Proc. Indian Acad. Sci. (Math. Sci.) Vol. 112, No. 2, May 2002, pp. 299–319.

A new trigonometric method of summation and its application to the degree of approximation

G DAS, ANASUYA NATH and B K RAY*

Department of Mathematics, Utkal University, Vani Vihar, Bhubaneswar 751 004, Orissa, India
*Department of Mathematics, Ravenshaw College, Cuttack 753 003, Orissa, India

MS received 22 June 1998; revised 11 June 2001

Abstract. The object of the present investigation is to introduce a new trigonometric method of summation which is both regular and Fourier effective and determine its status with reference to other methods of summation (see §2–§4) and also give an application of this method to determine the degree of approximation in a new Banach space of functions conceived as a generalized Hölder metric (see §5).

Keywords. Nörlund mean; (R_2) method; Abelian theorems; Hölder metric; Banach space; Fourier effective.

1. Introduction and definition

The object of the present investigation is to introduce a new trigonometric method of summation which is both regular and Fourier effective and determine its status with reference to other methods of summation (see §2–§4) and also give an application of this method to determine the degree of approximation in a new Banach space of functions conceived as a generalized Hölder metric (see §5).

We know ([11], p. 5) that

$$\sum_{k=1}^{\infty} \frac{\cos kx}{k} = -\log\left(2\sin\frac{x}{2}\right), \quad 0 < x < 2\pi.$$

Integrating twice in succession, we get for $0 < x < 2\pi$

$$\sum_{k=1}^{\infty} \frac{1-\cos kx}{k^3} = -\int_0^x \mathrm{d}y \int_0^y \log\left(2\sin\frac{1}{2}t\right)\mathrm{d}t$$
$$= \frac{x^2}{2}\log\frac{1}{2}csc\frac{1}{2}x + H(x) \tag{1.1}$$

where

$$H(x) = -\frac{1}{2}\int_0^x \frac{1}{2}y^2\cot\frac{1}{2}y\,\mathrm{d}y + x\int_0^x \frac{1}{2}t\cot\frac{1}{2}t\,\mathrm{d}t. \tag{1.2}$$

From (1.1), we get

$$\frac{1}{\log\frac{1}{2}\,csc\,\frac{1}{2}x}\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\sin\frac{1}{2}kx}{\frac{1}{2}kx}\right)^2 = 1 + \frac{2H(x)}{x^2\log\frac{1}{2}\,csc\,\frac{1}{2}x}. \tag{1.3}$$

By simple calculation, it can be shown that

$$H(x) \simeq \frac{3}{4}x^2 \quad \text{as} \quad x \to 0^+. \tag{1.4}$$

From (1.3) and (1.4), it follows that as $x \to 0^+$

$$\frac{1}{\log \frac{1}{2}csc\frac{1}{2}x} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\sin \frac{1}{2}kx}{\frac{1}{2}kx} \right)^2 = 1 + O\left(\frac{1}{\log(1/x)} \right). \tag{1.5}$$

We rewrite (1.5) as follows:

$$\sum_{k=0}^{\infty} \beta_k = 1 + O\left(\frac{1}{\log(1/x)} \right) \text{ as } x \to 0^+ \tag{1.6}$$

where

$$\beta_0(x) = \frac{1}{\log \frac{1}{2}csc(1/x)}, \qquad \beta_k(x) = \beta_0(x) \left(\frac{\sin \frac{1}{2}kx}{\frac{1}{2}kx} \right)^2 \frac{1}{k}, \quad k \gg 1. \tag{1.7}$$

The expression given in (1.6) motivates to introduce a new method of summation as follows: Given an infinite series $\sum_{n=0}^{\infty} a_n$ with the sequence of partial sums $\{S_n\}$, we define a trigonometric mean of $\{S_n\}$ by

$$F(h) = \sum_{k=0}^{\infty} \beta_k(h) S_k \tag{1.8}$$

provided the series on the right is convergent for all positive small h.

If further $\lim_{h\to 0^+} F(h) = s$ then we say that the sequence $\{S_n\}$ is (T) summable to s. We may take $h = 2/n$ and write

$$F\left(\frac{2}{n}\right) = \sum_{k=0}^{\infty} \beta_k \left(\frac{2}{n}\right) S_k \tag{1.9}$$

which is a discrete version of the (T) mean and we may denote the method as (T^*). Though continuous method always implies its discrete version, the converse is not necessarily true.

A summability method which appears similar to (1.8) is the familiar (R_2) method which is defined by the mean (see [3], p. 89)

$$t(h) = \frac{2h}{\pi} \sum_{k=0}^{\infty} \left(\frac{\sin kh}{kh} \right)^2 S_k \tag{1.10}$$

where the coefficient of S_0 in the sum is interpreted as h.

We now recall the definition of two familiar methods which we need in the sequel.

The Nörlund or (N, p) mean is defined by the mean (see [3], p. 64)

$$t_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} S_k \tag{1.11}$$

where $\{p_n\}$ is a sequence of constants, real or complex, such that $P_n = p_0 + p_1 + \cdots + p_n \neq 0\ (n \geq 0)$, $P_n = p_n = 0\ (n < 0)$. In the special case in which $p_n = A_n^{\alpha-1}$, where A_n^{δ} is the coefficient of x^n in the power series expansion of $(1-x)^{-\delta-1}$ for $|x| < 1$, the Nörlund mean (N, p) reduces to the familiar (C, α) mean, where $\alpha \neq -1, -2, \ldots$

We write $p_n \in \mathscr{F}$, if

$$p_n > 0, \qquad \frac{p_{n+1}}{p_n} \leq \frac{p_{n+2}}{p_{n+1}} \leq 1 \tag{1.12}$$

for $n = 0, 1, 2, \ldots$.

The logarithmic or (L) mean is defined by (see [3], p. 81)

$$\frac{1}{-\log(1-x)} \sum_{n=1}^{\infty} \frac{S_n}{n} x^n, \qquad 0 < x < 1. \tag{1.13}$$

2. Abelian theorems

We prove

Theorem 1. *Let $p_n \in \mathscr{F}$. Then*

$$(N,\ p) \subset (T).$$

In particular

$$(C, \alpha) \subset (T) \qquad (0 < \alpha \leq 1).$$

As a consequence of this theorem, we conclude that (T) method is regular and Fourier effective since the method $(C, \alpha), \alpha > 0$ is regular and Fourier effective. It is though possible to establish regularity and Fourier effectiveness of (T) method directly.

Kuttner (see [3], p. 365) shows that $(R_2) \subset (A)$. It is known that (see [3], p. 81) $(A) \subset (L)$. We also prove

Theorem 2.

$$(T) \subset (L).$$

3. Lemmas for Theorem 1

We require the following lemmas for the proof of Theorem 1.

Lemma 1. *(*[3]*, Theorem* 22*). Let $p_n \in \mathscr{F}$ and let the sequence of constants $\{C_n\}$ be defined by the identity*

$$\left(\sum_{n=0}^{\infty} p_n x^n\right)^{-1} = \sum_{n=0}^{\infty} C_n X^n, \quad C_{-1} = 0. \tag{3.1}$$

Then

(i) $C_0 > 0, \quad C_n \leq 0\,(n = 1, 2, 3, \dots)$
(ii) $\sum_{n=0}^{\infty} C_n x^n$ is absolutely convergent for $|x| \leq 1$.
(iii) $\sum_{n=0}^{\infty} C_n > 0$ except when $\sum_{n=0}^{\infty} p_n = \infty$ in which case $\sum_{n=0}^{\infty} C_n = 0$.

Lemma 2. (*Das* [1], *Lemmas* 3 *and* 4). *Let* $p_n \in \mathscr{F}$. *Then*

(i) $C_n^{(1)} = C_0 + C_1 + \cdots + C_n \geq 0$ *and monotonic non-increasing*
(ii) $\sum_{n=m+1}^{\infty} |C_{n-v}| \leq C_{m-v}^{(1)}$,
(iii) $\sum_{v=0}^{\infty} P_v \sum\limits_{n=m+1}^{\infty} |C_{n-v}| \leq m + 1$.

Lemma 3. *Let* $p_n \in \mathscr{F}$. *Let* δ *be a sufficiently small positive number so that* $(\sin\theta/\theta)^2$ *is decreasing for* $0 < \theta < \delta$, *and let* $M = (2\delta/h),\ h > 0$. *Then*

$$\sum_{k=v}^{M} \frac{C_{k-v}}{k} \left(\frac{\sin \frac{1}{2}kh}{\frac{1}{2}kh}\right)^2 \geq 0.$$

Proof. We first note that for $0 < |\theta| < \delta,\ \sin\theta/\theta$. By Abel's transformation

$$\begin{aligned}
&\sum_{k=v}^{M} \frac{C_{k-v}}{k} \left(\frac{\sin \frac{1}{2}kh}{\frac{1}{2}kh}\right)^2 \\
&= \sum_{k=v}^{M-1} \left[\left(\frac{1}{k} - \frac{1}{k+1}\right)\left(\frac{\sin \frac{1}{2}kh}{\frac{1}{2}kh}\right)^2 + \frac{1}{k+1}\Delta_k \left(\frac{\sin \frac{1}{2}kh}{\frac{1}{2}kh}\right)^2\right] C_{k-v}^{(1)} \\
&\quad + \frac{1}{M}\left(\frac{\sin \frac{1}{2}Mh}{\frac{1}{2}Mh}\right)^2 C_{M-v}^{(1)}.
\end{aligned} \tag{3.2}$$

If $|h|$ is chosen small enough the expression $\Delta_k \left(\frac{\sin \frac{1}{2}kh}{\frac{1}{2}kh}\right)^2$ is non-negative when $v \leq k \leq M$ and hence Lemma 3 follows from (3.2) by an appeal to Lemma 2.

Proof of Theorem 1. Using the inverse transformation of (3.1), we get

$$S_n = \sum_{v=0}^{n} C_{n-v} P_v t_v.$$

Hence from (1.8), we obtain

$$\begin{aligned}
F(h) &= \sum_{k=0}^{\infty} \beta_k(h) \sum_{v=0}^{k} C_{k-v} P_v t_v \\
&= \sum_{n=0}^{\infty} P_v t_v \sum_{k=v}^{\infty} C_{k-v} \beta_k(h) = \sum_{v=0}^{\infty} d_v(h) t_v
\end{aligned} \tag{3.3}$$

where

$$d_v(h) = P_v \sum_{k=v}^{\infty} C_{k-v} \beta_k(h). \tag{3.4}$$

To prove Theorem 1, we need only to establish the regularity of the transformation given by (3.3). The necessary and sufficient conditions for the transformation (3.3) to be regular (see [3], p. 49) are:

$$\text{(i)} \qquad \sum_{v=0}^{\infty} d_v(h)| = O(1) \text{ as } h \to 0^+ \tag{3.5}$$

$$\text{(ii)} \qquad \sum_{v=0}^{\infty} d_v(h) \to 1 \text{ as } h \to 0^+ \tag{3.6}$$

$$\text{(iii)} \qquad d_v(h) \to 0 \text{ as } h \to 0^+ (\text{fixed } v). \tag{3.7}$$

Since by Lemma 1, $\Sigma|C_n| < \infty$, it follows that

$$d_v(h) = 0\left(\sum_{n=0}^{\infty} |C_n|\right) \frac{1}{\log \frac{1}{2} csc \frac{1}{2} h} = o(1) \quad \text{as } h \to 0^+$$

which proves (3.7). Since

$$\sum_{v=0}^{k} P_v C_{k-v} = 1 \text{ (for all } k) \tag{3.8}$$

it follows from the definition of $\beta_k(h)$ and (1.5) that

$$\begin{aligned}\sum_{v=0}^{\infty} d_v(h) &= \sum_{k=0}^{\infty} \beta_k(h) \sum_{v=0}^{k} P_v C_{k-v} = \sum_{k=0}^{\infty} \beta_k(h) \\ &= \beta_0(h)\left[1 + \sum_{k=1}^{\infty} \beta_k(h)\right] = 1 + O\left(\frac{1}{\log(1/h)}\right) \to 1 \\ &\quad \text{as } h \to 0^+\end{aligned}$$

which ensures (3.6).

Putting $M = (2\delta/h)$ for some $0 < \delta < 1$, we write

$$\begin{aligned}\Sigma &= \sum_{v=0}^{\infty} |d_v(h)| = \sum_{v=0}^{\infty} \left|\left(\sum_{k=v}^{M} + \sum_{k=M+1}^{\infty}\right) P_v C_{k-v} \beta_k(h)\right| \\ &\le \sum_{v=0}^{\infty} P_v \left|\sum_{k=v}^{M} C_{k-v}\beta_k(h)\right| + \sum_{v=0}^{\infty} P_v \left|\sum_{k=M+1}^{\infty} C_{k-v}\beta_k(h)\right| \\ &= \Sigma_1 + \Sigma_2, \text{ say.}\end{aligned} \tag{3.9}$$

Now in the sum Σ_1, $v \le M$ and so considering $v \ge M + 1$ terms as empty we have by Lemma 3, and by identity (3.8)

$$\begin{aligned}\Sigma_1 &= \sum_{v=0}^{M} P_v \left|\sum_{k=v}^{M} C_{k-v}\beta_k(h)\right| \\ &= \sum_{v=0}^{M} P_v \sum_{k=v}^{M} C_{k-v}\beta_k(h) = \sum_{k=0}^{M} \beta_k(h) \sum_{v=0}^{k} P_v C_{k-v} \\ &= \sum_{k=0}^{M} \beta_k(h) = 1 + O\left(\frac{1}{\log(1/h)}\right)\end{aligned}$$

using (1.5). Lastly by Lemma 2(iii)

$$\begin{aligned}\Sigma_2 &\le \sum_{v=0}^{\infty} P_v \sum_{k=M+1}^{\infty} |C_{k-v}|\,\beta_k(h) = O(1)\beta_0(h) \sum_{v=0}^{\infty} P_v \sum_{k=M+1}^{\infty} \left|\frac{C_{k-v}}{k}\right| \\ &= O(1)\frac{\beta_0(h)}{M+1} \sum_{v=0}^{\infty} P_v \sum_{k=M+1}^{\infty} |C_{k-v}| \\ &= O(1)\beta_0(h) = O(1).\end{aligned}$$

This completes the proof of Theorem 1.

4. Lemmas for Theorem 2

We need the following lemma for the proof of Theorem 2.

Lemma 4. ([3], p. 366, *Theorem* 258) *If* $\Sigma c_n(1 - \cos nx)$ *is convergent for all* x *in an internal* (α, β)*, then* Σc_n *is convergent.*

Additional notations. For $0 < r < 1$ and $0 < \theta \le \pi$, we write

$$\begin{aligned}\Delta(r,\theta) &= 1 - 2r\cos\theta + r^2 \\ P(r,\theta) &= \frac{1}{2} + \sum_{n=1}^{\infty} r^n \cos n\theta = \frac{1}{2}\frac{1-r^2}{\Delta(r,\theta)} \\ P''(r,\theta) &= \frac{\partial^2 P(r,\theta)}{\partial\theta^2} = -\sum_{n=1}^{\infty} n^2 r^n \cos n\theta.\end{aligned}$$

It is easily seen that

$$P''(r,\theta) = -\frac{r(1-r^2)[(1+r^2)\cos\theta - 2r(1+\sin^2\theta)]}{\Delta^3(r,\theta)}. \tag{4.1}$$

It can be easily proved that

$$P''(r,\theta) = O(1)\begin{cases}(1-r)^{-3} \\ (1-r)\theta^{-4}\end{cases}. \tag{4.2}$$

Proof of Theorem 2. By the hypothesis $S_n \to S(T)$ we may assume without any loss of generality that $S_0 = 0$, $S = 0$.

Thus we show that

$$G(h) = \frac{h^2}{2}\log\left(\frac{1}{2}csc\frac{1}{2}h\right)F(h) = \sum_{k=1}^{\infty}\frac{S_k}{k^3}(1-\cos kh)$$

which is convergent for small $h > 0$ and

$$G(h) = o\left(h^2 \log\frac{1}{2}\ csc\ \frac{1}{2}h\right). \tag{4.3}$$

By Lemma 4 $\sum_{n=1}^{\infty} S_n/n^3$ is convergent and hence we can write the series for $G(h)$ as

$$\frac{1}{2}\alpha_0 + \sum_{n-1}^{\infty} \alpha_n \cos\, nh \tag{4.4}$$

where

$$\alpha_n = -n^{-3} S_n \;\text{ for }\; n \geq 1, \quad \alpha_0 = 2\sum_{n=1}^{\infty} \frac{S_n}{n^3} = -2\sum_{n=1}^{\infty} \alpha_n. \tag{4.5}$$

At this stage by adopting an argument similar to those used in ([3], p. 368) it can be shown that Theorem 2 is true generally, if it holds in the following two particular cases:

(I) The series (4.4) for $G(h)$ is a Fourier series

(II) The series (4.4) converges uniformly to zero in a neighborhood of h that includes the origin.

Case I. If the series (4.4) is a Fourier series, then there exists an even periodic function $\phi(h)$ with period 2π of which (4.4) is the Fourier series, so that

$$\phi(h) = G(h) = o\left(h^2 \log \frac{1}{h}\right) \qquad \text{when } 0 < h \leq \delta \tag{4.6}$$

for some $0 < \delta < \pi$.

Now using (4.5), we get

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{S_n}{n} r^n &= -\sum_{n=1}^{\infty} n^2 \alpha_n r^n \\ &= -\frac{2}{\pi}\int_0^{\pi} \phi(\theta)\left(\sum_{n=1}^{\infty} n^2 r^n \cos\, n\theta\right) \mathrm{d}\theta \\ &= \frac{2}{\pi}\int_0^{\pi} \phi(\theta) P''(r,\theta)\mathrm{d}\theta. \end{aligned}\tag{4.7}$$

Using (4.7), we write

$$\sum_{n=1}^{\infty} \frac{S_n}{n} r^n = \frac{2}{\pi}\left[\int_0^{1-r} + \int_{1-r}^{\delta} + \int_{\delta}^{\pi}\right] \phi(\theta) P''(r,\theta)\mathrm{d}\theta.$$

Now using (4.6) and (4.2), we obtain

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{S_n}{n} r^n &= O(1)(1-r)^{-3}\int_0^{1-r} \theta^2 \log \frac{1}{\theta}\mathrm{d}\theta \\ &\quad + o(1)(1-r)\int_{1-r}^{\delta} \theta^{-2} \log \frac{1}{\theta}\mathrm{d}\theta + 0(1)(1-r)\int_{\delta}^{\pi} |\phi(\theta)|\mathrm{d}\theta \\ &= o\left(\log \frac{1}{1-r}\right) + O(1-r) \\ &= o\left(\log \frac{1}{1-r}\right) \text{ as } r \to 1- \end{aligned}$$

which proves that $S_n = o(L)$ and hence Theorem 2 holds in Case I.

Case II. We first note that, after simple calculation

$$r^n = \frac{2}{\pi}\int_0^\pi P(r,\theta)\cos\, n\theta\, \mathrm{d}\theta$$
$$= -\frac{2}{\pi}\int_0^\pi P''(r,\theta)\frac{1-\cos n\theta}{n^2}\mathrm{d}\theta, \quad n \geq 1$$

from which it follows that

$$\sum_{n=1}^{\infty}\frac{S_n}{n}\, r^n = -\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{S_n}{n^3}\int_0^\pi P''(r,\theta)(1-\cos\, n\theta)\mathrm{d}\theta. \tag{4.8}$$

Since the series

$$\frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty}\alpha_n \cos\, n\theta$$

converges uniformly to zero for small θ, say for $|\theta| \leq \theta_0$ it follows that

$$\sum_{n=1}^{\infty}\frac{S_n}{n^3}\int_0^{\theta_0}(1-\cos n\theta)P''(r,\theta)\,\mathrm{d}\theta = 0. \tag{4.9}$$

For the proof of Theorem 2, it remains to show that, for fixed θ_0

$$\sum_{n=1}^{\infty}\frac{S_n}{n^3}\int_{\theta_0}^{\pi}(1-\cos n\theta)P''(r,\theta)\,\mathrm{d}\theta = o\left(\log\frac{1}{1-r}\right). \tag{4.10}$$

Since $\int_{\theta_0}^{\pi} P''(r,\theta)\,\mathrm{d}\theta = O(1)$ it is clear that

$$\sum_{n=1}^{\infty}\frac{S_n}{n^3}\int_{\theta_0}^{\pi} P''(r,\theta)\,\mathrm{d}\theta = O(1). \tag{4.11}$$

By simple calculation, it can be seen that $P''(r,\theta)$, $P'''(r,\theta)$ and $P^{(\mathrm{iv})}(r,\theta)$ are uniformly bounded in the interval $[\theta_0,\pi]$ and hence by integration by parts twice, we obtain for $n \geq 1$.

$$J_n = \int_{\theta_0}^{\pi}\cos n\theta\, P''(r,\theta)\,\mathrm{d}\theta$$
$$= \frac{-P''(r,\theta_0)\sin n\theta_0}{n} - \frac{P'''(r,\theta_0)\cos n\theta_0}{n^2}$$
$$+ \frac{1}{n^2}\int_{\theta_0}^{\pi} P^{(\mathrm{iv})}(r,\theta)\cos n\theta\,\mathrm{d}\theta$$
$$= -P''(r,\theta_0)\frac{\sin n\theta_0}{n} + O\left(\frac{1}{n^2}\right). \tag{4.12}$$

As $S_n = o(n^3)$, we have

$$\sum_{n=1}^{\infty}\frac{S_n}{n^3}J_n = -P''(r,\theta_0)\sum_{n=1}^{\infty}\frac{S_n}{n^4}\sin n\theta_0 + O(1)\sum_{n=1}^{\infty}\frac{1}{n^2}.$$

As $\sum_{n=1}^{\infty}(S_n/n^3)\cos n\theta$ converges uniformly for $|\theta| \le \theta_0$, the series $\sum_{n=1}^{\infty}(S_n/n^4)\sin n\,\theta_0$ is convergent and hence

$$\sum_{n=1}^{\infty}\frac{S_n}{n^3}J_n = O(1). \tag{4.13}$$

Now it follows from (4.11) and (4.13) that

$$\sum_{n=1}^{\infty}\frac{S_n}{n^3}\int_{\theta_0}^{\pi}(1-\cos n\theta)P''(r,\theta)\,\mathrm{d}\theta = O(1)$$

and which further ensures (4.10). This completes the proof of Theorem 2.

5. An application

Let $C_{2\pi}$ denote the Banach space of all 2π-periodic and continuous functions defined on $[-\pi,\pi]$ under the supremum norm. Modulus of continuity of f is defined by

$$w(f,\delta) = \sup_{|h|\le\delta}|f(x+h)-f(x)|.$$

For $0<\alpha\le 1$, let

$$H_\alpha = \left\{f\in C_{2\pi} : w(f,\delta) = O(\delta^\alpha)\right\}.$$

It is known [7] that H_α is a Banach space with the norm defined by the Hölder metric:

$$\|f\|_{(\alpha)} = \sup_{-\pi\le x\le\pi}|f(x)| + \sup_{\substack{x,h\\ h\ne 0}}\frac{|f(x+h)-f(x)|}{|h|^\alpha}, \quad 0<\alpha\le 1.$$

For $f\in L^p[-\pi,\pi]$, $p\ge 1$ the integral modulus of continuity is defined by

$$w_p(f,\delta) = \sup_{|h|\le\delta}\|f(\cdot+h)-f(\cdot)\|_p.$$

It is known ([10], p. 45) that $w_p(f,\delta)\to 0$ as $\delta\to 0$.

A generalized Hölder metric has been provided in [2] as follows:

$$\|f\|_{(\alpha,p)} = \|f\|_p + \sup_{h\ne 0}\frac{\|f(\cdot+h)-f(\cdot)\|_p}{|h|^\alpha}, \qquad \alpha>0$$

$$\|f\|_{(0,p)} = \|f\|_p.$$

The norm is taken with respect to x throughout.

The class of functions H^w has been defined by Leindler [4] as

$$H^w = \{f\in C_{2\pi} : w(f,\delta) = O(w(\delta))\}$$

where w is a modulus of continuity, that is, w is a positive non-decreasing continuous function with the property: $w(o)=o$, $w(\delta_1+\delta_2)\le w(\delta_1)+w(\delta_2)$. The degree of approximation problems have been studied in H^w space by Leindler [4], Totik [8,9], Mazhar and Totik [6] and Mazhar [5]. We now provide a further generalization of H^w space in the following section.

5.1 *Introducing a generalized Hölder metric*

Let $w : [0, 2\pi) \to R$, be an arbitrary function with $w(t) > 0$ for $0 < t \leq 2\pi$ and $\lim_{t\to 0^+} w(t) = w(0) = 0$ and we denote

$$A(f; w) = \sup_{t\neq 0} \frac{\|f(\cdot + t) - f(\cdot)\|_p}{w(|t|)}, \qquad p \geq 1.$$

We define

$$H_p^{(w)} = \left\{f \in L^p[0, 2\pi] : 1 \leq p < \infty : A(f; w) < \infty\right\}$$

and

$$\|f\|_p^{(w)} = \|f\|_p + A(f; w).$$

It can be easily verified that $\|f\|_p^{(w)}$ is a norm on $H_p^{(w)}$. To prove the completeness of the space $H_p^{(w)}$ we bank upon the completeness of $L^p (p \geq 1)$. If we take $w(t) = t^\alpha$, $0 < \alpha \leq 1$ then $H_p^{(w)}$ reduces to $H(\alpha, p)$ space (with the norm $\|f\|_p^{(w)}$ replaced by $\|f\|_{\alpha,p}$) introduced earlier by Das, Ghosh and Ray [2]. If $w(t)/t$ tends to zero as $t \to 0^+$ then $f'(x)$ exists and is zero everywhere, and f is constant.

Given the spaces $H_p^{(w)}$ and $H_p^{(v)}$, if $w(t)/v(t)$ is non-decreasing we have

$$H_p^{(w)} \subseteq H_P^{(v)} \subseteq L^p, \qquad p \geq 1$$

since

$$\|f\|_p^{(v)} \leq \max\left(1, \frac{w(2\pi)}{v(2\pi)}\right) \|f\|_p^{(w)}.$$

We now investigate the degree of approximation of 2π periodic function $f \in H(w, p)$ by (T^*) mean of Fourier series of f at x which is defined as

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Let $S_n(x; f)$ and $W_n^*(x; f)$ respectively denote the n-th partial sum and (T^*) mean of the above series. Let F denote the class of real valued functions w defined over $[0, 2\pi]$ satisfying the following conditions:

(i) $w(t) > 0$ for $0 < t \leq 2\pi$
(ii) $w(t)$ is non-decreasing over $[0, 2\pi]$ and
(iii) $\lim_{t\to 0} w(t) = w(0) = 0$.

We use the following notations:

$$\phi_x(t) = \frac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\}$$

$$K_n(t) = \sum_{k=1}^{\infty} \frac{\sin^2 k/n}{k^3} \sin\left(k + \frac{1}{2}\right) t \tag{5.1}$$

$$G_n(t) = \sum_{k=1}^{n} \frac{\sin^2 k/n}{k^3} \sin\left(k + \frac{1}{2}\right) t \tag{5.2}$$

$$H_n(t) = \sum_{k=n+1}^{\infty} \frac{\sin^2 k/n}{k^3} \sin\left(k + \frac{1}{2}\right) t \tag{5.3}$$

$$l_n(x) = W_n^*(x, f) - f(x) \tag{5.4}$$

$$\theta(n) = \sum_{k=1}^{\infty} \beta_k(2/n) = \frac{1}{\log \frac{1}{2}\, csc\, 1/n} \sum_{k=1}^{\infty} \frac{1}{k} \left(\frac{\sin k/n}{k/n}\right)^2. \tag{5.5}$$

We prove the following theorem.

Theorem 3. *Let $F = \{u : u : [0, \pi) \to [0, \infty), \lim_{t\to 0^+} u(t) = 0 = u(0)$ and u is non-decreasing$\}$. Let $v, w, \in F$ such that $(w/v) \in F$. If $f \in H_p^{(w)}$, $p \geq 1$, then*

$$\|W_n^*(\cdot, f) - f\|_p^{(v)} = O(1)\frac{1}{\log n}\int_{\pi/n}^{\pi} \frac{w(t)}{t v(t)} \mathrm{d}t \tag{5.6}$$

where $W_n^*(x; f)$ is the (T^*) transformation of the Fourier series of f at x. Further, there exists a function $g \in H_p^{(w)}$ such that for some positive constant C

$$\|W^*(\cdot, g) - g\|_p^{(v)} \geq \frac{C}{\log n}\int_{\pi/n}^{\pi} \frac{w(t)}{t v(t)} \mathrm{d}t \tag{5.7}$$

provided that $w(t)/v(t)$ is a modulus of continuity.

We need the following lemmas for the proof of Theorem 3.

Lemma 5. *Let $f \in H_p^{(w)}$, $p \geq 1$. Then for* $0 < t \leq \pi$

(i) $$\|\phi_{\cdot+y} - \phi_{\cdot}(t)\|_p = O(1)\begin{cases} w(t) \\ w(|y|) \end{cases}.$$

If further $w(t)$ and $v(t)$ are defined as in Theorem 3 *then*

(ii) $$\|\phi_{\cdot+y}(t) - \phi_{\cdot}(t)\|_p = O(1)v(|y|)(w(t)/v(t)).$$

Proof. We omit the proof of (i) as it is an easy consequence of the hypothesis. Using (i), we get

$$\|\phi_{\cdot+y}(t) - \phi_{\cdot}(t)\|_p = O(w(t)) = O\left(v(t)\frac{w(t)}{v(t)}\right) = O\left(v(|y|)\frac{w(t)}{v(t)}\right)$$

for $t \leq |y|$, as v is non-decreasing.

If $t \geq |y|$, then

$$\frac{w(t)}{v(t)} \geq \frac{w(|y|)}{v(|y|)} \quad \left(\text{as } \frac{w(t)}{v(t)} \text{ is non-decreasing}\right)$$

so that

$$\|\phi_{\cdot+y}(t) - \phi_{\cdot}(t)\|_p = O(w(|y|)) = O(1)v(|y|)\frac{w(t)}{v(t)}.$$

This completes the proof of (ii).

Lemma 6

$$\sum_{k=n+1}^{\infty} \frac{\sin(k+\frac{1}{2})t}{k^3} = O(n^{-3}t^{-1}) \tag{5.8}$$

$$\sum_{k=n+1}^{\infty} \frac{2\sin kt \cos(2k/n)}{k^3} = O\left[n^{-3}\left(t - \frac{2}{n}\right)^{-1}\right], \quad \frac{2}{n} < t \leq \pi \tag{5.9}$$

$$\sum_{k=n+1}^{\infty} \frac{2\cos kt \cos(2k/n)}{k^3} = O\left[n^{-3}\left(t - \frac{2}{n}\right)^{-1}\right], \quad \frac{2}{n} < t \leq \pi \tag{5.10}$$

$$G_n(t) = O(n^{-2}) \tag{5.11}$$

$$G_n(t) = O(tn^{-1}) \tag{5.12}$$

$$H_n(t) = O\left[n^{-3}\left(t - \frac{2}{n}\right)^{-1}\right], \quad \frac{2}{n} < t \leq \pi \tag{5.13}$$

$$H_n(t) = O(t^{\delta}n^{\delta-2}), \quad 0 < \delta < 1 \tag{5.14}$$

$$K_n(t) = O(n^{-2}\log n) \tag{5.15}$$

$$K_n(t) = O(n^{-2}) + O\left[n^{-3}\left(t - \frac{2}{n}\right)^{-1}\right], \quad \frac{2}{n} < t \leq \pi \tag{5.16}$$

$$K_n(t) = O(tn^{-1}) + O(t^{\delta}n^{\delta-2}), \quad 0 < \delta < 1 \tag{5.17}$$

$$\theta(n) - 1 = O\left(\frac{1}{\log n}\right) \tag{5.18}$$

Proof. As k^{-3} is monotonic decreasing, (5.8) follows at once. We have

$$\begin{aligned}\sum_{k=n+1}^{\infty} \frac{2\sin kt \cos(2k/n)}{k^3} &= \sum_{k=n+1}^{\infty} \frac{\sin k[t+(2/n)]}{k^3} + \sum_{k=n+1}^{\infty} \frac{\sin k[t-(2/n)]}{k^3} \\ &= O\left[n^{-3}\left(t + \frac{2}{n}\right)^{-1}\right] + O\left[n^{-3}\left(t - \frac{2}{n}\right)^{-1}\right] \\ &= O\left[n^{-3}\left(t - \frac{2}{n}\right)^{-1}\right], \quad \text{when } t > \frac{2}{n}\end{aligned}$$

which ensures (5.9). We omit the proof of (5.10) as its proof is similar to that of (5.9).

Since $[\sin(k/n)/(k/n)]^2$ is monotonic non-increasing in k for $k \leq n$, we get

$$\begin{aligned}|G_n(t)| &= \frac{1}{n^2}\left|\sum_{k=1}^{n}\left(\frac{\sin k/n}{k/n}\right)^2 \frac{\sin(k+1/2)t}{k}\right| \\ &\leq \frac{1}{n^2}\left(\frac{\sin 1/n}{1/n}\right)^2 \max_{1<M,M'<n}\left|\sum_{M}^{M'}\frac{\sin(k+1/2)t}{k}\right| \\ &= O(n^{-2})\end{aligned}$$

as the last sum in bounded and this proves (5.11).

Proof of (5.12) follows from

$$|G_n(t)| \leq \frac{t}{n^2}\sum_{k=1}^{n}\frac{(k+1/2)}{k} = O(tn^{-1}).$$

Writing

$$\begin{aligned}H_n(t) = \frac{1}{2}\sum_{k=n+1}^{\infty}\frac{\sin(k+1/2)t}{k^3} - \frac{1}{4}\cos\frac{1}{2}t\sum_{k=n+1}^{\infty}\frac{2\sin kt\cos 2k/n}{k^3} \\ - \frac{1}{4}\sin\frac{1}{2}t\sum_{k=n+1}^{\infty}\frac{2\cos kt\cos 2k/n}{k^3}\end{aligned}$$

and using the estimates (5.8)–(5.10) we obtain

$$\begin{aligned}H_n(t) &= O(n^{-3}t^{-1}) + O\left[n^{-3}\left(t-\frac{2}{n}\right)^{-1}\right] \\ &= O\left[n^{-3}\left(t-\frac{2}{n}\right)^{-1}\right], t > \frac{2}{n}\end{aligned}$$

and this proves (5.13). Using the fact that for $0 < \delta < 1$

$$\left|\sin\left(k+\frac{1}{2}\right)t\right| = \left|\sin\left(k+\frac{1}{2}\right)t\right|^{\delta}\left|\sin\left(k+\frac{1}{2}\right)t\right|^{1-\delta} = O(k^{\delta}t^{\delta})$$

we get

$$H_n(t) = O\left(t^{\delta}\right)\sum_{k=n+1}^{\infty}k^{\delta-3} = O\left(t^{\delta}n^{\delta-2}\right)$$

and this proves (5.14). By putting $x = 2/n$ in (1.5) we have

$$\sum_{k=1}^{\infty}\frac{\sin^2 k/n}{k^3} = O\left(\frac{\log n}{n^2}\right)$$

from which inequality (5.15) follows at once as

$$|K_n(t)| \leq \sum_{k=1}^{\infty}\frac{\sin^2 k/n}{k^3}.$$

As $K_n(t) = G_n(t) + H_n(t)$ the inequality (5.16) follows from (5.11) and (5.13). For the proof of (5.17) we use the estimates (5.12) and (5.14). Inequality (5.18) follows from (1.5).

Lemma 7. *Let $u(t)$ be a positive non-decreasing function for $t > 0$. Then for $0 < \delta \leq 1$ and $n \geq 2$*

$$n^{\delta} \int_0^{\pi/n} u(t) t^{\delta-1} \mathrm{d}t \leq A \int_{\pi/n}^{\pi} \frac{u(t)}{t} \mathrm{d}t$$

where A is some positive constant.

Proof. The result follows by combining the following easily verifiable inequalities.

For $0 < \eta < \pi$

$$\int_0^{\eta} u(t) t^{\delta-1} \mathrm{d}t \leq \eta^{\delta} u(\eta)$$

and

$$\int_{\eta}^{\pi} \frac{u(t)}{t} \mathrm{d}t \geq u(\eta) \frac{(\pi - \eta)}{\pi}.$$

Lemma 8. *Let*

$$g(x) = \sum_{k=1}^{\infty} \left[\eta\left(\frac{1}{k}\right) - \eta\left(\frac{1}{k+1}\right) \right] (1 - \cos kx) \tag{5.19}$$

where $\eta(t)$ is any modulus of continuity. Then

(i) (5.19) *is a Fourier series for some function in $C_{2\pi}$*

(ii) $g \in H_p^{(w)}$, $p \geq 1$

(iii) $\| W_n^*(\cdot; g) - g \|_p^{(w)} \geq C \int_{\pi/n}^{\pi} \frac{\eta(t)}{t} \mathrm{d}t$,

where $W_p^*(x; g)$ is the (T^*) transform of (5.19) *and C is some positive constant.*

Proof. Since the series for $g(x)$ is uniformly convergent, part (i) of Lemma 8 is obvious.

We observe that

$$g(x) - g(x+h) = \sum_{k=1}^{\infty} \left[\eta\left(\frac{1}{k}\right) - \eta\left(\frac{1}{k+1}\right) \right] (\cos k(x+h) - \cos kx)$$

It can be shown as in Lemma 5 of Totik [8] that $g \in H^w$ and hence $g \in H_p^{(w)}$, $p \geq 1$. Now

$$\begin{aligned} W_n^*(x, g) - g(x) &= \sum_{k=1}^{\infty} \beta_k \left(\frac{2}{n}\right) \left\{ g(x) - \sum_{v=k+1}^{\infty} \left[\eta\left(\frac{1}{v}\right) \right.\right. \\ &\quad \left.\left. - \eta\left(\frac{1}{v+1}\right) \right] (1 - \cos vx) \right\} - g(x) \\ &= \sum_{k=1}^{\infty} \beta_k \left(\frac{2}{n}\right) \left\{ g(x) - \sum_{v=k+1}^{\infty} \left[\eta\left(\frac{1}{v}\right) \right.\right. \\ &\quad \left.\left. - \eta\left(\frac{1}{v+1}\right) \right] (1 - \cos vx) \right\} - g(x) \end{aligned}$$

$$= g(x)[\theta(n) - 1] - \sum_{k=1}^{\infty} \beta_k(2/n) \sum_{v=k+1}^{\infty} \left[\eta\left(\frac{1}{v}\right) - \eta\left(\frac{1}{v+1}\right)\right](1 - \cos vx)$$
$$= f_1(n,x) - f_2(n,x), \quad \text{say.}$$

Now

$$\|W_n^*(\cdot;g) - g\|_p^{(w)} \geq \|W_n(\cdot;g) - g\|_p$$
$$= \sup_{\psi\in L^q} \left|\int_0^{\pi} (W_n^*(x,g) - g(x))\psi(x)\,\mathrm{d}x;\ \|\psi\|_q \geq 1\right|$$
$$\geq \left|\int_0^{\pi} (W_n^*(x,g) - g(x))\,\mathrm{d}x\right|$$
$$= \left|\int_0^{\pi} (f_1(n,x) - f_2(n,x)\,\mathrm{d}x\right| \tag{5.20}$$

by Theorem 9.4 (see [10], p. 19) and choosing $\psi(x) = 1$.

From (1.3) and (1.4), it follows that as $n \to \infty$

$$\theta(n) - 1 \simeq \frac{3}{2\log\frac{1}{2}csc1/n}. \tag{5.21}$$

Therefore

$$\int_0^{\pi} f_1(n,x)\,\mathrm{d}x \simeq \frac{3\pi}{2\log\frac{1}{2}csc1/n} \sum_{k=1}^{\infty}\left[\eta\left(\frac{1}{k}\right) - \eta\left(\frac{1}{k+1}\right)\right]$$
$$= \frac{3\pi}{2\log\frac{1}{2}csc1/n}\eta(1). \tag{5.22}$$

Now

$$\int_0^{\pi} f_2(n,x)\,\mathrm{d}x = \frac{\pi n^2}{\log\frac{1}{2}csc1/n}\sum_{k=1}^{\infty}\frac{(\sin k/n)^2}{k^3}\eta\left(\frac{1}{k+1}\right) \tag{5.23}$$
$$\geq \frac{\pi n^2}{\log\frac{1}{2}csc1/n}\sum_{k=1}^{n}\frac{(\sin k/n)^2}{k^3}\eta\left(\frac{1}{k+1}\right)$$
$$\geq \frac{\pi n^2}{\log\frac{1}{2}csc1/n}\left(\frac{2}{\pi}\right)^2\sum_{k=1}^{n}\frac{k^2/n^2}{k^3}\eta\left(\frac{1}{k+1}\right)$$
$$= \frac{4}{\pi\log\frac{1}{2}csc1/n}\sum_{k=1}^{n}\frac{1}{k}\eta\left(\frac{1}{k+1}\right). \tag{5.24}$$

From (5.20), (5.21) and (5.24), it follows that

$$\|W_n^*(\cdot;g) - g\|_p^{(w)} \geq \frac{1}{\log\frac{1}{2}csc1/n}\left|\frac{4}{\pi}\sum_{k=1}^{n}\frac{1}{k}\eta\left(\frac{1}{k+1}\right) - \frac{3\pi\eta(1)}{2}\right|$$
$$\simeq \frac{4}{\pi\log n}\sum_{k=1}^{n}\frac{1}{k}\eta\left(\frac{1}{k+1}\right) \tag{5.25}$$

in case

$$\sum_{k=1}^{\infty} \frac{1}{k}\eta\left(\frac{1}{k+1}\right) = \infty.$$

Next, we consider the case when

$$\sum_{k=1}^{\infty} \frac{1}{k}\eta\left(\frac{1}{k+1}\right) < \infty.$$

The right side sum of (5.23) is the $(R, 2)$ transform (discrete version) of the series $\sum_{k=1}^{\infty} \frac{1}{k}\eta[1/(k+1)]$.

It is known ([3], p. 89) that (R, k) method is regular for $k > 1$ and hence

$$\lim_{\eta\to\infty} \sum_{k=1}^{\infty} \left(\frac{\sin k/n}{k/n}\right)^2 \frac{1}{k}\eta\left(\frac{1}{k+1}\right) = \sum_{k=1}^{\infty} \frac{1}{k}\eta\left(\frac{1}{k+1}\right). \tag{5.26}$$

From (5.23) and (5.26), it follows that

$$\int_0^{\pi} f_2(n, x)\,\mathrm{d}x \simeq \frac{\pi}{\log \frac{1}{2} csc 1/n} \sum_{k=1}^{\infty} \frac{1}{k}\eta\left(\frac{1}{k+1}\right). \tag{5.27}$$

Collecting the results from (5.20), (5.22) and (5.27), we get

$$\begin{aligned} \|W_n^*(\cdot; g) - g\|_p^{(w)} &\geq \frac{\pi}{\log \frac{1}{2} csc 1/n} \left| \frac{3}{2}\eta(1) - \sum_{k=1}^{\infty} \frac{1}{k}\eta\left(\frac{1}{k+1}\right)\right| \\ &\geq \frac{\pi}{2\log \frac{1}{2} csc 1/n} \sum_{k=1}^{\infty} \frac{1}{k}\eta\left(\frac{1}{k+1}\right). \end{aligned} \tag{5.28}$$

Assuming as we may that

$$\frac{3}{4}\eta(1) > \sum_{k=1}^{\infty} \frac{1}{k}\eta\left(\frac{1}{k+1}\right).$$

From (5.25) and (5.28), it follows that

$$\|W_n^*(\cdot; g) - g\|_p^{(w)} \geq \frac{C}{\log n} \sum_{k=1}^{n} \frac{1}{k}\eta\left(\frac{1}{k+1}\right) \tag{5.29}$$

where C is some positive constant.

Since

$$\sum_{k=1}^{n} \frac{1}{k}\eta\left(\frac{1}{k+1}\right) \geq A \int_{\pi/n}^{\pi} \frac{\eta(t)}{t}\mathrm{d}t \quad \text{as} \quad n \to \infty$$

for some positive constant A, the lemma follows.

Proof of Theorem 3. We know ([10], p. 50) that

$$S_k(x;f) - f(x) = \frac{2}{\pi}\int_0^{\pi} \phi_x(t)\frac{\sin(k+\frac{1}{2})t}{2\sin\frac{1}{2}t}\mathrm{d}t, \quad k \geq 0.$$

Using the definition of (T^*) transformation, we have

$$\begin{aligned}
l_n(x) &= W_n^*(x;f) - f(x)\\
&= \sum_{k=0}^{\infty} \beta_k\left(\frac{2}{n}\right) S_k(f;x)\\
&= \sum_{k=0}^{\infty} \beta_k\left(\frac{2}{n}\right)\left\{\frac{2}{\pi}\int_0^{\pi}\phi_x(t)\frac{\sin(k+\frac{1}{2})t}{2\sin t/2}\mathrm{d}t + f(x)\right\} - f(x)\\
&= \frac{2}{\pi}\int_0^{\pi}\frac{\phi_x(t)}{2\sin\frac{1}{2}t}\left\{\sum_{k=0}^{\infty}\beta_k\left(\frac{2}{n}\right)\sin(k+\frac{1}{2})t\right\}\mathrm{d}t\\
&\quad + f(x)\left\{\sum_{k=0}^{\infty}\beta_k\left(\frac{2}{n}\right) - 1\right\}\\
&= \frac{2}{\pi}\int_0^{\pi}\frac{\phi_x(t)}{2\sin\frac{1}{2}t}\left\{\sum_{k=1}^{\infty}\beta_k\left(\frac{2}{n}\right)\sin(k+\frac{1}{2})t\right\}\mathrm{d}t\\
&\quad + \frac{1}{\pi}\beta_0\left(\frac{2}{n}\right)\int_0^{\pi}\phi_x(t)\,\mathrm{d}t + f(x)\left\{\beta_0\left(\frac{2}{n}\right) + \theta(n) - 1\right\}\\
&= \frac{2}{\pi}n^2\beta_0\left(\frac{2}{n}\right)\int_0^{\pi}\frac{\phi_x(t)}{2\sin\frac{1}{2}t}K_n(t)\,\mathrm{d}t + \frac{1}{\pi}\beta_0\left(\frac{2}{n}\right)\int_0^{\pi}\phi_x(t)\mathrm{d}t\\
&\quad + f(x)\left\{\beta_0\left(\frac{2}{n}\right) + \theta(n) - 1\right\}. \qquad (5.30)
\end{aligned}$$

Using (5.30), we get

$$\begin{aligned}
l_n(x+y) - l_n(x) &= \frac{2}{\pi}n^2\beta_0\left(\frac{2}{n}\right)\int_0^{\pi}\frac{\phi_{x+y}(t)-\phi_x(t)}{2\sin\frac{1}{2}t}K_n(t)\,\mathrm{d}t\\
&\quad + \frac{1}{\pi}\beta_0\left(\frac{2}{n}\right)\int_0^{\pi}(\phi_{x+y}(t)-\phi_x(t))\,\mathrm{d}t\\
&\quad + (f(x+y)-f(x))\left\{\beta_0\left(\frac{2}{n}\right)+\theta(n)-1\right\}.
\end{aligned}$$

Hence by generalized Minkowski's inequality

$$\begin{aligned}
\|l_n(\cdot+y) - l_n(\cdot)\|_p &\leq \frac{2}{\pi}\beta_0\left(\frac{2}{n}\right)\int_0^{\pi}\frac{\|\phi_{\cdot+y}(t)-\phi_{\cdot}(t)\|_p}{2\sin\frac{1}{2}t}|K_n(t)|\,\mathrm{d}t\\
&\quad + \frac{1}{\pi}\beta_0\left(\frac{2}{n}\right)\int_0^{\pi}\|\phi_{\cdot+y}(t)-\phi_{\cdot}(t)\|_p\,\mathrm{d}t\\
&\quad + \|f(\cdot+y)-f(\cdot)\|_p\left|\beta_0\left(\frac{2}{n}\right)+\theta(n)-1\right|\\
&= I_n + J_n + K_n, \quad \text{say.} \qquad (5.31)
\end{aligned}$$

We write

$$I_n = \frac{2}{\pi}\beta_0\left(\frac{2}{n}\right)\left[\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right] \frac{\|\phi_{\cdot+y}(t) - \phi_{\cdot}(t)\|_p}{2\sin\frac{1}{2}t}\,|K_n(t)|\,\mathrm{d}t$$
$$= I_n^{(1)} + I_n^{(2)}, \quad \text{say.} \tag{5.32}$$

Using Lemma 5(ii) and (5.17)

$$I_n^{(1)} = O(1)\frac{n^2}{\log n}v(|y|)\int_0^{\pi/n}\frac{w(t)}{tv(t)}\left(\frac{t}{n} + t^{\delta}n^{\delta-2}\right)\mathrm{d}t$$
$$= O(1)v(|y|)\left[\frac{n}{\log n}\int_0^{\pi/n}\frac{w(t)}{v(t)}\mathrm{d}t + \frac{n^{\delta}}{\log n}\int_0^{\pi/n}\frac{w(t)}{v(t)}t^{\delta-1}\mathrm{d}t\right]. \tag{5.33}$$

Using Lemma 5(ii) and (5.16)

$$I_n^{(2)} = O(1)\frac{n^2}{\log n}v(|y|)\left[\int_{\pi/n}^{\pi}\frac{w(t)}{tv(t)}n^{-2}\,\mathrm{d}t + \int_{\pi/n}^{\pi}\frac{w(t)}{tv(t)(t-2/n)}n^{-3}\mathrm{d}t\right]$$
$$= O(1)\frac{v(|y|)}{\log n}\int_{\pi/n}^{\pi}\frac{w(t)}{tv(t)}\mathrm{d}t \tag{5.34}$$

since

$$\left(t - \frac{2}{n}\right)^{-1} = O(n) \ \text{ for } \ t > \frac{\pi}{n}.$$

By Lemma 5(ii)

$$J_n = O(1)\beta_0(2/n)v(|y|)\int_0^{\pi}\frac{w(t)}{v(t)}\mathrm{d}t$$
$$= O(1)\frac{v(|y|)}{\log n}. \tag{5.35}$$

Lastly by the hypothesis $\|f(\cdot + y) - f(\cdot)\|_p = 0(w(|y|))$ and hence by (5.18), we get

$$K_n = O(1)\frac{w(|y|)}{\log n} = O\left(\frac{v(|y|)}{\log n}\right)\frac{w(|y|)}{v(|y|)} = O(1)\frac{v(|y|)}{\log n}. \tag{5.36}$$

Using the estimates (5.33)–(5.36) in (5.31), we obtain

$$\sup_{y\neq 0}\frac{\|l_n(\cdot + y) - l_n(\cdot)\|_p}{v(|y|)}$$
$$= O(1)\left[\frac{n}{\log n}\int_0^{\pi/n}\frac{w(t)}{v(t)}\mathrm{d}t + \frac{n^{\delta}}{\log n}\int_0^{\pi/n}\frac{w(t)}{v(t)}t^{\delta-1}\mathrm{d}t\right]$$
$$+ O\left(\frac{1}{\log n}\right)\int_{\pi/n}^{\pi}\frac{w(t)}{tv(t)}\,\mathrm{d}t + O\left(\frac{1}{\log n}\right). \tag{5.37}$$

By Lemma 5(i) $\|\phi \cdot (t)\|_p = 0(w(t))$. So proceeding in the above lines, we have

$$\begin{aligned}
\|l_n(\cdot)\|_p &\leq \frac{2n^2}{\pi}\beta_0\left(\frac{2}{n}\right)\left[\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right]\|\phi.(t)\|_p \frac{|K_n(t)|}{2\sin t/2}\mathrm{d}t \\
&\quad + \frac{1}{\pi}\beta_0\left(\frac{2}{n}\right)\int_0^{\pi}\|\phi.(t)\|_p\,\mathrm{d}t + \|f\|_p\left|\beta_0\left(\frac{2}{n}\right) + \theta(n) - 1\right| \\
&= O(1)\frac{n}{\log n}\int_0^{\pi/n} w(t)\,\mathrm{d}t \\
&\quad + O(1)\frac{n^{\delta}}{\log n}\int_{\pi/n}^{\pi} w(t)t^{\delta-2}\mathrm{d}t + O\left(\frac{1}{\log n}\right).
\end{aligned} \tag{5.38}$$

Collecting the results of (5.37) and (5.38) and making use of the fact that

$$w(t) = \frac{w(t)}{v(t)}v(t) \leq v(\pi)\frac{w(t)}{v(t)}, \quad 0 < t \leq \pi$$

we get

$$\begin{aligned}
\|l_n(\cdot)\|_p^{(v)} &= \|l_n(\cdot)\|_p + \sup_{y\neq 0}\frac{\|l_n(\cdot + y) - l_n(\cdot)\|_p}{v(|y|)} \\
&= O(1)\left[\frac{n}{\log n}\int_0^{\pi/n}\frac{w(t)}{v(t)}\mathrm{d}t + \frac{n^{\sigma}}{\log n}\int_0^{\pi/n}\frac{w(t)}{v(t)}t^{\delta-1}\mathrm{d}t\right. \\
&\quad \left. + \frac{1}{\log n}\int_{\pi/n}^{\pi}\frac{w(t)}{tv(t)}\mathrm{d}t + \frac{1}{\log n}\right]
\end{aligned}$$

which by Lemma 7 (putting $u(t) = w(t)/v(t)$) is of the order

$$O(1)\frac{1}{\log n}\int_{\pi/n}^{\pi}\frac{w(t)}{tv(t)}\mathrm{d}t.$$

This completes the first part of Theorem 3. The second part follows from Lemma 8 by taking $\eta(t) = w(t)/v(t)$.

COROLLARY 1

Let $0 \leq \beta \leq \alpha < 1$ *and* $f \in H(\alpha, p)$, $p \geq 1$. *Then*

$$\|W_n^*(\cdot; f) - f\|_{\beta,p} = O\left(\frac{1}{\log n}\right)$$

Proof. Put $w(t) = t^{\alpha}$, $v(t) = t^{\beta}$, $0 \leq \beta < \alpha \leq 1$. We observe that

$$\int_{\pi/n}^{\pi}\frac{w(t)}{tv(t)}\mathrm{d}t = \int_{\pi/n}^{\pi} t^{\alpha-\beta-1}\mathrm{d}t = O(1)$$

and hence results follows from first part of Theorem 3.

COROLLARY 2

Let $0 \leq \beta < \alpha \leq 1$ *and let* $\gamma, \delta \in R$. *Suppose*

$$w(t) = \frac{t^{\alpha}}{\left(\log \frac{2\pi}{t}\right)^{\delta}}, \quad v(t) = \frac{t^{\beta}}{\left(\log \frac{2\pi}{t}\right)^{\gamma}}, \quad 0 < t \leq \pi.$$

If $f \in H_p^{(w)}$, $p \geq 1$ *then*

$$\|W_n^*(\cdot; f) - f\|_p^{(v)} = \begin{cases} O\left(\frac{1}{\log n}\right), & \text{when } \alpha > \beta \text{ for all } \gamma, \delta \in R \\ O\left(\frac{1}{(\log n)^{\delta-\gamma}}\right), & \text{when } \alpha = \beta > 0 \text{ and } \gamma - \delta > -1 \\ O\left(\frac{\log \log n}{n}\right), & \text{when } \alpha = \beta \text{ and } \gamma - \delta = -1 \\ O\left(\frac{1}{\log n}\right), & \text{when } \alpha = \beta, \gamma - \delta < -1 \end{cases}$$

Proof. Since

$$\int_{\pi/n}^{\pi} \frac{w(t)}{t v(t)} \mathrm{d}t = \int_{\pi/n}^{\pi} t^{\alpha-\beta-1} \left(\log \frac{2\pi}{t}\right)^{\gamma-\delta} \mathrm{d}t$$
$$= \begin{cases} O(1), & \alpha > \beta \text{ and } \gamma, \delta \in R \\ O(1)(\log, n)^{\gamma-\delta+1}, & \alpha = \beta, \gamma - \delta > -1 \\ O(1) \log \log n, & \alpha = \beta, \gamma - \delta = -1 \\ O(1), & \alpha = \beta, \gamma - \delta < -1 \end{cases}$$

the result follows from the first part of Theorem 3.

Acknowledgement

The authors are grateful to the referee for his valuable suggestions and criticisms which led to the improvement of the paper.

References

[1] Das G, Tauberian theorems for absolute Nörlund summability, *Proc. London Math. Soc.* **19**(3) (1969) 357–384

[2] Das G, Ghosh T and Ray B K, Degree of approximation of functions by their Fourier series in the generalized Hölder metric, *Proc. Indian Acad. Sci., (Math. Sci.),* **106(2)** (1996) 139–153

[3] Hardy G H, Divergent Series (Oxford: Clarendon Press) (1949)

[4] Leindler L, Generalizations of Prössdorf's theorems, *Studia Scientiarum Mathematicarum Hungarica* **14** (1979) 431–439

[5] Mazhar S M, Approximation by logarithmic means of a Fourier series, in: Approximation Theory VI (eds) C K Chui, L L Schumaker and J D Ward (Academic Press) **2** (1989) 421–424

[6] Mazhar S M and Totik V, Approximation of continuous functions by T-means of Fourier series, *J. Approximation Theory*, **60** (1990) 174–182

[7] Prössdorf S, Zur Konvergenz der Fourier Reihen Hölder Steliger Funktionen, *Math. Nachr.* **69** (1975) 7–14

[8] Totik V, On the strong approximation by the (C, α) means of Fourier series I, *Anal. Math.* **6** (1980) 57–85

[9] Totik V, Strong approximation by the de la valle Poussin and Abel means of Fourier series, *J. Indian Math. Soc.* **45** (1981) 85–108

[10] Zygmund A, Trigonometric Series, (New York: Cambridge University Press) (1968) 2nd edn, Vol. I