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A basic inequality for submanifolds in locally conformal almost cosymplectic manifolds

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Abstract. For submanifolds tangent to the structure vector field in locally conformal almost cosymplectic manifolds of pointwise constant φ-sectional curvature, we establish a basic inequality between the main intrinsic invariants of the submanifold on one side, namely its sectional curvature and its scalar curvature; and its main extrinsic invariant on the other side, namely its squared mean curvature. Some applications including inequalities between the intrinsic invariant δ_M and the squared mean curvature are given. The equality cases are also discussed.

Keywords. Locally conformal almost cosymplectic manifold; invariant submanifold; semi-invariant submanifold; δ_M-invariant; squared mean curvature.

1. Introduction

To find simple relationships between the main extrinsic invariants and the main intrinsic invariants of a submanifold is one of the basic interests in the submanifold theory. Recently, Chen [4] introduced a well-defined intrinsic invariant δ_M for a Riemannian manifold M. Let M be an n-dimensional Riemannian manifold. For each point $p \in M$, let $(\inf K)(p) = \inf\{K(\pi) : \text{plane sections } \pi \subset T_pM\}$. Then, $\delta_M(p)$ is given by

$$\delta_M(p) = \tau(p) - (\inf K)(p), \tag{1}$$

where τ is the scalar curvature of M (see also [6]).

In [3], Chen first established the following basic inequality involving the intrinsic invariant δ_M and the squared mean curvature for n-dimensional submanifolds M in a real space form $R(c)$ of constant sectional curvature c:

$$\delta_M \le \frac{n^2(n-2)}{2(n-1)}\|H\|^2 + \frac{1}{2}(n+1)(n-2)c. \tag{2}$$

It was remarked in [7], that the above inequality is also true for anti-invariant submanifolds in complex space forms $\widetilde{M}(4c)$. In [5], he proved a general inequality for an arbitrary submanifold of dimension > 2 in a complex space form. By applying this inequality, he showed that (2) is also valid for arbitrary submanifolds in complex hyperbolic space

$CH^m(4c)$. He also established the basic inequality for a submanifold in a complex projective space CP^m.

A submanifold normal to the structure vector field ξ of a contact manifold is anti-invariant. Its simplest possible proof is given in [13].Thus C-totally real submanifolds in a Sasakian manifold are anti-invariant, as they are normal to ξ. An inequality similar to (2) for C-totally real submanifolds in a Sasakian space form $\tilde{M}(c)$ of constant φ-sectional curvature c is given in [8]. In [9], for submanifolds in a Sasakian space form $\tilde{M}(c)$ tangential to the structure vector field ξ, a basic inequality along with some applications are presented.

On the other hand, there is an interesting class of almost contact metric manifolds which are locally conformal to almost cosymplectic manifolds. These manifolds are called locally conformal almost cosymplectic manifolds (see [10,11]).

Thus motivated sufficiently, in this paper, we study submanifolds tangent to the structure vector field ξ in locally conformal almost cosymplectic manifolds of pointwise constant φ-sectional curvature. The paper is organized as follows. In §2, necessary details about submanifolds and locally conformal almost cosymplectic manifolds are reviewed. In §3, for submanifolds tangent to the structure vector field ξ in locally conformal almost cosymplectic manifolds of pointwise constant φ-sectional curvature, we establish a basic inequality between the main intrinsic invariants of the submanifold on one side, namely its sectional curvature function K and its scalar curvature function τ; and its main extrinsic invariant on the other side, namely its mean curvature function $\|H\|$. In the last section, we present some applications including inequalities between the intrinsic invariant δ_M and the extrinsic invariant $\|H\|$. The equality cases are also discussed.

2. Preliminaries

Let $\tilde{M}$ be a $(2m+1)$-dimensional almost contact manifold [2] endowed with an almost contact structure (φ, ξ, η), that is, φ is a $(1,1)$ tensor field, ξ is a vector field and η is 1-form such that $\varphi^2 = -I + \eta \otimes \xi$ and $\eta(\xi) = 1$. Then, $\varphi(\xi) = 0$ and $\eta \circ \varphi = 0$. The almost contact structure is said to be *normal* if the induced almost complex structure J on the product manifold $\tilde{M} \times \mathbb{R}$ defined by $J(X, \lambda \mathrm{d}/\mathrm{d}t) = (\varphi X - \lambda\xi, \eta(X)\,\mathrm{d}/\mathrm{d}t)$ is integrable, where X is tangent to $\tilde{M}$, t the coordinate of $\mathbb{R}$ and λ a smooth function on $\tilde{M} \times \mathbb{R}$. The condition for being normal is equivalent to vanishing of the torsion tensor $[\varphi, \varphi] + 2\mathrm{d}\eta \otimes \xi$, where $[\varphi, \varphi]$ is the Nijenhuis tensor of φ.

Let g be a compatible Riemannian metric with (φ, ξ, η), that is, $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$ or equivalently, $\Phi(X, Y) \equiv g(X, \varphi Y) = -g(\varphi X, Y)$ and $g(X, \xi) = \eta(X)$ for all $X, Y \in T\tilde{M}$. Then, $\tilde{M}$ becomes an almost contact metric manifold equipped with an almost contact metric structure (φ, ξ, η, g).

If the fundamental 2-form Φ and 1-form η are closed, then $\tilde{M}$ is said to be *almost cosymplectic manifold.* A normal almost cosymplectic manifold is *cosymplectic* [2]. $\tilde{M}$ is called a *locally conformal almost cosymplectic manifold* [11] if there exists a 1-form ω such that $\mathrm{d}\Phi = 2\omega \wedge \Phi$, $\mathrm{d}\eta = \omega \wedge \eta$ and $\mathrm{d}\omega = 0$.

A necessary and sufficient condition for a structure to be normal locally conformal almost cosymplectic is [10]

$$\left(\tilde{\nabla}_X \varphi\right) Y = f(g(\varphi X, Y)\xi - \eta(Y)\varphi X), \tag{3}$$

where $\tilde{\nabla}$ is the Levi–Civita connection of the Riemannian metric g and $\omega = f\eta$. From formula (3) it follows that

$$\tilde{\nabla}_X \xi = f(X - \eta(X)\xi). \tag{4}$$

A plane section σ in $T_p\tilde{M}$ of an almost contact metric manifold $\tilde{M}$ is called a *φ-section* if $\sigma \perp \xi$ and $\varphi(\sigma) = \sigma$. $\tilde{M}$ is of *pointwise constant φ-sectional curvature* if at each point $p \in \tilde{M}$, the sectional curvature $\tilde{K}(\sigma)$ does not depend on the choice of the φ-section σ of $T_p\tilde{M}$, and in this case for $p \in \tilde{M}$ and for any φ-section σ of $T_p\tilde{M}$, the function c defined by $c(p) = \tilde{K}(\sigma)$ is called the *φ-sectional curvature* of $\tilde{M}$. A locally conformal almost cosymplectic manifold $\tilde{M}$ of dimension ≥ 5 is of pointwise constant φ-sectional curvature if and only if its curvature tensor $\tilde{R}$ is of the form [11]

$$\begin{aligned}
\tilde{R}(X, Y, Z, W) = {} & \frac{c - 3f^2}{4}\{g(X, W)g(Y, Z) - g(X, Z)g(Y, W)\} \\
& + \frac{c + f^2}{4}\{g(X, \varphi W)g(Y, \varphi Z) - g(X, \varphi Z)g(Y, \varphi W) \\
& - 2g(X, \varphi Y)g(Z, \varphi W)\} \\
& - \left(\frac{c + f^2}{4} + f'\right)\{g(X, W)\eta(Y)\eta(Z) \\
& - g(X, Z)\eta(Y)\eta(W) \\
& + g(Y, Z)\eta(X)\eta(W) - g(Y, W)\eta(X)\eta(Z)\},
\end{aligned} \tag{5}$$

where f is the function such that $\omega = f\eta$, $f' = \xi f$; and c is the pointwise φ-sectional curvature of $\tilde{M}$.

Let M be an n-dimensional submanifold of a manifold $\tilde{M}$ equipped with a Riemannian metric g. The Gauss and Weingarten formulae are given respectively by $\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)$ and $\tilde{\nabla}_X N = -A_N X + \nabla_X^{\perp} N$ for all $X, Y \in TM$ and $N \in T^{\perp}M$, where $\tilde{\nabla}$, ∇ and $\nabla^{\perp}$ are respectively the Riemannian, induced Riemannian and induced normal connections in $\tilde{M}$, M and the normal bundle $T^{\perp}M$ of M respectively, and h is the second fundamental form related to the shape operator A by $g(h(X, Y), N) = g(A_N X, Y)$.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of the tangent space T_pM. The mean curvature vector $H(p)$ at $p \in M$ is

$$H(p) \equiv \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i). \tag{6}$$

The submanifold M is *totally geodesic* in $\tilde{M}$ if $h = 0$, and *minimal* if $H = 0$. We put

$$h_{ij}^r = g(h(e_i, e_j), e_r) \quad \text{and} \quad \|h\|^2 = \sum_{i,j=1}^{n} g(h(e_i, e_j), h(e_i, e_j)).$$

Now, we recall the following algebraic lemma for further use.

Lemma 2.1. [3] *If $a_1, \ldots, a_{n+1}, a$ are $n + 2$ $(n \geq 1)$ real numbers such that*

$$\left(\sum_{i=1}^{n+1} a_i\right)^2 = n\left(\sum_{i=1}^{n+1} a_i^2 + a\right),$$

then $2a_1a_2 \geq a$, with equality holding if and only if $a_1 + a_2 = a_3 = \cdots = a_{n+1}$.

3. A basic inequality

Let M be a submanifold of an almost contact metric manifold. For a vector field X on M, we put

$$\varphi X = PX + FX, \qquad PX \in TM,\ FX \in T^{\perp}M.$$

Thus, P is an endomorphism of the tangent bundle of M and satisfies $g(X, PY) = -g(PX, Y)$ for all $X, Y \in TM$. For a plane section $\pi \subset T_pM$ at a point $p \in M$,

$$\alpha(\pi) = g(e_1, Pe_2)^2 \qquad \text{and} \qquad \beta(\pi) = (\eta(e_1))^2 + (\eta(e_2))^2$$

are real numbers in the closed unit interval [0, 1], which are independent of the choice of the orthonormal basis $\{e_1, e_2\}$ of π. Moreover, if the structure vector field ξ is tangential to M, then we write the orthogonal direct decomposition $TM = \{\xi\} \oplus \{\xi\}^{\perp}$.

Now, we prove the following:

Theorem 3.1. *Let M be an $(n+1)$-dimensional $(n \geq 2)$ submanifold isometrically immersed in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that the structure vector field ξ is tangential to M. Then, for each point $p \in M$ and each plane section $\pi \subset T_pM$, we have*

$$\begin{aligned}\tau - K(\pi) \leq \frac{(n+1)^2(n-1)}{2n} \|H\|^2 - \frac{1}{2}(n+2)(n-1) f^2 - (n - \beta(\pi)) f' \\ + \frac{(c+f^2)}{8}(3\|P\|^2 - 6\alpha(\pi) + 2\beta(\pi) + (n+1)(n-2)).\end{aligned} \tag{7}$$

The equality in (7) *holds at $p \in M$ if and only if there exists an orthonormal basis $\{e_1, \ldots, e_{n+1}\}$ of T_pM and an orthonormal basis $\{e_{n+2}, \ldots, e_{2m+1}\}$ of $T_p^{\perp}M$ such that* (a) $\pi = \mathrm{Span}\{e_1, e_2\}$ *and* (b) *the forms of shape operators $A_r \equiv A_{e_r}$, $r = n+2, \ldots, 2m+1$, become*

$$A_{n+2} = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & (\lambda+\mu) I_{n-1} \end{pmatrix}, \tag{8}$$

$$A_r = \begin{pmatrix} h_{11}^r & h_{12}^r & 0 \\ h_{12}^r & -h_{11}^r & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix}, \qquad r = n+3, \ldots, 2m+1. \tag{9}$$

Proof. The Gauss equation for M,

$$\begin{aligned}\tilde{R}(X, Y, Z, W) = R(X, Y, Z, W) - g(h(X, W), h(Y, Z)) \\ + g(h(X, Z), h(Y, W))\end{aligned}$$

becomes

$$\begin{aligned} R(X,Y,Z,W) = {} & \frac{c-3f^2}{4}\{g(X,W)g(Y,Z)-g(X,Z)g(Y,W)\} \\ & + \frac{c+f^2}{4}\{g(X,PW)g(Y,PZ)-g(X,PZ)g(Y,PW) \\ & - 2g(X,PY)g(Z,PW)\} \\ & - \left(\frac{c+f^2}{4}+f'\right)\{g(X,W)\eta(Y)\eta(Z) \\ & - g(X,Z)\eta(Y)\eta(W) \\ & + g(Y,Z)\eta(X)\eta(W)-g(Y,W)\eta(X)\eta(Z)\} \\ & + g(h(X,W),h(Y,Z))-g(h(X,Z),h(Y,W)) \end{aligned} \tag{10}$$

for vector fields X, Y, Z and W tangent to M; and thus the scalar curvature and the mean curvature of M at p satisfy

$$\begin{aligned} 2\tau = {} & n(n+1)\left(\frac{c-3f^2}{4}\right)+\frac{3(c+f^2)}{4}\|P\|^2 \\ & - 2n\left(\frac{c+f^2}{4}+f'\right)+(n+1)^2\|H\|^2-\|h\|^2, \end{aligned} \tag{11}$$

where $\|P\|^2$ is defined by

$$\|P\|^2 = \sum_{i,j=1}^{n+1} g(e_i, Pe_j)^2$$

for any local orthonormal basis $\{e_1, e_2, \ldots, e_{n+1}\}$ for T_pM. We introduce

$$\begin{aligned} \rho = {} & 2\tau - \frac{(n+1)^2(n-1)}{n}\|H\|^2 - n(n+1)\left(\frac{c-3f^2}{4}\right) \\ & - \frac{3\left(c+f^2\right)}{4}\|P\|^2 + 2n\left(\frac{c+f^2}{4}+f'\right). \end{aligned} \tag{12}$$

From (11) and (12), we get

$$(n+1)^2\|H\|^2 = n(\|h\|^2+\rho). \tag{13}$$

Let p be a point of M and let $\pi \subset T_pM$ be a plane section at p. We choose an orthonormal basis $\{e_1, e_2, \ldots, e_{n+1}\}$ for T_pM and $\{e_{n+2}, \ldots, e_{2m+1}\}$ for the normal space $T_p^{\perp}M$ at p such that $\pi = \operatorname{Span}\{e_1, e_2\}$ and the mean curvature vector $H(p)$ is parallel to e_{n+2}, then from eq. (13) we get

$$\left(\sum_{i=1}^{n+1} h_{ii}^{n+2}\right)^2 = n\left(\sum_{i=1}^{n+1}(h_{ii}^{n+2})^2 + \sum_{i\neq j}(h_{ij}^{n+2})^2 + \sum_{r=n+3}^{2m+1}\sum_{i,j=1}^{n+1}(h_{ij}^r)^2 + \rho\right). \tag{14}$$

Using Lemma 2.1, from (14) we obtain

$$h_{11}^{n+2}h_{22}^{n+2} \geq \frac{1}{2}\left\{\sum_{i\neq j}(h_{ij}^{n+2})^2 + \sum_{r=n+3}^{2m+1}\sum_{i,j=1}^{n+1}(h_{ij}^r)^2 + \rho\right\}. \tag{15}$$

From (10), we also have

$$K(\pi) = \frac{c-3f^2}{4} + \frac{3\left(c+f^2\right)}{4}\alpha(\pi) - \left(\frac{c+f^2}{4} + f'\right)\beta(\pi)$$
$$+ h_{11}^{n+2}h_{22}^{n+2} - (h_{12}^{n+2})^2 + \sum_{r=n+3}^{2m+1}(h_{11}^r h_{22}^r - (h_{12}^r)^2). \tag{16}$$

Thus, we have

$$K(\pi) \geq \frac{c-3f^2}{4} + \frac{3(c+f^2)}{4}\alpha(\pi) - \left(\frac{c+f^2}{4} + f'\right)\beta(\pi) + \frac{1}{2}\rho$$
$$+ \sum_{r=n+2}^{2m+1}\sum_{j>2}\{(h_{1j}^r)^2 + (h_{2j}^r)^2\} + \frac{1}{2}\sum_{i\neq j>2}(h_{ij}^{n+2})^2$$
$$+ \frac{1}{2}\sum_{r=n+3}^{2m+1}\sum_{i,j>2}(h_{ij}^r)^2 + \frac{1}{2}\sum_{r=n+3}^{2m+1}(h_{11}^r + h_{22}^r)^2, \tag{17}$$

or

$$K(\pi) \geq \frac{c-3f^2}{4} + \frac{3(c+f^2)}{4}\alpha(\pi) - \left(\frac{c+f^2}{4} + f'\right)\beta(\pi) + \frac{1}{2}\rho. \tag{18}$$

In view of (12) and (18), we obtain (7).

If the equality in (7) holds, then the inequalities given by (15) and (17) become equalities. In this case, we have

$$h_{1j}^{n+2} = 0,\ h_{2j}^{n+2} = 0,\ h_{ij}^{n+2} = 0, \quad i \neq j > 2;$$
$$h_{1j}^r = h_{2j}^r = h_{ij}^r = 0,\ r = n+3, \ldots, 2m+1;\ i, j = 3, \ldots, n+1;$$
$$h_{11}^{n+3} + h_{22}^{n+3} = \cdots = h_{11}^{2m+1} + h_{22}^{2m+1} = 0. \tag{19}$$

Furthermore, we may choose e_1 and e_2 so that $h_{12}^{n+2} = 0$. Moreover, by applying Lemma 2.1, we also have

$$h_{11}^{n+2} + h_{22}^{n+2} = h_{33}^{n+2} = \cdots = h_{n+1\ n+1}^{n+2}. \tag{20}$$

Thus, choosing a suitable orthonormal basis $\{e_1, \ldots, e_{2m+1}\}$, the shape operator of M becomes of the form given by (8) and (9). The converse is straightforward. ∎

4. Some applications

A submanifold M of an almost contact metric manifold $\tilde{M}$ with $\xi \in TM$ is called a *semi-invariant submanifold* [1] of $\tilde{M}$ if $TM = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \{\xi\}$, where $\mathcal{D} = TM \cap \varphi(TM)$

and $\mathcal{D}^{\perp} = TM \cap \varphi(T^{\perp}M)$. In fact, the condition $TM = \mathcal{D} \oplus \mathcal{D}^{\perp} \oplus \{\xi\}$ implies that the endomorphism P is an *f-structure* [14] on M with rank $(P) = \dim(\mathcal{D})$. A semi-invariant submanifold of an almost contact metric manifold becomes an *invariant* or *anti-invariant submanifold* according as the anti-invariant distribution $\mathcal{D}^{\perp}$ is {0} or invariant distribution $\mathcal{D}$ is {0} [1,14].

Theorem 4.1. *Let M be an $(n+1)$-dimensional $(n > 1)$ submanifold isometrically immersed in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that the structure vector field ξ is tangential to M. If $c < -f^2$, then*

$$\delta_M \leq \frac{(n+1)^2 (n-1)}{2n} \|H\|^2 - \frac{1}{2}(n+2)(n-1) f^2 - nf' + \frac{1}{2}(n+1)(n-2)\frac{(c+f^2)}{4}. \tag{21}$$

The equality in (21) *holds if and only if M is a semi-invariant submanifold with* rank $(P) = 2$ *and* $\beta(\pi) = 0$.

Proof. Since $c < -f^2$, in order to estimate δ_M, we minimize $3\|P\|^2 - 6\alpha(\pi) + 2\beta(\pi)$ in (7). For an orthonormal basis $\{e_1, \dots, e_{n+1}\}$ of T_pM with $\pi = \mathrm{Span}\{e_1, e_2\}$, we write

$$\|P\|^2 - 2\alpha(\pi) = \sum_{i,j=3}^{n+1} g(e_i, \varphi e_j)^2 + 2\sum_{j=3}^{n+1}\{(g(e_1, \varphi e_j)^2 + g(e_2, \varphi e_j)^2\}.$$

Thus, we see that the minimum value of $3\|P\|^2 - 6\alpha(\pi) + 2\beta(\pi)$ is zero, provided $\pi = \mathrm{Span}\{e_1, e_2\}$ is orthogonal to ξ and $\mathrm{Span}\{\varphi e_j \mid j = 3, \dots, n\}$ is orthogonal to the tangent space T_pM. Thus, we have (21) with equality case holding if and only if M is semi-invariant such that $\mathrm{rank}(P) = 2$ with $\beta = 0$. ∎

Theorem 4.2. *Let M be an $(n+1)$-dimensional $(n > 1)$ submanifold isometrically immersed in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature c such that $\xi \in TM$. If $c > -f^2$, then*

$$\delta_M \leq \frac{(n+1)^2 (n-1)}{2n} \|H\|^2 - \frac{1}{2}(n+2)(n-1) f^2 - (n-1) f' + \frac{1}{2}n(n+2)\frac{\left(c+f^2\right)}{4}. \tag{22}$$

The equality in (22) *holds if and only if M is an invariant submanifold and $\beta = 1$.*

Proof. Since $c > -f^2$, in order to estimate δ_M, we maximize $3\|P\|^2 - 6\alpha(\pi) + 2\beta(\pi)$ in (7). We observe that the maximum of $3\|P\|^2 - 6\alpha(\pi) + 2\beta(\pi)$ is attained for $\|P\|^2 = n$, $\alpha(\pi) = 0$ and $\beta(\pi) = 1$, that is, M is invariant and $\xi \in \pi$. Thus, we obtain (22) with equality case if and only if M is invariant with $\beta = 1$. ∎

Theorem 4.3. *Let M be an $(n+1)$-dimensional $(n>1)$ submanifold isometrically immersed in a $(2m+1)$-dimensional normal locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature $c>-f^2$ such that $\xi \in TM$ and*

$$\delta_M = \frac{(n+1)^2(n-1)}{2n}\|H\|^2 - \frac{1}{2}(n+2)(n-1)f^2 - (n-1)f' + \frac{1}{2}n(n+2)\frac{(c+f^2)}{4}.$$

Then, M is a totally geodesic locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c.

Proof. In view of Theorem 4.2, M is an odd-dimensional invariant submanifold of the almost cosymplectic manifold $\tilde{M}(c)$. For every point $p \in M$, we can choose an orthonormal basis $\{e_1 = \xi, e_2, \dots, e_{n+1}\}$ for T_pM and $\{e_{n+2}, \dots, e_{2m+1}\}$ for $T_p^{\perp}M$ such that A_r $(r = n+2, \dots, 2m+1)$ take the forms (8) and (9). Since M is an invariant submanifold of a normal locally conformal almost cosymplectic manifold, it is minimal and $A_r\varphi + \varphi A_r = 0$, $r = n+2, \dots, 2m+1$ [12]. Thus all the shape operators take the form

$$A_r = \begin{pmatrix} c_r & d_r & 0 \\ d_r & -c_r & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix}, \qquad r = n+2, \dots, 2m+1. \tag{23}$$

Since, $A_r\varphi e_1 = 0$, $r = n+2, \dots, 2m+1$, from $A_r\varphi + \varphi A_r = 0$, we get $\varphi A_r e_1 = 0$. Applying φ to this equation, we obtain $A_r e_1 = \eta(A_r e_1)\xi = \eta(A_r e_1)e_1$; and thus $d_r = 0$, $r = n+2, \dots, 2m+1$. This implies that $A_r e_2 = -c_r e_2$. Applying φ to both sides, in view of $A_r\varphi + \varphi A_r = 0$, we get $A_r\varphi e_2 = c_r\varphi e_2$. Since φe_2 is orthogonal to ξ and e_2 and φ has maximal rank, the principal curvature c_r is zero. Hence, M becomes totally geodesic. As in Proposition 1.3 on p. 313 of [14], it is easy to show that M is a locally conformal almost cosymplectic manifold of pointwise constant φ-sectional curvature c. ■

Now for the case $c = -f^2$, we have the following pinching result.

COROLLARY 4.4

Let M be an $(n+1)$-dimensional $(n>1)$ submanifold isometrically immersed in a $(2m+1)$-dimensional locally conformal almost cosymplectic manifold $\tilde{M}(c)$ of pointwise constant φ-sectional curvature $c=-f^2$ such that $\xi \in TM$. Then, we have

$$\delta_M \le \frac{(n+1)^2(n-1)}{2n}\|H\|^2 - \frac{1}{2}(n+2)(n-1)f^2 - (n-1)f', \quad f' > 0,$$

$$\delta_M \le \frac{(n+1)^2(n-1)}{2n}\|H\|^2 - \frac{1}{2}(n+2)(n-1)f^2 - nf', \quad f' < 0.$$

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