Proc. Indian Acad. Sci. (Math. Sci.) Vol. 112, No. 2, May 2002, pp. 321–330.

The heat kernel and Hardy's theorem on symmetric spaces of noncompact type

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MS received 10 August 2001; revised 19 January 2002

Abstract. For symmetric spaces of noncompact type we prove an analogue of Hardy's theorem which characterizes the heat kernel in terms of its order of magnitude and that of its Fourier transform.

Keywords. Hardy's theorem; uncertainty principles; symmetric spaces.

1. Introduction

A theorem of Hardy's states that if f is a complex valued measurable function on $\mathbb{R}$ and $\mathcal{F}(f)(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) \mathrm{e}^{-iyx}\, \mathrm{d}x$ is its Fourier transform with $|f(x)| \leq C\mathrm{e}^{-ax^2}$, $|\mathcal{F}(f)(y)| \leq C\mathrm{e}^{-by^2}$, where $C, a, b > 0$, are constants, then: (i) for $ab = \frac{1}{4}$, $f(x) = C\mathrm{e}^{-ax^2}$, (ii) for $ab > \frac{1}{4}$, $f = 0$, and (iii) for $ab < \frac{1}{4}$, there exist infinitely many linearly independent functions satisfying the above inequalities (see [5]). Because of (ii) we can view Hardy's theorem as a mathematical uncertainty principle, that *a nonzero function and its Fourier transform cannot simultaneously be sharply localized* (here very rapid decay at ∞ is interpreted as 'sharp localization'). Recently Hardy's theorem has been extended to several classes of noncommutative groups and also to the context of eigenfunction expansion (see [1,6,7,13,14,16]). These analogues of Hardy's theorem focus on parts (ii) and (iii) above. Despite the great interest of part (i) of the theorem, an obvious difficulty of extending this result to the new contexts lies in identifying the correct analogue of the function e^{-x^2}. A possibility is opened up by viewing the result by saying that if $h_t(x) = t^{-1/2}\mathrm{e}^{-x^2/4t}$ denotes the heat kernel, then (i) implies that f is actually a constant multiple of h_t for some $t > 0$.

From the above point of view, in this paper, we consider the heat kernel associated to the Laplace–Beltrami operator on symmetric spaces of noncompact type, in terms of the decay of a function and its Helgason–Fourier transform (Theorem 3.2). We show that a characterization of the heat kernel is available with an appropriate decay condition on the function. This was motivated by a remark made by Varadarajan some years ago, in

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connection with the paper [16], suggesting that even in the case of symmetric spaces of noncompact type, the heat kernel should play an important role.

This paper is organized as follows: In §2 we fix notation and describe the necessary background material from the theory of noncompact semisimple Lie groups along with a result we need about the entire functions. The last result is an extension of the complex analytic lemma needed to prove Hardy's theorem on the real line. In §3 we prove the main result.

2. Notation and preliminaries

In this section we set up the notation that we subsequently employ and recall some basic facts from the theory of semisimple Lie groups. Our discussion of the latter will be brief and we refer the reader to [9,11,12] for details.

Let G be a connected, noncompact, real semisimple Lie group with finite center and K be a fixed maximal compact subgroup of G. Let $\mathcal{G}$, $\mathcal{K}$ denote the Lie algebras of G and K respectively. Let B be the Cartan Killing form on $\mathcal{G}$ and $\mathcal{G} = \mathcal{K} \oplus \mathcal{P}$ is the Cartan decomposition of $\mathcal{G}$. It is known that $B|_{\mathcal{P}\times\mathcal{P}}$ is positive definite, thus it gives an inner product and hence a norm on $\mathcal{P}$. Let $\mathcal{A}$ be a fixed maximal Abelian subspace of $\mathcal{P}$. Let Σ denotes the set of nonzero roots corresponding to $(\mathcal{G}, \mathcal{A})$ and Σ_+ the set of positive roots with respect to some ordering. Let W denotes the Weyl group associated to Σ. Let $\mathcal{A}_+$ be the positive Weyl chamber and $\overline{\mathcal{A}_+}$ be its closure. By $\mathcal{A}^*_+$ (the correct notation should be $(\mathcal{A}^*)_+$ but as $\mathcal{A}_+$ is not a vector space we hope this does not cause any confusion) and by $\overline{\mathcal{A}^*_+}$ we denote the similar cones in $\mathcal{A}^*$ (the space of real linear functionals on $\mathcal{A}$). Given $\lambda \in \mathcal{A}^*$, we denote by λ_+, the Weyl translate of λ in $\overline{\mathcal{A}^*_+}$. Let A be the analytic subgroup of G with Lie algebra $\mathcal{A}$. A is closed in G and $\exp : \mathcal{A} \to A$ is an isomorphism. We define $A_+ = \exp \mathcal{A}_+$. If $\overline{A_+} = \exp \overline{\mathcal{A}_+}$ denotes the closure of A_+ in G then one gets the polar decomposition $G = K\overline{A_+}K$, that is, each $x \in G$ can be uniquely written as $x = k_1(x)a(x)k_2(x)$ with $k_1(x), k_2(x) \in K$, $a(x) \in \overline{A_+}$. If $\mathcal{G}_\alpha$ denotes the root space corresponding to $\alpha \in \Sigma$ with $m_\alpha = \dim \mathcal{G}_\alpha$ then one can choose a Haar measure $\mathrm{d}x$ on G such that relative to the polar decomposition it is given by $\mathrm{d}x = \text{Const.} J(a)\mathrm{d}k_1\, \mathrm{d}a\, \mathrm{d}k_2$ where $J(a) = J(\exp H) = \Pi_{\alpha\in\Sigma_+} \left(\mathrm{e}^{\alpha(H)} - \mathrm{e}^{-\alpha(H)}\right)^{m_\alpha}$ and $\mathrm{d}a$ is a Haar measure on A. If $\rho = \frac{1}{2}\Sigma_{\alpha\in\Sigma_+} m_\alpha \alpha$, then one has the trivial estimate

$$J(a) = J(\exp H) \le C\mathrm{e}^{2\rho(H)} \quad (H \in \overline{\mathcal{A}_+}). \tag{2.1}$$

Using the polar decomposition we denote

$$|x|_G = |k_1(x)a(x)k_2(x)| = B(\log a(x), \log a(x))^{1/2}.$$

For $\lambda \in \mathcal{A}^*$ we define $H_\lambda \in \mathcal{A}$ to be the unique vector such that

$$\lambda(H) = B(H, H_\lambda) = \langle H, H_\lambda\rangle \quad \text{for all } H \in \mathcal{A}.$$

For $\lambda, \mu \in \mathcal{A}^*$ we thus have $\langle\lambda, \mu\rangle = B(H_\lambda, H_\mu)$, which defines an inner product on $\mathcal{A}^*$ (the norm of λ with respect to this inner product is denoted by $|\lambda|_{\mathcal{A}^*}$). The bilinear extension of $\langle . , .\rangle$ to $\mathcal{A}^*_{\mathbb{C}}$ (the space of all complex valued real linear functionals on $\mathcal{A}$) is also denoted by $\langle . , .\rangle$.

Let $\mathcal{N} = \oplus_{\alpha\in\Sigma_+}\mathcal{G}_\alpha$. Then $\mathcal{N}$ is a nilpotent subalgebra of $\mathcal{G}$. Let N be the analytic subgroup of G with Lie algebra $\mathcal{N}$. Then N is closed in G. Let $G = KAN$ be the Iwasawa decomposition of G, then we have the projection mappings $\kappa : G \to K$, $a : G \to A$, $\eta : G \to N$ such that $x = \kappa(x)a(x)\eta(x) = \kappa(x) \exp\ H(x)\eta(x)$ where $H(x) = \log a(x) \in \mathcal{A}$. If M denotes the centralizer of A in K then $P = MAN$ is a minimal parabolic subgroup of G. For λ in $\mathcal{A}^*_{\mathbb{C}}$ we define a representation χ_λ of P by $\chi_\lambda(man) = \mathrm{e}^{(i\lambda+\rho)(\log a)}$. From this representation we get by induction a representation π_λ of G acting on the Hilbert space $L^2(K/M)$ and the action is given by

$$(\pi_\lambda(g)f)(k) = \mathrm{e}^{-(i\lambda+\rho)H(g^{-1}k)}\ f(\kappa(g^{-1}k)),$$

$g \in G$, $f \in L^2(K/M)$, $k \in K$ ($L^2(K/M)$ is regarded as the space of functions which are right invariant under the action of M). It is known that π_λ is unitary if and only if $\lambda \in \mathcal{A}^*$. These representations π_λ are called *spherical principal series representations* or *class-1 principal series representations.* Note that $\pi_\lambda|K$ are given by left translation on $L^2(K/M)$ and hence by Peter Weyl theorem $\pi_\lambda|K$ contains the trivial representation of K only once. In other words the K-fixed vectors are constant functions. Let $\{v_0, v_1, \dots\}$ be an orthonormal basis of $L^2(K/M)$ consisting of K-finite vectors with v_0 as the constant function one. Let ϕ_λ be the elementary spherical function corresponding to $\lambda \in \mathcal{A}^*_{\mathbb{C}}$, that is, for $\lambda \in \mathcal{A}^*_{\mathbb{C}}$

$$\phi_\lambda(x) = \langle \pi_\lambda(x)v_0, v_0\rangle = \int_K \mathrm{e}^{-(i\lambda+\rho)H(x^{-1}k)}\ \mathrm{d}k = \int \mathrm{e}^{(i\lambda-\rho)H(xk)}\ \mathrm{d}k.$$

The following properties of ϕ_λ are crucial for us and can be found in [9,11].

PROPOSITION 2.1

(i) *$\phi_\lambda(x)$ is K-biinvariant in $x \in G$ and W-invariant in $\lambda \in \mathcal{A}^*_{\mathbb{C}}$.*
(ii) *$\phi_\lambda(x)$ is a C^∞ function in x and a holomorphic function in λ.*
(iii) *We have*

$$\mathrm{e}^{-\rho(H)} \le \phi_0(\exp H) \le C(1 + \|H\|)^{m'}\mathrm{e}^{-\rho(H)},$$

for $H \in \overline{\mathcal{A}_+}$ and some constant $C > 0$, where m' is the number of short positive roots (or equivalently, the number of indivisible positive roots).

(iv) *We have*

$$0 < \phi_{i\lambda}(\exp H) \le \mathrm{e}^{\lambda(H)}\phi_0(\exp H),$$

for $H \in \overline{\mathcal{A}_+}$, $\lambda \in \overline{\mathcal{A}^*_+}$.

In the case of right K-invariant functions it is easy to see that the group Fourier transform $\hat{f}(\pi) = 0$ if $\pi\big|_K$ does not contain the trivial representation of K, as a subrepresentation. So the support of the Plancheral measure, in this case, is the set of spherical principal series representation, $\{\pi_\lambda/\lambda \in \mathcal{A}^*\}$. More precisely, for $f \in C_c^\infty(G/K)$ and $\hat{f}(\pi_\lambda) = \int_G f(x)\pi_\lambda(x)\mathrm{d}x$ we have

$$\int_G |f(x)|^2\mathrm{d}x = \text{Constant} \int_{\mathcal{A}^*/W} \|\hat{f}(\pi_\lambda)\|^2_{\mathrm{HS}}|c(\lambda)|^{-2}\mathrm{d}\lambda,$$

where $\|\hat{f}(\pi_\lambda)\|_{\mathrm{HS}}$ stands for the Hilbert–Schmidt norm of the operator $\hat{f}(\pi_\lambda)$ and c is the Harish-Chandra c-function. Then the Fourier transform extends as an isometry from $L^2(G/K\,\mathrm{d}x)$ onto $L^2(\mathcal{A}^*/W, |c(\lambda)|^{-2}\,\mathrm{d}\lambda, B_2[L^2(K/M)])$, where $B_2[L^2(K/M)]$ is the Hilbert space of Hilbert–Schmidt operators on $L^2(K/M)$. It is known that

$$|c(\lambda)|^{-2} \leq (1+|\lambda|)^{\dim \mathcal{N}},$$

that is, the Plancherel measure is of at most polynomial growth.

Let G/K be the Riemannian symmetric space equipped with a G-invariant Riemannian metric and Δ the Laplace–Beltrami operator on G/K. Then there exists a unique family of smooth function h_t, $t > 0$, with the following properties:

(i) h_t is K-biinvariant, for each $t > 0$.
(ii) For each $t > 0, h_t$ is a smooth nonnegative function on G/K with $\int_{G/K} h_t(x)\,\mathrm{d}x = 1$.
(iii) $h_t * h_s = h_{t+s}$ $t, s > 0$. Here '*' denotes the group convolution.
(iv) For each $t > 0$, $\partial h_t(x)/\partial t = \Delta_x h_t(x)$, that is $h_t(x)$ as a function of t and x satisfies the 'heat equation'.
(v) For each $\phi \in C_c^\infty(G/K)$, define

$$u^\phi(t,x) = h_t * \phi(x)$$

then u^ϕ also satisfies the heat equation and $u^\phi(t,x) \to \phi(x)$ as $t \to 0$, for every $x \in G/K$ (see [8]).

It is known that the heat kernel h_t is given by

$$h_t(x) = C\int_{\mathcal{A}^*} \mathrm{e}^{-t(|\lambda|^2+|\rho|^2)}\phi_\lambda(x)|c(\lambda)|^{-2}\,\mathrm{d}\lambda,$$

that is, $\hat{h_t}(\lambda) = \mathrm{e}^{-t(|\lambda|^2+|\rho|^2)}$. In [3], Theorem 3.1, (i) it has been proved that, for any $t_0 > 0$ there exists $C > 0$ such that

$$h_t(\exp H) \leq Ct^{n/2}\mathrm{e}^{-t|\rho|^2_{\mathcal{A}^*}-\rho(H)-(\|H\|^2/4t)}(1+\|H\|^2)^{(n'-a')/2}, \tag{2.2}$$

where $t \leq t_0$, $H \in \overline{\mathcal{A}_+}$, $n' = \dim G/K$, $a' = \dim \mathcal{A}$ (see also [4] and [8]).

The remaining part of this section is devoted to prove a lemma about the entire functions.

Lemma 2.2. *Suppose $f : \mathbb{C}^n \to \mathbb{C}$ is an entire function and satisfies the following:*

(i) $|f(z)| \leq C_1\mathrm{e}^{a\|\mathrm{Im}\, z\|^2}(1+\|z\|)^m$,
(ii) $|f(x)| \leq C_2\mathrm{e}^{-a\|x\|^2}$ *for* $x \in \mathbb{R}^n$,

where C_1, C_2, $a > 0$ are constants and $m \geq 0$ is an integer. Then $f(z) = C\mathrm{e}^{-a\sum_{j=1}^n z_j^2}$, where $z = (z_1, z_2, \ldots z_n) \in \mathbb{C}^n$ and C is a constant.

Proof. The proof proceeds by induction on n. Assume for the moment, the result is true for $n = 1$. By the induction hypothesis, if f is as in the statement of the lemma, the function $g(z_1, \ldots, z_{n-1}) = f(z_1, \ldots, z_{n-1}, 0)$ on $\mathbb{C}^{n-1}$ is given by $g(z_1, \ldots, z_{n-1}) =$

$C\mathrm{e}^{-a\sum_{j=1}^{n-1} z_j^2}$, $(z_1, \dots, z_{n-1}) \in \mathbb{C}^{n-1}$. For each fixed $\tilde{x} = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$, the function $h_{\tilde{x}}(z) = f(x_1, \dots, x_{n-1}, z)$ is an entire function in $z \in \mathbb{C}$ and satisfies

$$|h_{\tilde{x}}(z)| \leq C_1 \mathrm{e}^{a(\mathrm{Im}\ z)^2}(1+|z|)^m, \qquad |h_{\tilde{x}}(x)| \leq C_2 \mathrm{e}^{-ax^2},$$

for $z \in \mathbb{C}$ and $x \in \mathbb{R}$. By our assumption for $n = 1$, $h_{\tilde{x}}(z) = C(\tilde{x})\mathrm{e}^{-az^2}$, $z \in \mathbb{C}$, where $C(\tilde{x})$ is a constant. Comparing with g we get $C(\tilde{x}) = C_0 \mathrm{e}^{-a\sum_{j=1}^{n-1} x_j^2}$. Thus for the entire function f we have $f(x_1, \dots, x_{n-1}, x_n) = C_0 \mathrm{e}^{-a\sum_{j=1}^{n} x_j^2}$, $(x_1, \dots, x_n) \in \mathbb{R}^n$ and therefore $f(z_1, \dots, z_n) = C_0 \mathrm{e}^{-a\sum_{j=1}^{n} z_j^2}$, $(z_1, \dots, z_n) \in \mathbb{C}^n$. It remains to prove the case $n = 1$.

First we assume that f is even. Define $\phi(z) = f(\sqrt{z})$, $z \in \mathbb{C}$, then ϕ is an entire function and (A) $|\phi(z)| \leq C\mathrm{e}^{a|z|}(1+|z|)^s$, where $s = m/2$, (B) $|\phi(x)| \leq C\mathrm{e}^{-ax}$ for $x \geq 0$. For each α such that $0 < \alpha < \pi$ defined as in page 115 of [5]

$$w(z, \alpha) = w(r, \theta, \alpha) = \exp\left[aiz \, \frac{\mathrm{e}^{-i\frac{\alpha}{2}}}{\sin(\frac{\alpha}{2})} \right],$$

$z = r\mathrm{e}^{i\theta} \in G_\alpha = \{r\mathrm{e}^{i\theta} \,/\, r > 0,\ 0 < \theta < \alpha\}$. Then,

(a) $|w(r, 0, \alpha)| = \mathrm{e}^{ar}$

(b) $|w(r, \alpha, \alpha)| = \mathrm{e}^{-ar}$

(c) $\lim_{\alpha\to\pi} w(z, \alpha) = \mathrm{e}^{az}$.

Now consider the function $F(z) = w(z, \alpha)\, \phi(z)/(i+z)^s$ for $z = r\mathrm{e}^{i\theta} \in G_\alpha$. Then F is analytic in this domain and continuous on the closure. Now one can apply the Phragmen–Lindelöf principle. For $\theta = 0$ using (a) and (B) we have

$$|F(x)| = \frac{|w(x, 0, \alpha)||\phi(x)|}{|i+x|^s} \leq \text{Const.},$$

and on $\{z = r\mathrm{e}^{i\alpha}/r > 0\}$ using (b) and (A) we have

$$|F(r\mathrm{e}^{i\alpha})| \leq C \left(\frac{(1+r)}{|i + r\mathrm{e}^{i\alpha}|} \right)^s \leq \text{Const.},$$

where the constants are independent of α. By Phragmen–Lindelöf principle $|F(z)| \leq$ Const. on G_α. Let $\alpha \to \pi$, we get

$$\left| \frac{\phi(z)\mathrm{e}^{az}}{(i+z)^m} \right| \leq \text{Const.},$$

for every z in the upper half plane. Similarly we can show that for all z on the lower half plane

$$\left| \frac{\phi(z)\mathrm{e}^{az}}{(z-i)^s} \right| \leq \text{Const.}$$

Putting the above together we get $|\phi(z)\mathrm{e}^{az}| \leq C(1+|z|)^m$ for all $z \in \mathbb{C}$, which implies that $\phi(z)\mathrm{e}^{az}$ is a polynomial $Q(z)$ of at most degree m. Hence $f(z) = C\mathrm{e}^{-az^2}Q(z^2)$. But condition (ii) of the hypothesis forces Q to be a constant. If f is odd consider $g(z) = f(z)/z$, then g is an even entire function and as above we get $f(z) = C\mathrm{e}^{-az^2} zQ(z^2)$ for some polynomial Q. But again by condition (ii), $f = 0$ as $zQ(z^2)$ cannot be a nonzero constant. Finally the general case follows on decomposing f into its even and odd parts.

3. Hardy's theorem

Here we prove our main theorem. First, we prove a result for the group Fourier transform then we use it to prove an analogue of Hardy's theorem in terms of the Helgason–Fourier transform.

Theorem 3.1. *Suppose $f : G \to \mathbb{C}$ is a right K-invariant measurable function such that*

(i) $|f(x)| \le C e^{-a|x|_G^2} \phi_0(x)(1 + |x|_G)^r$,

(ii) $\|\hat{f}(\pi_\lambda)\|_{\mathrm{HS}} \le C e^{-b|\lambda|^2_{\mathcal{A}^*}}$,

where C, a, b are positive constants and $r \ge 0$. If $ab = \frac{1}{4}$ then for $\lambda \in \mathcal{A}^*$ we have $\langle \hat{f}(\pi_\lambda)v_0, v_0\rangle = C\, e^{-|\lambda|^2/4a}$ and $\langle \hat{f}(\pi_\lambda)v_0, v_m\rangle = 0$ if $m \ne 0$.

Remark. It is interesting to note that even if we assume the function to be only right K-invariant, the decay condition with $ab = \frac{1}{4}$ forces it to be K-biinvariant, just as in the Euclidean case. The fact that f becomes 0 if $ab > \frac{1}{4}$ continues to be true here also and follows from [16].

First, we give an example to show that, if we allow f to have slightly less decay than in Theorem 3.1 then there are many functions available.

Example. Let $G = SL(2, \mathbb{C})$. Then

$$A = \left\{ \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} : t \in \mathbb{R} \right\} \text{ and } \mathcal{A} = \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} : t \in \mathbb{R} \right\}.$$

Then each $\lambda \in \mathbb{R}$ can be identified with an element of $\mathcal{A}^*$ via the map $\begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \mapsto \lambda t$. With this identification the elementary spherical functions are given by

$$\phi_\lambda \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} = \frac{2 \sin \lambda t}{\lambda \, \sinh 2t}.$$

Also $|\lambda|_{\mathcal{A}^*} = |\lambda|/4$ and $\left\| \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right\|_G = 4|t|$ and the Plancheral measure is given by Const. $\lambda^2 \, d\lambda$ (see [17]). We will show that there exist many functions f such that

(a) f is K-biinvariant

(b) $|f(x)| \le C e^{-\frac{1}{16}|x|_G^2} e^{-\alpha(\log x^+)}$ for all $x \in A$ and for any $\alpha < \rho (= 2)$.

(c) $|\hat{f}(\lambda)| \le C e^{-4|\lambda|^2_{\mathcal{A}^*}}$.

Let $C_c^\infty(\mathbb{R})^{\text{even}} = \{f \in C_c^\infty(\mathbb{R}) : f(x) = f(-x)\}$ and $C_c^\infty(\beta, \beta) = \{f \in C_c^\infty(\mathbb{R})^{\text{even}} : f(x) = 0 \text{ if } |x| \ge \beta\}$, $\beta > 0$. Let $\psi \in C_c^\infty(-\beta, \beta)^{\text{even}}$. Let $\mathcal{F}(\psi)$ be its Euclidean Fourier transform. Define $f(x) = \mathbf{A}^{-1}(\psi) * h_4(x)$ where $\mathbf{A} : C_c^\infty((SL(2,\mathbb{C})||SU(2)) \to C_c^\infty(\mathbb{R})^{\text{even}}$ is the Abel transform (by Paley–Wiener theorem and the fact that $\mathcal{F}(\mathbf{A}(f))(\lambda) = \hat{f}(\phi_\lambda)$, $\mathbf{A}^{-1}$ exists). Then f is a $SU(2)$-biinvariant function and is in the L^p-Schwartz class for $0 < p \le 2$ (see [9,2] for definition of L^p

Schwartz class functions). Since $\hat{f}(\lambda) = \mathcal{F}(\psi)(\lambda)\hat{h_4}(\lambda)$, condition (c) is easily verified. By the spherical Fourier inversion we have

$$\begin{aligned} f\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} &= C \int_{\mathbb{R}} \mathcal{F}(\psi)(\lambda) e^{-\lambda^2/4} \frac{\sin \lambda t}{\lambda \sinh 2t} \lambda^2 \, d\lambda \\ &= \frac{C}{\sinh 2t} \int_{\mathbb{R}} \mathcal{F}(\psi')(\lambda)\, e^{-\lambda^2/4} \sin \lambda t \, d\lambda \\ &= \frac{C}{\sinh 2t} (\psi' *_E h)(t), \end{aligned}$$

where ψ' is the derivative of ψ and is an odd function, $h(t) = e^{-t^2}$ and '$*$'$_E$ is the Euclidean convolution. Thus for large t

$$\begin{aligned} \left| f\begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \right| &= C \left| \frac{(\psi' *_E h)(t)}{\sinh 2t} \right| \\ &\le C \frac{e^{-t^2} e^{2\beta t}}{|\sinh 2t|} \\ &\le C\, e^{-t^2} e^{(\beta-1)2t} \\ &= C\, e^{-\frac{1}{16}|t|_G^2} e^{-(1-\beta)2t}. \end{aligned}$$

Since $\rho = 2$ and $\alpha < \rho$ we can choose a β such that $2(1-\beta) > \alpha$. With this choice (b) is satisfied.

Proof of Theorem. Notice that because of (i) f is in every L^p and hence $\hat{f}$ makes sense. Let $\{v_0, v_1, \ldots\}$ be an orthonormal basis of $L^2(K/M)$ consisting of K-finite vectors where v_0 is the constant function. Let $\Phi^\lambda_{n,m}(x) = \langle \pi_\lambda(x) v_n, v_m \rangle_{L^2(K/M)}$ and

$$F_{n,m}(\lambda) = \int_G f(x)\, \Phi^\lambda_{n,m}(x)\, dx,$$

be the matrix coefficients of $\hat{f}(\pi_\lambda)$. Since f is right K-invariant $F_{n,m}(\lambda) = 0$ for $\lambda \in \mathcal{A}^*$, if $n \neq 0$. We denote $F_{0,m}(\lambda)$ by $F_m(\lambda)$. Now for $\lambda = \lambda_R + i\lambda_I \in \mathcal{A}_{\mathbb{C}^*}$ we have

$$|\Phi^\lambda_{0,m}(\exp H)| \le e^{(\lambda_I)_+(H)}\, \phi_0(\exp H) \tag{3.1}$$

for $H \in \overline{\mathcal{A}_+}$.

Now for $\lambda = \lambda_R + i\lambda_I \in \mathcal{A}^*_{\mathbb{C}}$,

$$\begin{aligned} |F_m(\lambda)| \le C \int_K \int_{\overline{\mathcal{A}_+}} \int_K & |f(k_1 \exp H k_2)|\, e^{(\lambda_I)_+(H)} \\ & \times (1 + \|H\|)^{m'} e^{-\rho(H)} e^{2\rho(H)}\, dk_1\, dH\, dk_2 \\ & \text{(by (2.1), Proposition 2.1 and (3.1))} \end{aligned}$$

$$\leq C \int_{\overline{\mathcal{A}^+}} e^{-a\|H\|^2}\, e^{(\lambda_I)_+(H)}\, \phi_0(\exp H)\, (1+\|H\|)^{r+m'}\, e^{\rho(H)}\, dH$$

(by (i))

$$\leq C \int_{\overline{\mathcal{A}_+}} e^{-a\|H\|^2}\, e^{(\lambda_I)_+(H)}\, (1+\|H\|)^s\, dH$$

(by (iii) of Proposition 2.1, where $s = r + 2m'$)

$$\leq C\, e^{\frac{1}{4a}\|H_{\lambda_I}\|^2} \int_{\mathcal{A}} e^{-a\|H-\frac{1}{2a}H_{\lambda_I}\|^2}\, (1+\|H\|)^s\, dH$$

$$= C\, e^{\frac{1}{4a}\|H_{\lambda_I}\|^2}\, P(\|H_{\lambda_I}\|) \tag{3.2}$$

where P is a polynomial. Thus $F_m(\lambda)$ is well-defined for all $\lambda \in \mathcal{A}^*_{\mathbb{C}}$ and by standard arguments, defines an entire function on $\mathbb{C}^n$. From (ii) it follows that

$$|F_m(\lambda)| \leq C\, e^{-b|\lambda|^2_{\mathcal{A}^*}}. \tag{3.3}$$

By (3.2) and (3.3), if $ab = \frac{1}{4}$, that is, $b = \frac{1}{4a}$, then F_m satisfies (i) and (ii) of Lemma 2.2 and hence

$$F_m(\lambda) = c_m\, e^{-\frac{1}{4a}(\sum_{j=1}^n \lambda_j^2)} \tag{3.4}$$

where $n = \dim \mathcal{A}$. For $\lambda \in \mathcal{A}^*$, $\lambda_j = B(H_\lambda, H_{e_j})$ and for $\lambda \in \mathcal{A}^*_{\mathbb{C}}$, $\lambda_j = (\lambda_R)_j + i(\lambda_I)_j$ and $\{H_{e_j}/j = 1, 2, \ldots n\}$ is an orthonormal basis with respect to B on $\mathcal{A}$. But

$$F_m(\lambda) = \int_G f(x)\, \langle \pi_\lambda(x)v_0, v_m\rangle\, dx$$

$$= \int_G f(x) \left(\int_{K/M} e^{-(i\lambda+\rho)(H(x^{-1}k))}\, \overline{v_m(k)}\, dk \right) dx.$$

Note that for $\lambda = i\rho$, the inner integral vanishes if $m \neq 0$. Hence $F_m(i\rho) = 0$, thus (3.4) shows that c_m is 0 if $m \neq 0$ which completes the proof of the theorem.

For a sufficiently nice function f on $G/K = X$, its Helgason–Fourier transform $\tilde{f}$ is a function defined on $\mathcal{A}^*_{\mathbb{C}} \times K/M$, given by

$$\hat{f}(\pi_\lambda)(1)(kM) = \tilde{f}(\lambda, kM) = \int_{G/K} f(x)\, e^{-(i\lambda+\rho)(H(x^{-1}k))}\, dx$$

where dx is a G invariant measure on G/K (see [10]). The relation between the Helgason–Fourier transform and the group Fourier transform is given by $\tilde{f}(\lambda, k) = \hat{f}(\pi_\lambda)(v_0)(k)$ where v_0 is the essentially unique K-fixed vector. Thus it follows that

$$\int_{K/M} |\tilde{f}(\lambda, k)|^2\, dk = \langle \hat{f}(\pi_\lambda)(1), \hat{f}(\pi_\lambda)(1)\rangle_{L^2(K/M)} = \|\hat{f}(\pi_\lambda)\|^2_{\mathrm{HS}}. \tag{3.5}$$

Finally, the Helgason–Fourier transform $f(x) \to \tilde{f}(\lambda, b)$ extends to an isometry of $L^2(G/K)$ onto $L^2(\mathcal{A}^*_+ \times k/M, |c(\lambda)|^{-2}\, d\lambda\, db)$, where db is a K-invariant measure on K/M. Moreover,

$$\int_{G/K} f_1(x)\overline{f_2(x)}dx = \text{Const.} \int_{\mathcal{A}^*_+ \times K/M} \tilde{f}_1(\lambda, b)\overline{\tilde{f}_2(\lambda, b)}|c(\lambda)|^{-2}\, d\lambda\, db$$

(see [10]). If we think of a function on G/K as a right K-invariant function on G then the following theorem is, in view of (2.2) and (3.5), an easy corollary of Theorem 3.1.

Theorem 3.2. *Suppose $f : X \to \mathbb{C}$ is measurable and satisfies*

(i) $|f(x)| \leq Ch_t(x)$,
(ii) $|\tilde{f}(\lambda, k)| \leq C\, \mathrm{e}^{-t|\lambda|^2}\mathcal{A}^*$,

where $t \leq t_0$ and $t_0 > 0$. Then f is a constant multiple of h_t.

Remark. After our work was finished we came to know about [15], where a result similar to Theorem 3.1 has been proved.

Acknowledgement

We are grateful to A Sitaram and S C Bagchi for several useful discussions.

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