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A geometric characterization of arithmetic varieties

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Abstract. A result of Belyi can be stated as follows. Every curve defined over a number field can be expressed as a cover of the projective line with branch locus contained in a rigid divisor. We define the notion of geometrically rigid divisors in surfaces and then show that every surface defined over a number field can be expressed as a cover of the projective plane with branch locus contained in a geometrically rigid divisor in the plane. The main result is the characterization of arithmetically defined divisors in the plane as geometrically rigid divisors in the plane.

Keywords. Equisingular; geometrically rigid.

1. Introduction

This paper is an attempt to generalize a result of Belyi (see [1]).

Theorem (Belyi). *Let C be a smooth projective curve over an algebraic number field and T a finite set of closed points in C. There is a finite morphism $f : C \to \mathbb{P}^1$ so that the image $f(T)$ and the branch locus of f are contained in the set of three points* $\{0, 1, \infty\}$.

We note that this gives a completely geometric characterization of algebraic curves over number fields, since any deformation of a triple of points in $\mathbb{P}^1$ is in fact trivialized by an automorphism of $\mathbb{P}^1$.

A naive generalization of this could require a surface over a number field to be expressible as a cover of $\mathbb{P}^2$ that is étale outside four general lines; however, as Kollár pointed out, this fails since the fundamental group of the complement of four general lines in $\mathbb{P}^2$ is abelian, whereas many surfaces have non-abelian fundamental groups. Thus one needs to look at more general divisors in $\mathbb{P}^2$. The problem is that these divisors have non-trivial flat deformations. We need to find an algebraic notion that restricts the possible deformations. Thus, in §1 we define the notion of *geometrically rigid* divisors on a surface.

Let C be any collection of 4 or less lines in general position in $\mathbb{P}^2$. From the definitions in §1 it follows easily that, C is geometrically rigid. Moreover, it is equally clear that collections of five or more lines in general position in $\mathbb{P}^2$ are *not* geometrically rigid. Geometrically rigid divisors in $\mathbb{P}^2$ (and hence their singular loci) are defined over $\overline{\mathbb{Q}}$ (see Lemma 7):

Theorem 1. *Let C be any divisor in $\mathbb{P}^2$ defined over $\mathbb{C}$ which is geometrically rigid. There is an automorphism g of $\mathbb{P}^2$ so that $g(C)$ is defined over $\overline{\mathbb{Q}}$.*

Now, if C is a curve of degree 1 or 2 in $\mathbb{P}^2$, then C is geometrically rigid but a general curve of degree 3 or more is not. In spite of this we will see that there are many geometrically rigid divisors in $\mathbb{P}^2$. In fact (see the end of §3),

Theorem 2. *Let C be any divisor in $\mathbb{P}^2$ defined over $\overline{\mathbb{Q}}$, and T be a finite set of points in $\mathbb{P}^2$ defined over $\overline{\mathbb{Q}}$. There is a geometrically rigid divisor D in $\mathbb{P}^2$ so that $C \subset D$ and T is contained in the singular locus of D.*

These results give a geometric characterization of reduced algebraic subschemes of $\mathbb{P}^2$ that are defined over $\overline{\mathbb{Q}}$. As an easy corollary we have a generalization of Belyi's characterization to the case of surfaces.

COROLLARY 3

Let S be a smooth projective surface, C a divisor in S and T a finite set of points in S.

Assume that S, C and T are defined over $\overline{\mathbb{Q}}$, then there is a geometrically rigid divisor D in $\mathbb{P}^2$ and a finite morphism $f : S \to \mathbb{P}^2$ so that the image of C and the branch locus of f are contained in D; moreover, the image of T is contained in the singular locus of D.

Conversely, suppose there is a tuple (S, C, T) as above over $\mathbb{C}$ and a finite morphism $f : S \to \mathbb{P}^2$ so that the image of C and the branch locus of f are sub-divisors of a geometrically rigid divisor D and the image of T is contained in the singular locus of D. Then the tuple (S, C, T) is isomorphic to (the base-change to $\mathbb{C}$ of) a tuple (S_0, C_0, T_0) which is defined over $\overline{\mathbb{Q}}$.

It is reasonably clear that these results should be extendable *mutatis mutandi* to higher dimension.

2. Geometricrigidity

Throughout the paper we work with schemes of finite type over a field of characteristic zero.

Let A be a smooth family of divisors in a smooth surface S; in other words let $\mathcal{C} \subset \mathcal{S} = A \times S$ be a divisor with A smooth. More generally, we can consider the case of non-constant ambient spaces by only assuming that $\mathcal{S} \to A$ is a smooth projective morphism. We are interested in *topologically trivial* families $p : (\mathcal{S}, \mathcal{C}) \to A$. Over the field of complex numbers this can be characterized by saying that any point $a \in A$ has an analytic neighborhood U so that the pair $(U \times S, p^{-1}U)$ is homeomorphic over U to $U \times (S, p^{-1}(a))$. The geometric notion of *equisingular* families results in topologically trivial families.

Remark 1. The notion of equisingularity was first defined and studied by Zariski in a series of papers [2,3]. Theorem 7.4 in [3] proves the equivalence of his definition with that studied here. Alternatively, one can directly prove Lemmas 4, 6 and 7 using his definitions. We require a specialized application of Zariski's results which we develop in this section.

A special case is that of a *family of divisors with normal crossings* which is characterized by the following properties:

1. The divisor $\mathcal{C}$ is a divisor with normal crossings in $\mathcal{S}$.
2. Each component of $\mathcal{C}$ is smooth over A.
3. The critical locus of $\mathcal{C} \to A$ is étale over A.

In particular, each component of the critical locus of $\mathcal{C} \to A$ meets and is contained in exactly two components of $\mathcal{C}$.

Now let $\mathcal{S}_n \to \cdots \to \mathcal{S}_0 = \mathcal{S}$ be a sequence of blow-ups with irreducible reduced centres $A_k \subset \mathcal{S}_k$ such that $A_k \to A$ is finite étale. Moreover, let $\mathcal{C}_k$ denote the reduced union of the total transform of $\mathcal{C}$ in $\mathcal{S}_k$ and the exceptional locus of $\mathcal{S}_k \to \mathcal{S}$. We further assume that either,

1. A_k is contained in the critical locus of $\mathcal{C}_k \to A$ or,
2. A_k is contained in $\mathcal{C}_k$ but misses the critical locus of $\mathcal{C}_k \to A$ entirely or,
3. A_k lies in the complement of $\mathcal{C}_k$ in $\mathcal{S}_k$.

While the latter two are irrelevant to the desingularization it is useful to allow these to simplify the proofs. If $\mathcal{C}_n$ is family of divisors with normal crossings, then we call such a sequence of blow-ups a *simultaneous desingularization* of the family of divisors $\mathcal{C} \to S$. If such a sequence of blow-ups exists then we say that the family is *simultaneously desingularizable* or *equisingular*. In order to understand how one arrives at this definition we state

Lemma 4. *Fix a ground field k of characteristic zero. Let $\mathcal{S} \to A$ be a smooth family of projective surfaces of a reduced scheme A. Let $\mathcal{C} \subset \mathcal{S}$ be a reduced divisor. There is an open dense subset U of A over which $\mathcal{C}$ is an equisingular family.*

Proof. We can replace A by its smooth locus and further operate on each component individually; thus we can assume that A is smooth and irreducible. Now, consider the reduced critical locus of $\mathcal{C} \to A$. This is a closed subscheme B of $\mathcal{C}$ which is generically finite over A. Thus the locus where $B \to A$ is not étale is a proper closed subscheme of A. We can replace A by the complement of this closed subscheme. Now we can take $A_1 = B$ and perform a blow-up of $\mathcal{S}$ along A_1 to obtain $\mathcal{S}_1$. Since A_1 is étale over A the resulting family $\mathcal{S}_1 \to A$ is smooth. Let $\mathcal{C}_1$ denote the (reduced) union of the strict transform of $\mathcal{C}$ in $\mathcal{S}_1$ and the exceptional locus of the blow-up. We can now inductively construct the sequence $\mathcal{S}_n$ as above. By the embedded desingularization of curves in characteristic zero, there is an n so that the generic fibre of $\mathcal{C}_n \to A$ is a divisor with normal crossings; i.e. each irreducible component (not geometrically irreducible component) of this generic fibre is smooth over the function field of A and at most two of them meet at any singular point (which is closed over the function field of A) and this meeting is transversal. Now replace A by the open subset where the critical locus of $\mathcal{C}_n \to A$ is étale and each component of $\mathcal{C}_n \to A$ is smooth. It follows that $\mathcal{C}_n \to A$ is a family of divisors with normal crossings in $\mathcal{S}_n \to A$. □

One point that is important from our perspective is the fact that U is defined over k since all schemes are of finite type over k. We also note the following lemma.

Lemma 5. *Let B_k be the image of the critical locus B_n of $\mathcal{C}_n \to A$ in $\mathcal{S}_k$ for each k. Then $B_n \to B_k$ and $B_k \to A$ are étale. Any component of B_k that meets A_k is actually A_k. Let $\mathcal{D}_k$ be a union of components of $\mathcal{C}_k$. If $\mathcal{D}_k$ and a component of B_k meet then the latter is contained in the former. Finally, the critical locus of $\mathcal{D}_k \to A$ is a union of components of B_k.*

Proof. We prove the statements by downward induction on k; we start at $k = n$ where this is true by the definition of a family of divisors with normal crossings. Now suppose that the result is proved for B_{k+1} and for all divisors of the form $\mathcal{D}_{k+1}$. Let E_k be the exceptional locus of $\mathcal{S}_{k+1} \to \mathcal{S}_k$. Then E_k is contained in $\mathcal{C}_{k+1}$ by the definition of $\mathcal{C}_{k+1}$. The map $E_k \to A$ factors through $A_k \to A$ which is étale.

Let Y be the union of those connected components of B_k which meet A_k; in particular, this includes those components which contain points where $B_{k+1} \to B_k$ is not an isomorphism. Let X be the inverse image of Y in B_{k+1}; by the induction hypothesis $X \to A$ is étale. Moreover, each component of X meets E_k. By choosing $\mathcal{D}_{k+1} = E_k$ we see that X is contained in E_k by the induction hypothesis. Thus, the morphism $X \to A$ is étale and factors through $A_k \to A$. It follows that $Y = A_k$. Thus B_k is the disjoint union of A_k and components disjoint from A_k. The remaining components descend isomorphically from components of B_{k+1} and $B_{k+1} \to A$ is étale by induction. Hence $B_k \to A$ is étale.

Let $\mathcal{D}_k$ be a union of irreducible components of $\mathcal{C}_k$ and suppose that $\mathcal{D}_k$ meets A_k. Let $\mathcal{D}_{k+1}$ be its strict transform in $\mathcal{S}_{k+1}$. Then $\mathcal{D}_{k+1}$ must meet E_k; let Z be any component of $\mathcal{D}_{k+1} \cap E_k$. This is a divisor in E_k which is contained in the critical locus of $\mathcal{D}_{k+1} \cup E_k \to A$. By the induction hypothesis applied to $\mathcal{D}_{k+1} \cup E_k$ we see that Z is a component of B_{k+1}. Hence, $Z \to A$ is étale by induction, and the image of Z is A_k as above. Thus $\mathcal{D}_k$ contains A_k.

Finally, any critical point p of $\mathcal{D}_k \to A$ which is not the image of a critical point of $\mathcal{D}_{k+1} \to A$, would have to lie in A_k. Either (a) there are two points q and q' that lie in $\mathcal{D}_{k+1} \cap E_k$ over p, or (b) there is a point q in $\mathcal{D}_{k+1} \cap E_k$ where this interesection is not transvesal. In case (a), let Z and Z' be the components of $\mathcal{D}_{k+1} \cap E_k$ that contain q and q' respectively ($Z = Z'$ is a possibility). Then $Z \to A_k$ and $Z' \to A_k$ are étale as explained above. In particular, $\mathcal{D}_k \to A$ has critical points along A_k. In case (b), let Z be the component of $\mathcal{D}_{k+1} \cap E_k$ that contains q. The map $Z \to A_k$ is étale as above, hence Z is smooth. Thus the intersection of $\mathcal{D}_{k+1}$ and E_k is non-transversal everywhere along Z. Thus, in this case A_k is contained in the critical locus of $\mathcal{D}_k \to A$ again. Any critical point of $\mathcal{D}_k \to A$ is thus either contained in A_k which is contained in this critical locus or contained in the image of the critical locus of $\mathcal{D}_{k+1} \to A$ which is a union of components of B_k. Since A_k is contained in B_k in both cases (a) and (b), it follows that the critical locus of $\mathcal{D}_k \to A$ is a union of components of B_k. □

In particular, note that this means that A_k is a connected component of the critical locus of $\mathcal{C}_k \to A$ if it meets this locus; this strengthens condition (1) in the definition above. The fundamental lemma that we will use in our constructions is a corollary of the above lemma.

Lemma 6. *Let* $(\mathcal{S}, \mathcal{C}) \to A$ *be an equisingular family of divisors in a family of smooth projective surfaces over a smooth variety* A. *Let* $\mathcal{D} \subset \mathcal{C}$ *be a union of components of* $\mathcal{C}$, *then* $(\mathcal{S}, \mathcal{D}) \to A$ *is an equisingular family of divisors.*

Proof. Let $\mathcal{S}_n \to \cdots \to \mathcal{S}_0 = \mathcal{S}$ be a simultaneous desingularization of $\mathcal{C}$ as above. Let $\mathcal{D}_k$ be the reduced total transform of $\mathcal{D}$ in $\mathcal{S}_k$. Since $\mathcal{D}_n$ is a union of components of $\mathcal{C}_n$, it too is a relative divisor with normal crossings over A. By the above lemma we see that whenever $\mathcal{D}_k \to A$ has a critical point on A_k, then A_k is contained in this critical locus. Moreover, if $\mathcal{D}_k$ meets A_k then it contains it. Thus the given sequence of blow-ups is a simultaneous desingularization of $\mathcal{D}_k$. □

Let $C \subset S$ be a divisor. Let G be an algebraic group of automorphisms of S. Given a morphism $A \to G$, we can construct an equisingular family containing C as follows. Let $m : A \times S \to S$ denote the action of A on S and let $\mathcal{C} = m^{-1}(C)$. More generally, we say that a family $\mathcal{C} \subset A \times S$ is G *iso-trivial*, if it is associated with a G-torsor on S. In other words, each point $a \in A$ has an étale neighborhood $B \to A$ so that $\mathcal{C}_B = \mathcal{C} \times_A B$ is isomorphic over B to $m_B^{-1}(C_a)$ for some morphism $m_B : B \to G$. Any iso-trivial family is clearly equisingular.

We now define C to be a *geometrically rigid* divisor in S if this is the only way to construct equisingular deformations of C; i.e. for any equisingular family $\mathcal{C} \subset A \times S$ parametrized by a smooth connected variety A so that C is the fibre $p^{-1}(a)$ for some point a in A, there is an algebraic group G of automorphisms of S so that the family $\mathcal{C} \to A$ is G iso-trivial.

The following lemma follows easily from the construction of universal deformations of divisors and the flattening stratification.

Lemma 7. *Let S be smooth surface over an algebraically closed field k and C be a geometrically rigid divisor in S defined over an algebraically closed extension K of k. Then there is an automorphism g of S over K, so that $g(C)$ is the base change to K of a curve C_0 in S which is defined over k.*

As a consequence, geometric rigidity is a sufficient criterion to reduce the field of definition.

Proof. Let H be the Hilbert scheme of divisors in S over k. Let A be the closure of the (non-closed) point of H which corresponds to C. Then A is a scheme of finite type over k to which we can apply Lemma 4 above. Thus replacing A by an open subscheme U defined over k we have an equisingular family $\mathcal{C} \to A$ in $S \times A$ with generic fibre isomorphic to the given C.

By the geometric rigidity of S it follows that this family is isotrivial for some algebraic group G of automorphisms of S. Thus there is a finite étale cover $A' \to A$ so that the family is group-theoretically trivial over A'. Since k is algebraically closed there is a k-valued point of A'. The fibre of $\mathcal{C}$ at this point is then a 'model' of (S, C) which is defined over k. $\square$

In particular, we note that Theorem 1 follows.

3. Constructions

We now give inductive constructions of geometrically rigid divisors to prove Theorem 2.

Lemma 8. *Let D be a geometrically rigid divisor in $\mathbb{P}^2$ and let p, q be singular points of D. The divisor $D \cup \overline{pq}$ is geometrically rigid, where $\overline{pq}$ is the line joining the points p and q.*

Proof. Let $\mathcal{C} \to A$ be an equisingular deformation of $D \cup \overline{pq}$. We wish to construct a group-theoretic trivialization of this deformation over a finite étale cover of A.

Let $A_1 \to A$ (respectively $A_2 \to A$) be a component of the critical locus of $\mathcal{C} \to A$ which contains p (respectively contains q). These are étale covers of A by Lemma 5. Let $B \to A$ be a connected étale cover of A that dominates both covers; we have natural morphisms $P : B \to \mathbb{P}^2$ and $Q : B \to \mathbb{P}^2$ passing through p and q respectively. Let $\mathcal{L} \to B$ be the component of $\mathcal{C}_B = \mathcal{C} \times_A B$, that contains $\overline{pq}$. Then, the fibre of L over $b \in B$ consists of the line joining $P(b)$ and $Q(b)$. Let $\mathcal{D}_B$ be the union of the remaining components of $\mathcal{C}_B$. By Lemma 6, the familiy $\mathcal{D}_B \to B$ is an equisingular deformation of D.

Now, by the geometric rigidity of D, we see that $\mathcal{D}_B \to B$ is iso-trivial. In particular, we take a further étale cover (which we also denote by B by abuse of notation) so that the family $\mathcal{D}_B$ is group-theoretic. Now, $P(B)$ and $Q(B)$ continue to be part of the critical locus of $\mathcal{D}_B \to B$, thus by the connectedness of B the trivialization of the family must take them to $B \times \{p\}$ and $B \times \{q\}$ respectively. But then the same trivialization also takes L to $B \times \overline{pq}$. Thus we have a group-theoretic trivialization of $\mathcal{C}_B$. □

Starting with the geometrically rigid divisor Q of 4 lines in general position on $\mathbb{P}^2$, we look at all the divisors obtained by iterated application of the above lemma. The usual constructions of projective geometry that give the field operations for points on a line give the following result.

PROPOSITION 9

Let T be any finite set of points in $\mathbb{P}^2$ defined over $\mathbb{Q}$. There is a geometrically rigid divisor D consisting of lines so that T is contained in the singular locus of D.

Proof. Fixing the reference quadrilateral Q consisting of four general lines in $\mathbb{P}^2$ also fixes a coordinate system so that the lines are given by $X = 0, Y = 0, Z = 0$ and $X+Y+Z = 0$. The singular points of the quadrilateral are $(1 : 0 : 0)$, $(0 : 1 : 0)$, $(0 : 0 : 1)$, $(1 : -1 : 0)$, $(1 : 0 : -1)$ and $(0 : 1 : -1)$.

For any $t \in \mathbb{P}^2(\mathbb{Q})$ and a geometrically rigid divisor C_0 containing Q we will construct a larger geometrically rigid divisor that contains t. We can then construct D by starting with Q and successively adding each point of the finite set T.

Thus we can assume that T consists of just one point. Since at least one coordinate of t is non-zero we can assume that it takes the form $(u : v : 1)$ in these coordinates for some rational numbers u and v.

Now, suppose that we can add to C_0 and produce a geometrically rigid divisor C so that the singular locus of C contains $(u : 0 : 1)$ and $(0 : v : 1)$. We can then add to C the line L joining $(u : 0 : 1)$ and $(0 : 1 : 0)$, and the line M joining $(0 : v : 1)$ and $(1 : 0 : 0)$, again producing a geometrically rigid divisor $C \cup L \cup M$ by Lemma 8. Now the point t is the intersection point of L and M so it is a singular point of this divisor as required.

Similarly, if we can add to C_0 to produce a geometrically rigid divisor C containing $(v : 0 : 1)$ in its singular locus then the divisor $C \cup L$ is also geometrically rigid, where L is the line joining $(v : 0 : 1)$ and $(1 : -1 : 0)$. The point $(0 : v : 1)$ which is the point of intersection of L and the line $X = 0$, is a singular point of this divisor. Thus to prove the result, it is enough to construct for each rational number u a divisor C_u containing C_0 so that the point $(u : 0 : 1)$ is in the singular locus of C_u.

We write $u = p/q$, where q is a positive integer and p is some integer. Suppose we can construct a divisor C containing C_0 so that $(0 : p : 1)$ and $(0 : -q : 1)$ are singular points of C. Let L be the line joining $(1 : 0 : -1)$ and $(0 : -q : 1)$; as before $C \cup L$ is a geometrically rigid divisor. Moreover, $(1 : -q : 0)$ is a singular point of this divisor as it lies on L and the line $Z = 0$. Let M be the line joining $(0 : p : 1)$ and $(1 : -q : 0)$; as before the divisor $C \cup L \cup M$ is geometrically rigid. The point $(p/q : 0 : 1)$ is a singular point of this divisor as it lies on M and the line $Y = 0$.

Thus we have finally reduced to the problem of constructing for each integer p a geometrically rigid divisor C_p containing C_0 for which $(0 : p : 1)$ is a singular point. We will do this by induction on the absolute value of p. Let L_1 be the line joining $(0 : 1 : 0)$ and $(-1 : 0 : 1)$, L_2 the line joining $(-1 : 1 : 0)$ and $(0 : 0 : 1)$. By Lemma 8 the divisor $Q \cup L_1 \cup L_2$ is geometrically rigid. The point $(-1 : 1 : 1)$ is the intersection point of L_1 and L_2, hence it is a singular point of this divisor. Let M be the line joining this point to $(1 : 0 : 0)$. Then $Q \cup L_1 \cup L_2 \cup M$ is geometrically rigid. The point $(0 : 1 : 1)$ is the intersection point of M and the line $X = 0$. Thus we have produced C_p for every p less than 1 in absolute value.

Now, suppose that we have constructed a divisor C containing C_0 which has $(0 : p : 1)$ and $(0 : q : 1)$ in its singular locus. Let L_1 be the line joining $(0 : p : 1)$ with $(1 : 0 : 0)$ and L_2 be the line joining $(-1 : 0 : 1)$ with $(0 : 1 : 0)$. The point $(-1 : p : 1)$ is a singular point of the geometrically rigid divisor $C \cup L_1 \cup L_2$. Let M_1 be the line joining $(0 : 0 : 1)$ and $(-1 : p : 1)$; the point $(-1 : p : 0)$ is a singular point of the geometrically rigid divisor $C \cup L_1 \cup L_2 \cup M_1$. Let M_2 be the line joining $(-1 : p : 0)$ and $(0 : q : 1)$; then $(-1 : p+q : 1)$ is a singular point of the geometrically rigid divisor $C \cup L_1 \cup L_2 \cup M_1 \cup M_2$. Finally, we add the line M_3 joining $(1 : 0 : 0)$ and $(-1 : p+q : 1)$ to obtain a geometrically rigid divisor which has $(0 : p + q : 1)$ as a singular point.

For every $p > 1$ we apply the latter construction to the divisor $C_{p-1} \cup C_1$ each of which we have already constructed by the induction hypothesis and has singular points at $(0 : p - 1 : 1)$ and $(0 : 1 : 1)$. For $p < 1$ we apply the latter construction to C_{p+1} which has $(0 : p + 1 : 1)$ and $(0 : -1 : 1)$ as singular points. This provides the required construction and hence the result is proved. □

To construct points with coordinates in algebraic number fields we need to have curves of degree greater than one in geometrically rigid divisors.

Lemma 10. *Let D be a geometrically rigid divisor on a rational surface S and let T be a finite subset of the singular points of D. Let L be a divisor class on S so that the linear system $| L - T |$ has a unique element E. Then the divisor $D \cup E$ is geometrically rigid.*

Actually, we only need the regularity of S (i.e. $H^1(S, \mathcal{O}_S) = 0$) in the proof given below. Further generalizations even for irregular surfaces are possible.

Proof. Let $\mathcal{C} \to A$ be an equisingular deformation of the divisor $C = D \cup E$. Let B be the connected component of the critical locus of $\mathcal{C} \to A$ that contains T. This is finite étale over A by Lemma 5. By base change we may assume that $B \to A$ is an isomorphism. Thus, we can write $\mathcal{C} = \mathcal{D} \cup \mathcal{E}$ where $\mathcal{D}$ is the union of irreducible components of $\mathcal{C}$ that meet D and $\mathcal{E}$ is the union of the irreducible components of $\mathcal{C}$ that meet E. By Lemma 6, $\mathcal{D} \to A$ is an equisingular deformation of D. Thus by base change we have a group-theoretic trivialization of $\mathcal{D}$. Since B is contained in the critical locus of $\mathcal{D} \to A$, it is mapped into T by the trivialization. Thus, after applying this trivialization, $\mathcal{E} \to A$ becomes a family of divisors containing T.

Now, the divisor class L has no deformation since S is rational. Thus, the divisor class of every fibre of $\mathcal{E} \to A$ is in the class L. By assumption, E is a unique class containing T, thus $\mathcal{E} \to A$ is the trivial family. Hence the trivialization for $\mathcal{D} \to A$ in fact gives a trivialization of $\mathcal{E}$ and $\mathcal{C}$ as well. □

The above lemma allows us to apply the Lagrange interpolation formula to prove the following proposition.

PROPOSITION 11

Let T be a finite set of algebraic points on $\mathbb{P}^2$, then there is a geometrically rigid divisor D so that T is contained in the singular locus of D.

Proof. As in the proof of Proposition 9, given a geometrically rigid divisor C_0 which contains the reference quadrilateral Q and a point $t \in \mathbb{P}^2(\overline{\mathbb{Q}})$, we construct a larger divisor

$C \supset C_0$ so that t is in the singular locus of t. Since T is a finite set we can inductively add all the points $t \in T$ to obtain the required divisor D. Thus we can assume that T consists of one point t.

Again, as in the proof of Proposition 9 we can further reduce to the case where the point has the form $(u : 0 : 1)$ where u is an algebraic number. Let $f(T)$ be a monic polynomial with rational coefficients for which $f(u) = 0$; let n be the degree of f. Let F be the set of points $(k : f(k) : 1)$ for $k = 0, \ldots, n^2$. The curve E defined by $YZ^{n-1} = f(X/Z)Z^n$ passes through these $n^2 + 1$ points. Thus it is the unique curve of degree n that passes through these points. Let C be a divisor (containing the quadrilateral Q) constructed using Proposition 9 which contains F in its singular locus. Lemma 10 then asserts that $D = C \cup E$ is geometrically rigid. The point $(u : 0 : 1)$ is a point of intersection of E and the line $Y = 0$ which lies in Q; hence it is a singular point of D. □

Finally, any curve of degree n defined over $\overline{\mathbb{Q}}$ is uniquely determined in its divisor class by $n^2 + 1$ distinct $\overline{\mathbb{Q}}$-valued points on it.

Proof (of Theorem 2). Let C be any curve of degree n in $\mathbb{P}^2$ which is defined over $\overline{\mathbb{Q}}$. Let T be a collection of $n^2 + 1$ distinct points on this curve over $\overline{\mathbb{Q}}$. Let D be a geometrically rigid divisor in $\mathbb{P}^2$ that contains T in its singular locus. By Lemma 10 the divisor $D \cup C$ is geometrically rigid. Applying this argument to each component of a given divisor in $\mathbb{P}^2$ defined over $\overline{\mathbb{Q}}$, we have the result. □

4. Remarks and open problems

A similar collection of arguments can be used to obtain geometrically rigid configurations in $\mathbb{P}^n$ for $n \geq 3$. Projection arguments can be used to define the notion of equisingular deformations in higher (co-)dimensions. Arguments similar to the ones in the previous section can then be probably used to show:

Problem 1. For each k between 0 and $n - 1$, let T_k be a closed subscheme of $\mathbb{P}^n$ of pure dimension k that is defined over $\overline{\mathbb{Q}}$. Then there is a geometrically rigid divisor S_{n-1} in $\mathbb{P}^n$ so that if S_k is defined inductively as the singular locus of S_{k+1}, then S_k has pure dimension k and $T_k \subset S_k$.

Another possible generalization of Belyi's theorem is the following:

Problem 2. If C is a projective algebraic curve over a field of transcendence degree r there is a morphism $f : C \to \mathbb{P}^1$ for which the branch locus has cardinality less than or equal to $3 + r$.

Belyi's original arguments can be used to show that the branch locus can be assumed to be defined over the field of rational functions in r variables. However, there does not seem to be any obvious way to reduce the number of points to $3 + r$. The converse (that such a cover is defined over a field of transcendence degree at most r) follows from the fact that s-tuples of points in $\mathbb{P}^1$ have a moduli space of dimension $s - 3$.

Finally, it is clear from the above construction that the complexity of the configuration required to obtain rigidity is related to the height of the defining equation of a curve. Can this relation be explicitly used to define a notion of height?

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