zero proper ideal. Let $E = (R/I)^n \oplus R^m$ where $n \geq 1$ and $m \geq 0$. Then the functor GL_E is not representable.*

Proof. By Nakayama, $\mathfrak{m}I \neq I$, so we can base-change to $R/\mathfrak{m}I$ and assume that $\mathfrak{m}I = 0$, in particular, $I^2 = 0$. Suppose GL_E is represented by a group-scheme G over R. The restriction of G to R/I is the affine scheme $GL_{n+m,\, R/I}$ over R/I, and I is a nilpotent ideal. Hence G must be affine by Lemma 6, and finite-type over R by Lemma 7. By Lemma 5, the automorphisms which preserve $(R/I)^n \subset E$ are represented by a closed subgroup scheme $P \subset G$. Let $P = \mathrm{Spec}(A)$ where A is a finitely generated R-algebra.

The elements of the group $P(R)$ are matrices with the block form $\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}$ where $X \in GL_n(R/I)$, $Y \in Hom(R^m, (R/I)^n) = (R/I)^{mn}$, and $Z \in GL_m(R)$. Hence the elements $g \in P(R)$ which restrict to the identity in $P(R/I)$, that is, elements of the kernel of $P(R) \to P(R/I)$, are exactly the elements of the form $\begin{pmatrix} 1 & 0 \\ 0 & 1+W \end{pmatrix}$ where $W \in M_m(I)$ is an arbitrary matrix with all entries in I.

The restriction of P to R/I is the parabolic subgroup scheme $H \subset GL_{n+m,R/I}$ which preserves $(R/I)^n \subset (R/I)^{n+m}$, with coordinate ring

$$B = R/I\,[x_{i,j},\ y_{i,\beta},\ z_{\alpha,\beta},\ \det(x_{i,j})^{-1},\ \det(z_{\alpha,\beta})^{-1}]$$

where $1 \leq i, j \leq n$, and $1 \leq \alpha, \beta \leq m$. As $B = A/IA$ where $I^2 = 0$, by Lemma 7 we get that

$$A = R\,[x_{i,j},\ y_{i,\beta},\ z_{\alpha,\beta},\ \det(x_{i,j})^{-1},\ \det(z_{\alpha,\beta})^{-1}]/J$$

for some ideal $J \subset IR\,[x_{i,j},\ y_{i,\beta},\ z_{\alpha,\beta},\ \det(x_{i,j})^{-1},\ \det(z_{\alpha,\beta})^{-1}]$. Let $V \in M_n(I)$ be any arbitrary $n \times n$-matrix over I. We can define an R-algebra homomorphism $A \to R$ by

$$x_{i,j} \mapsto \delta_{i,j} + v_{i,j}, \qquad y_{i,\beta} \mapsto 0 \quad \text{and} \quad z_{\alpha,\beta} \mapsto \delta_{\alpha,\beta}.$$

Modulo I, this specializes to identity, hence this contradicts the above description of the kernel of $P(R) \to P(R/I)$. This contradiction proves the lemma. □

Now all the necessary preliminaries are in place for completing the proof of the main result.

Proof of Theorem 1. Suppose that E is not locally free. By first passing to the local ring of S at some point where E is not locally free and then going modulo a high power of the maximal ideal (see Remark 3), we can assume that $S = \mathrm{Spec}(R)$ where R is an artin local ring, and E is a finite R module which is not free. Let $0 \neq I \subset \mathfrak{m}$ be the ideal defined by E as in Lemma 2, where $\mathfrak{m}$ is the maximal ideal of R. Let $I = (a_1, \ldots, a_r)$ where r is the smallest number of generators needed to generate the ideal I. If $r \geq 2$, let $J = (a_1, \ldots, a_{r-1}) \subset I$. Then going modulo J (that is, by base-changing to R/J), we are reduced to the case where I is a principal ideal. By further going modulo $\mathfrak{m}I$, we can assume $\mathfrak{m}I = 0$. Hence by Lemma 4, E splits as a direct sum $R^m \oplus (R/I)^n$, where $n \geq 1$ as E is not free. Hence GL_E is not representable by Lemma 8, which completes the proof of the theorem. □

Example 9. The functor on commutative rings, defined by $R \mapsto (R/2R)^\times$ (the multiplicative group of units in the ring $R/2R$), is not representable by a scheme. This follows by taking $S = \mathrm{Spec}(\mathbb{Z})$ and $E = \mathbb{Z}/2\mathbb{Z}$ in Theorem 1. A shorter direct proof is also possible in this example, by using discrete valuation rings instead of artin local rings.

Direct proof. If a group scheme $G \to \mathrm{Spec}(\mathbb{Z})$ represents this functor, then the fiber of G over the closed point (2) will be $\mathbf{G}_{m,\mathbf{F}_2}$, while over the open complement $\mathrm{Spec}(\mathbb{Z}) - (2)$, the restriction of G will be trivial. Let U be an affine open neighborhood in G of the identity point $1 \in \mathbf{G}_{m,\mathbf{F}_2} \subset G$, and let $x \in \mathbf{G}_{m,\mathbf{F}_2}$ be a closed point other than 1 which is in U (the purpose of using an affine open U is to avoid any assumption about separatedness of G). The residue field $\kappa(x)$ at x is a finite extension of $\mathbf{F}_2$, hence separable over $\mathbf{F}_2$. Let A be the henselization of the local ring $\mathbb{Z}_{(2)}$ with respect to the residue field extension $\mathbf{F}_2 \subset \kappa(x)$. This is a discrete valuation ring of characteristic zero, with maximal ideal $2A$ as A is étale over $\mathbb{Z}_{(2)}$, and residue field $\kappa(x)$. Therefore, $G(\kappa(x)) = \kappa(x)^\times = (A/2A)^\times = G(A)$, and so x uniquely prolongs to an A-valued point of G, which we denote by x'. Note that $x' : \mathrm{Spec}\, A \to G$ factors through $U \subset G$. Therefore we have points 1 and x' of $U(A)$ which coincide over the generic point of A, but differ over the special point. This contradicts the separatedness of $U \to \mathrm{Spec}(\mathbb{Z})$. □

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