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Two-dimensional weak pseudomanifolds on eight vertices

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Abstract. We explicitly determine all the two-dimensional weak pseudomanifolds on 8 vertices. We prove that there are (up to isomorphism) exactly 95 such weak pseudomanifolds, 44 of which are combinatorial 2-manifolds. These 95 weak pseudomanifolds triangulate 16 topological spaces. As a consequence, we prove that there are exactly three 8-vertex two-dimensional orientable pseudomanifolds which allow degree three maps to the 4-vertex 2-sphere.

Keywords. Two-dimensional complexes; pseudomanifolds; degree of a map.

1. Introduction

Recall that a *simplicial complex* (in short, *complex*) is a collection of non-empty finite sets such that every non-empty subset of an element is also an element. For $i \geq 0$, the elements of size $i+1$ are called the i*-simplices* of the complex. For $i = 1, 2$, the i-simplices are also called the *edges* and *triangles* of the complex, respectively. For a complex X, the maximum of k such that X has a k-simplex is called the *dimension* of X. The union of all the simplices of a complex X is called the *vertex-set* of X and is denoted by $V(X)$. Elements of $V(X)$ are called *vertices* of X. A complex X is called *finite* if $V(X)$ is a finite set. A k-simplex $\{v_0, \ldots, v_k\}$ of a complex is also denoted by $v_0 \cdots v_k$.

If X_1 and X_2 are two complexes, then a simplicial map from X_1 to X_2 is a map $\varphi : V(X_1) \to V(X_2)$ such that $\sigma \in X_1$ implies $\varphi(\sigma) \in X_2$. A bijection $\pi : V(X_1) \to V(X_2)$ is called an *isomorphism* if both π and π^{-1} are simplicial. Two complexes X_1, X_2 are called (simplicially) *isomorphic* when such an isomorphism exists. We identify two complexes if they are isomorphic. An isomorphism from a complex X to itself is called an *automorphism* of X. All the automorphisms of X form a group, which is denoted by $\mathrm{Aut}(X)$. Two simplicial maps $f, g\colon X_1 \to X_2$ are said to be *equivalent* (denoted by $f \cong g$) if there exist $\varphi \in \mathrm{Aut}(X_1)$ and $\psi \in \mathrm{Aut}(X_2)$ such that $\psi \circ f \circ \varphi = g$.

A d-dimensional simplicial complex X is called a d-dimensional *weak pseudomanifold* if each simplex of X is contained in a d-simplex of X and each $(d-1)$-simplex of X is contained in exactly two d-simplices of X. A d-dimensional weak pseudomanifold X is called a *pseudomanifold* (without boundary) if for any pair σ, λ of d-simplices of X, there exists a sequence $\tau_1, \ldots, \tau_n$ of d-simplices of X, such that $\sigma = \tau_1$, $\lambda = \tau_n$ and $\tau_i \cap \tau_{i+1}$ is a $(d-1)$-simplex of X for $1 \leq i \leq n-1$. (P_1 and P_2, given in §2, are pseudomanifolds but Q_1 and Q_2 are not.)

A simplicial complex is usually thought of as a prescription for the construction of a topological space by pasting together geometric simplices (see §3 for finite complexes,

[8] and [11] for the general case). The space thus obtained from a complex K is called the *geometric carrier* of K and is denoted by $|K|$. We also say that K triangulates $|K|$.

For any set V on $d+2$ (≥ 2) elements, let K be the simplicial complex whose simplices are all the non-empty proper subsets of V. Then $|K|$ is homeomorphic to the sphere S^d. This complex is called the *standard d-sphere* and is denoted by $S^d_{d+2}(V)$ or simply by S^d_{d+2}. A finite complex X is called a *combinatorial d-sphere*, if $|X|$ is PL-homeomorphic to $|S^d_{d+2}|$ [10]. Clearly, a finite one-dimensional complex is a combinatorial 1-sphere if and only if it is a pseudomanifold. For $n \geq 3$, the combinatorial 1-sphere with n vertices is unique and is denoted by C_n. The complex C_n is also called an *n-cycle*. An n-cycle with edges $v_1v_2, \ldots, v_{n-1}v_n, v_nv_1$ is also denoted by $C_n(v_1, \ldots, v_n)$.

If v is a vertex of a simplicial complex X, then the *link* of v in X, denoted by $\mathrm{Lk}_X(v)$, is the complex whose simplices are those simplices τ of X such that $v \not\in \tau$ and $\{v\} \cup \tau$ is a simplex of X. The number of vertices in the link of v is called the *degree* of v and is denoted by $\deg(v)$. Clearly, the link of a vertex in a d-dimensional weak pseudomanifold is a $(d-1)$-dimensional weak pseudomanifold.

A finite simplicial complex X is called a *combinatorial d-manifold* if $|X|$ is a d-dimensional PL-manifold (without boundary), i.e., $\mathrm{Lk}_X(v)$ is a combinatorial $(d-1)$-sphere for each vertex v in X [7,10]. So, X is a combinatorial 2-manifold if the link of each vertex is a cycle. We also know (e.g., see [7]) that a finite simplicial complex K is a combinatorial 2-manifold if and only if $|K|$ is a two-dimensional topological manifold.

A vertex of a finite two-dimensional weak pseudomanifold is called *singular* if its link is not a cycle (and hence consists of more than one cycle). So, a two-dimensional weak pseudomanifold is not a combinatorial manifold if and only if it contains a singular vertex. (In each of P_1, P_2, Q_1 and Q_2, 7 is a singular vertex.)

A combinatorial 2-manifold X is called *d-equivelar* if each vertex of X has degree d. A combinatorial 2-manifold is called *equivelar* if it is d-equivelar for some d.

If the number of i-simplices of a d-dimensional finite complex X is $f_i(X)$ ($0 \leq i \leq d$), then the number $\chi(X) := \sum_{i=0}^{d}(-1)^i f_i(X)$ is called the *Euler characteristic* of X.

If K is a d-dimensional oriented pseudomanifold, then $[z_K]$ generates $H_d(K, \mathbb{Z})$, where $z_K := \sum_{\sigma^d \in K} 1 \cdot \sigma^d$ (summation is taken over all the positively oriented d-simplices). Let K and L be two oriented d-dimensional pseudomanifolds. If $\varphi\colon K \to L$ is a simplicial map then $\varphi^*_d\colon H_d(K, \mathbb{Z}) \to H_d(L, \mathbb{Z})$ is a homomorphism and hence there exists $m \in \mathbb{Z}$ such that $\varphi^*_d([z_K]) = m[z_L]$. This m is called the *degree* of φ and is denoted by $\deg(\varphi)$ [11].

Let K be a two-dimensional pseudomanifold and $\varphi\colon K \to S^2_4$ be a simplicial map. If $\varphi(u) \neq \varphi(v)$ for each edge uv of K then φ is called a *4-coloring* [12].

It is known (e.g., see [3,5]) that if the number of vertices of a two-dimensional weak pseudomanifold M is at most 6 then M is a combinatorial 2-manifold and M is isomorphic to $S_1, \ldots, S_4$ or R_1 (given in §2).

In [5], we have seen that there are exactly nine 7-vertex combinatorial 2-manifolds and four 7-vertex two-dimensional weak pseudomanifolds which are not combinatorial 2-manifolds. Among the four non-manifolds two are pseudomanifolds, which triangulate the *pinched sphere* (the space obtained by identifying two points of S^2).

In [6], we have determined all the equivelar combinatorial 2-manifolds on at most 11 vertices. There are 27 such equivelar combinatorial 2-manifolds.

Altshuler and Steinberg [2] showed that there are fourteen 8-vertex combinatorial 2-spheres. Cervone [4] showed that there are exactly six 8-vertex combinatorial 2-manifolds, which triangulate the Klein bottle. It is known (e.g., see [7,9]) that there does not exist any 8-vertex combinatorial 2-manifold of Euler characteristic -1. Here, we classify all the two-dimensional weak pseudomanifolds on 8 vertices. More explicitly, we prove:

Theorem 1.1. *There are exactly* 44 *distinct combinatorial* 2*-manifolds on* 8 *vertices, namely,* $S_{10}, \dots, S_{23}$, $R_5, \dots, R_{20}$, $T_2, \dots, T_8$, $K_1, \dots, K_6$ *and* D (*given in* §2).

Theorem 1.2. *There are exactly* 51 *distinct* 8*-vertex two-dimensional weak pseudomanifolds which are not combinatorial* 2*-manifolds, namely,* $P_3, \dots, P_{39}$, $Q_3, \dots, Q_{16}$ (*given in* §2).

COROLLARY 1.3

Let M *be a two-dimensional weak pseudomanifold on* n (≤ 8) *vertices.*

(i) *If* $|M|$ *is a manifold then* $|M|$ *is homeomorphic to the* 2*-sphere* (S^2)*, the real projective plane* $(\mathbb{R}P^2)$*, the torus* $(S^1 \times S^1)$*, the Klein bottle* (K) *or the space consisting of two disjoint* 2*-spheres.*

(ii) *If* $|M|$ *is not a manifold then* $|M|$ *is homeomorphic to the pinched sphere* (P), $\mathbb{R}P^2\#P$, $P\#P$, $\mathbb{R}P^2\#P\#P$, $K\#P$, $(S^1 \times S^1)\#P$*, the union of two* S^2*'s having one, two, three or four points in common or the union of* S^2 *and* $\mathbb{R}P^2$ *having three points in common* (*given at the end of* §2). (*Here,* $A\#B$ *denotes the connected sum of* A *and* B).

If $\varphi\colon K \to S^2_4$ is a simplicial map, where K is an oriented 8-vertex two-dimensional pseudomanifold, then $f_2(K) \leq 18$ and hence $\deg(\varphi) \leq 4$. Here we prove:

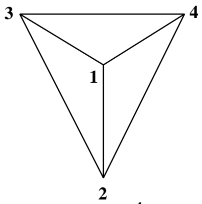
Theorem 1.4. *Let* $\varphi\colon K^2_n \to S^2_4$ *be a simplicial map, where* K^2_n *is a two-dimensional oriented pseudomanifold on* n *vertices. Let* f, g *and* h *be as in Example* 2.1. *If* $n \leq 8$ *then* $\deg(\varphi) \leq 3$ (*and hence* ≥ -3). *Equality is attained here if and only if* φ *is equivalent to* f, g *or* h.

Remark 1.5. Observe that $P_3, \dots, P_{39}$ (in Theorem 1.2) are pseudomanifolds, whereas $Q_3, \dots, Q_{16}$ are not pseudomanifolds. Among the pseudomanifolds, $P_3, \dots, P_{20}$, $P_{28}, \dots, P_{36}$ and P_{39} are orientable and $P_{21}, \dots, P_{27}$, P_{37} and P_{38} are non-orientable.

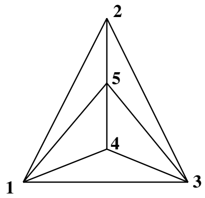
Remark 1.6. If M is an 8-vertex two-dimensional weak pseudomanifold, then it is easy to see that $\chi(M)$ lies between -1 and 4. If M is a combinatorial 2-manifold then, from Theorem 1.1, $\chi(M)$ is 0, 1, 2 or 4. However, by Theorem 1.2, there exist weak pseudomanifolds with Euler characteristic $-1, \dots, 3$.

Remark 1.7. Theorem 1.1, Theorem 1.2 and Proposition 3.1 together with the results in [3], which determine all the d-dimensional weak pseudomanifolds on at most $d + 4$ vertices, classify all the d-dimensional $(d \neq 3)$ weak pseudomanifolds on less than or equal to 8 vertices. Moreover, Theorem 1.1 together with Proposition 3.1, the results in [1], which classify all the combinatorial 3-manifolds on at most 8 vertices and the results in [3] classify all the combinatorial manifolds on less than or equal to 8 vertices.

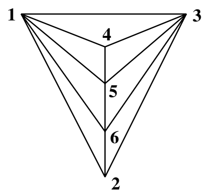
In §2 we present all the two-dimensional weak pseudomanifolds on at most 8 vertices. In §3 we give some definitions, constructions and results which we shall need later. In §4 we consider combinatorial manifolds and prove Theorem 1.1. In §5 we consider weak pseudomanifolds which are not combinatorial manifolds and prove Theorem 1.2. In §6 we prove Corollary 1.3 and Theorem 1.4.



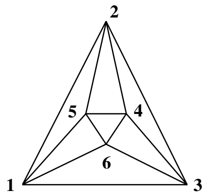
S_1 (3^4)



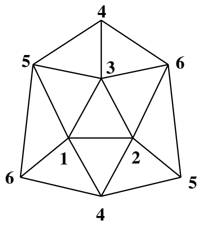
S_2 $(4^3.3^2)$



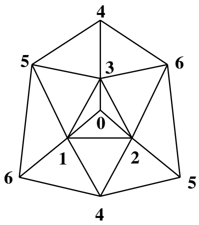
S_3 $(5^2.4^2.3^2)$



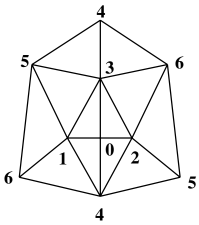
S_4 (4^6)



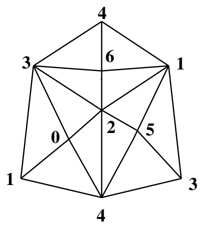
R_1 (5^6)



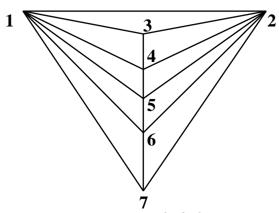
R_2 $(6.5^3.3^3)$



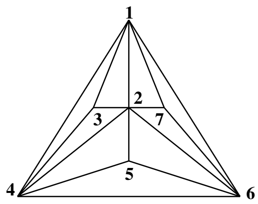
R_3 $(6.5^2.4^4)$



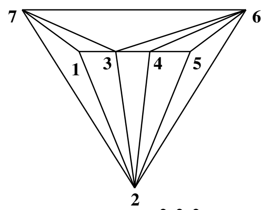
R_4 $(6^4.4^3)$



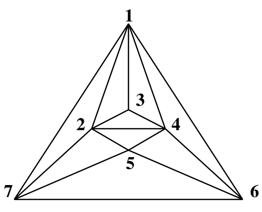
S_5 $(6^2.4^3.3^2)$



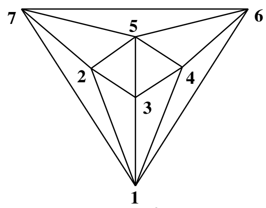
S_6 $(6.5^3.3^3)$



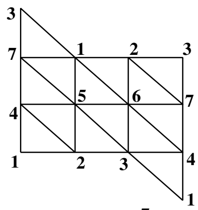
S_7 $(6.5^2.4^2.3^2)$



S_8 $(5^3.4^3.3)$



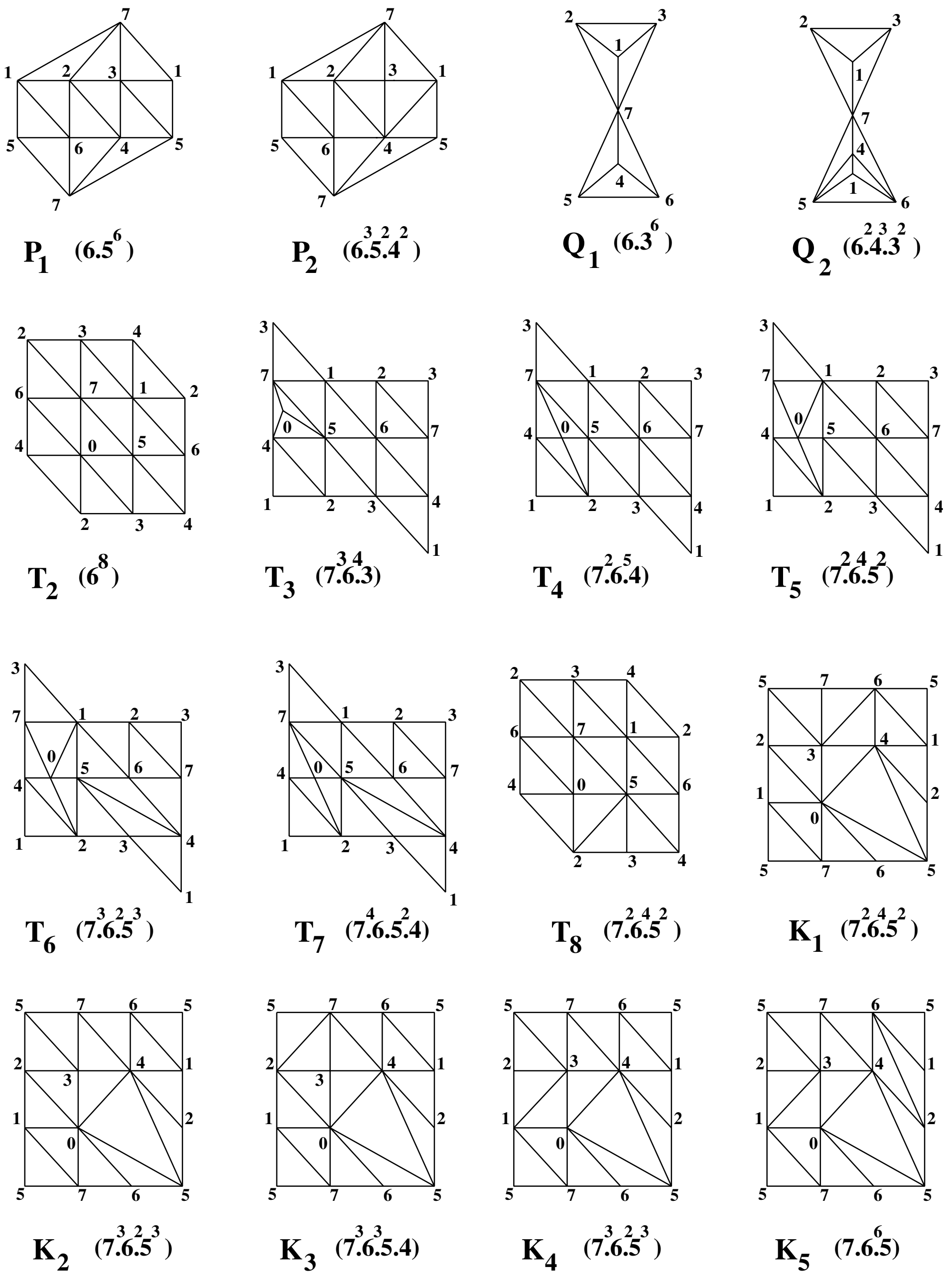
S_9 $(5^2.4^5)$



T_1 (6^7)

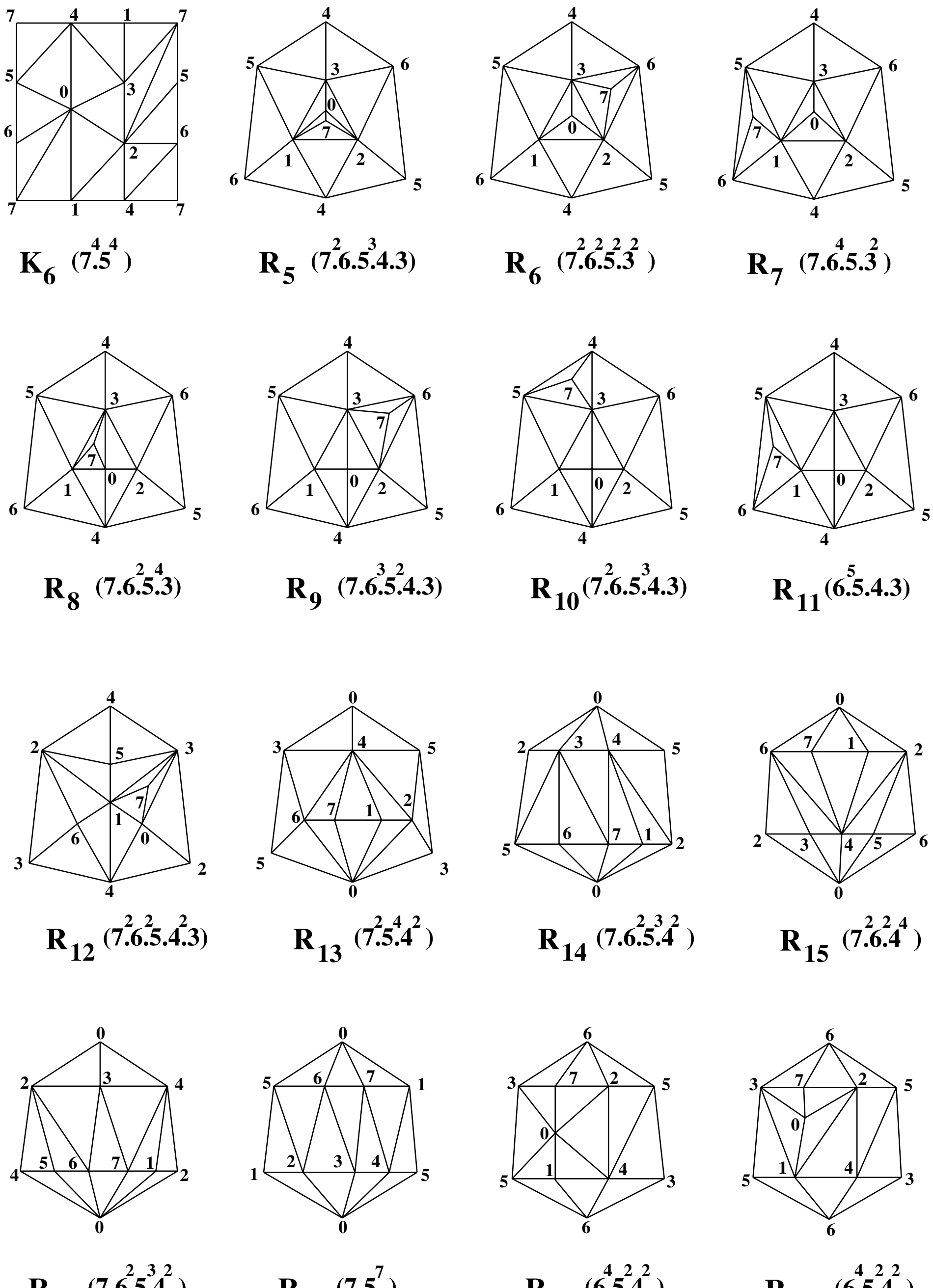
2. Examples

In Theorems 1.1 and 1.2 we have stated that there are 95 two-dimensional weak pseudomanifolds on 8 vertices. In this section we present all these 95 weak pseudomanifolds. We also present all the 18 two-dimensional weak pseudomanifolds on less than or equal to 7 vertices. The degree sequences are presented parenthetically below the figures. For $0 \leq i \leq 7$, i in the figures represents the vertex v_i. At the end of this section, we present

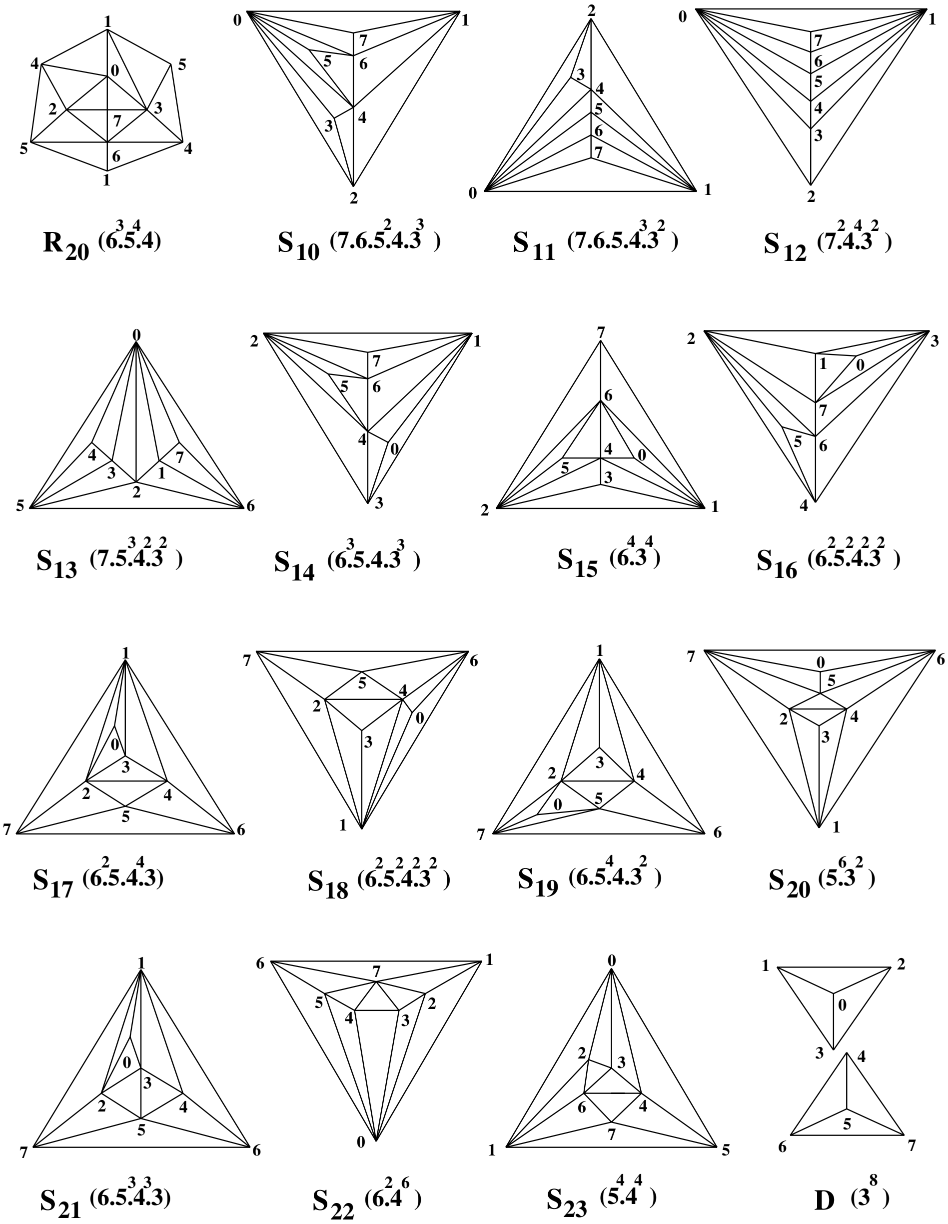


the geometric carriers of all the weak pseudomanifolds on 8 vertices. At the beginning we present three degree 3 maps (which we have mentioned in Theorem 1.4) to the 4-vertex 2-sphere.

Example 2.1. Let $S_1 = S^2_4(\{a, b, c, d\})$ with the positively oriented 2-simplices abc, acd, adb, bdc. Let S_{15}, P_{34} and P_{35} be as given below:

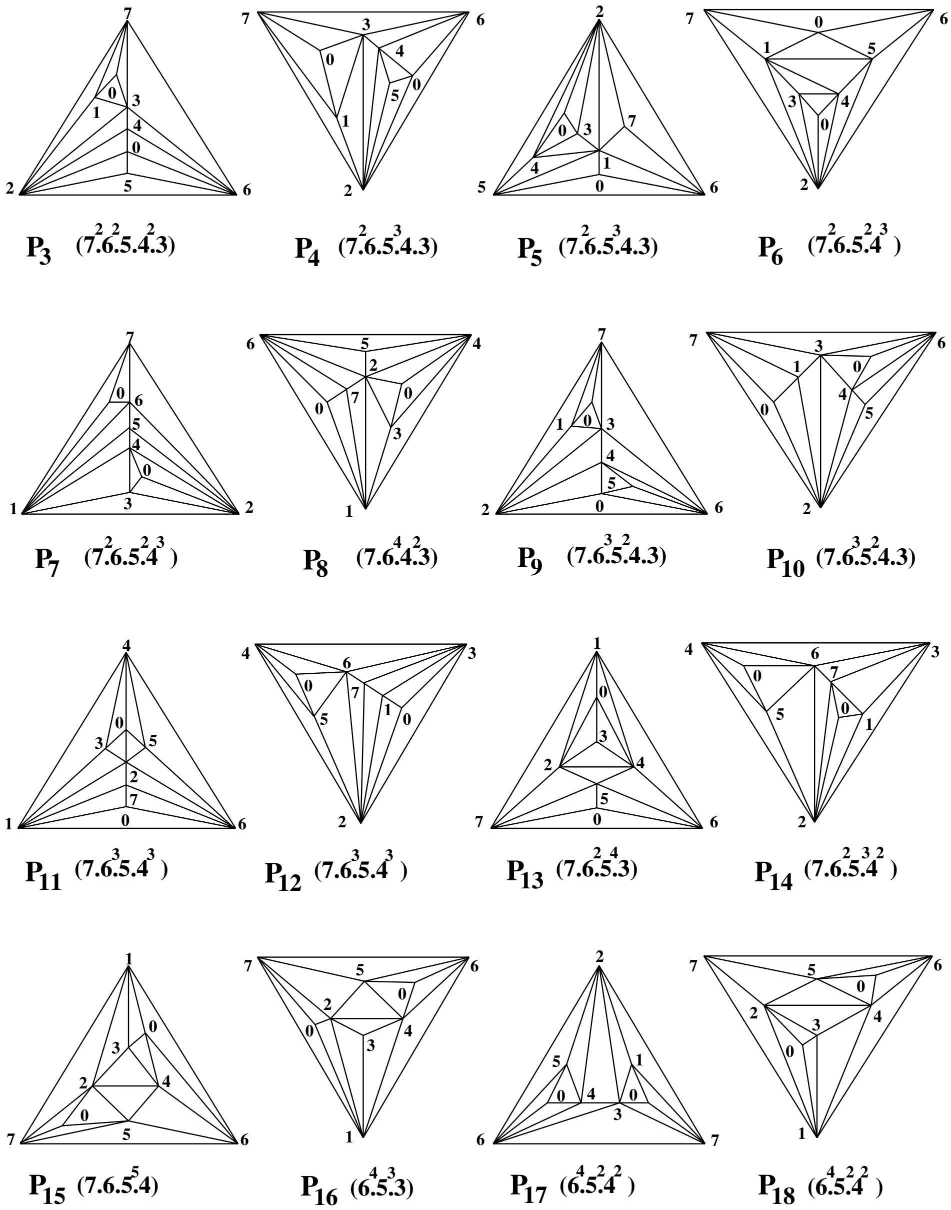


(a) Consider the orientation on S_{15} given by the positively oriented 2-simplices 176, 160, 064, 104, 143, 132, 234, 524, 546, 562, 267, 127. Let $f\colon S_{15} \to S_1$ be the simplicial map given by $f(1) = f(5) = a$, $f(0) = f(2) = b$, $f(4) = f(7) = c$ and $f(3) = f(6) = d$.



(b) Consider the orientation on P_{34} given by the positively oriented 2-simplices 475, 467, 765, 056, 015, 061, 162, 426, 024, 043, 453, 135, 173, 037, 072, 127. Let $g\colon P_{34} \to S_1$ be the simplicial map given by $g(0) = g(4) = a$, $g(1) = g(7) = b$, $g(2) = g(5) = c$ and $g(3) = g(6) = d$.

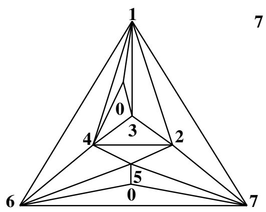
(c) Consider the orientation on P_{35} given by the positively oriented 2-simplices 523, 537, 572, 274, 704, 760, 673, 163, 130, 203, 102, 124, 146, 564, 506, 540. Let $h\colon P_{35} \to S_1$ be the simplicial map given by $h(1) = h(5) = a$, $h(2) = h(6) = b$, $h(3) = h(4) = c$ and $h(0) = h(7) = d$.



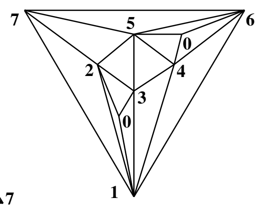
Then $\deg(f) = \deg(g) = \deg(h) = 3$. The map f is a 4-coloring but g and h are not.

3. Preliminaries

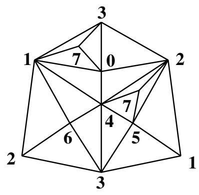
For a finite simplicial complex X, if n_i (> 0) is the number of vertices of degree d_i and $d_1 > d_2 > \cdots$, then $d_1^{n_1}.\ldots.d_k^{n_k}$ is called the *degree sequence* of X, where $\sum_{i=1}^{k} n_i$ is equal to the number of vertices of X.



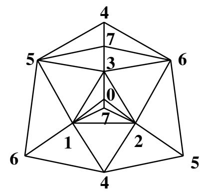
$\mathbf{P}_{19}$ $(6^3.5^4.4)$



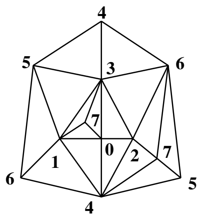
$\mathbf{P}_{20}$ $(6^3.5^4.4)$



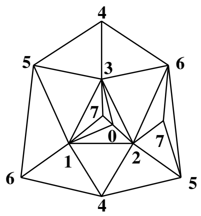
$\mathbf{P}_{21}$ $(7^4.6.5^2.4)$



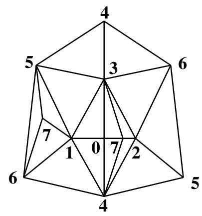
$\mathbf{P}_{22}$ $(7^3.6^3.5.4)$



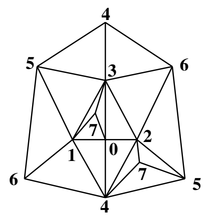
$\mathbf{P}_{23}$ $(7^3.6^2.5^3)$



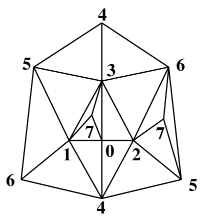
$\mathbf{P}_{24}$ $(7^3.6^3.5.4)$



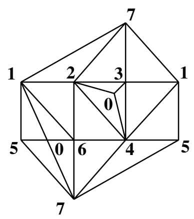
$\mathbf{P}_{25}$ $(7^3.6^3.5.4)$



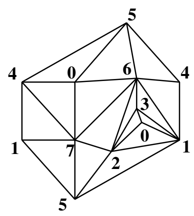
$\mathbf{P}_{26}$ $(7^2.6^4.5^2)$



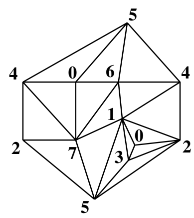
$\mathbf{P}_{27}$ $(7.6^6.5)$



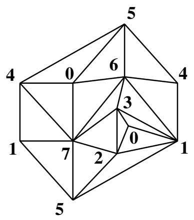
$\mathbf{P}_{28}$ $(7^4.6.5^2.4)$



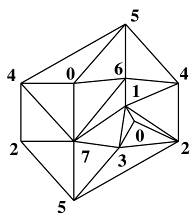
$\mathbf{P}_{29}$ $(7^3.6^3.5.4)$



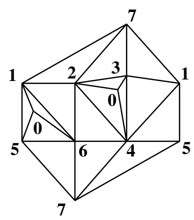
$\mathbf{P}_{30}$ $(7^3.6^3.5.4)$



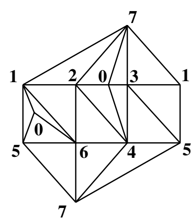
$\mathbf{P}_{31}$ $(7^3.6^2.5^3)$



$\mathbf{P}_{32}$ $(7^2.6^4.5^2)$



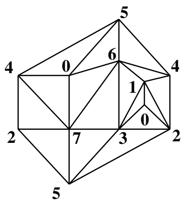
$\mathbf{P}_{33}$ $(7^2.6^4.5^2)$



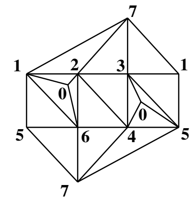
$\mathbf{P}_{34}$ $(7^2.6^4.5^2)$

If m_i (> 0) is the number of singular vertices of degree c_i in a two-dimensional weak pseudomanifold X and $c_1 > c_2 > \cdots$ then $c_1^{m_1}.\ldots.c_k^{m_k}$ is called the *singular degree sequence* of X, where $\sum_{i=1}^{k} m_i$ is equal to the number of singular vertices of X.

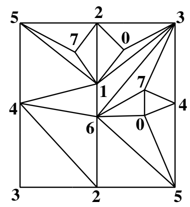
If X is a finite simplicial complex then one defines a geometric realization of X as follows: Let $V(X) = \{v_1, \ldots, v_n\}$. We choose a set of n points $\{x_1, \ldots, x_n\}$ in $\mathbb{R}^N$ (for some N) in such a way that a subset $S = \{x_{j_1} \ldots, x_{j_{i+1}}\}$ of $i + 1$ points is affinely independent if $\sigma = v_{j_1} \cdots v_{j_{i+1}}$ is a simplex of X. The convex set spanned by S is called the *geometric carrier* of σ or the *geometric simplex* corresponding to σ and denoted



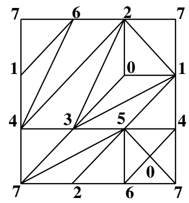
P_{35} $(7.6^6.5)$



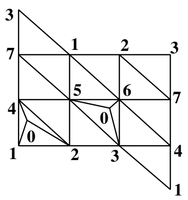
P_{36} (6^8)



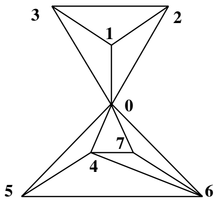
P_{37} $(7^2.6^6)$



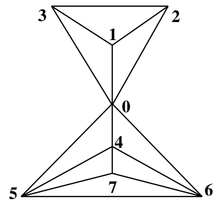
P_{38} $(7^2.6^6)$



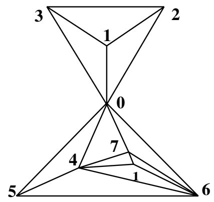
P_{39} $(7^2.6^6)$



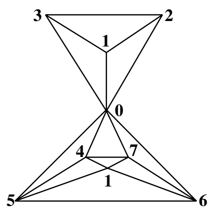
Q_3 $(7.4^2.3^5)$



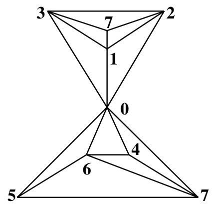
Q_4 $(6.4^3.3^4)$



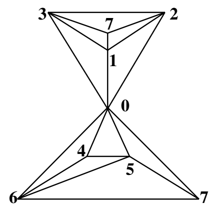
Q_5 $(7.6.5^2.4.3^3)$



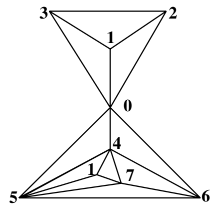
Q_6 $(7^2.4^4.3^2)$



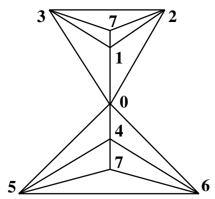
Q_7 $(7^2.4^4.3^2)$



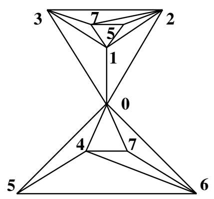
Q_8 $(7.6.4^5.3)$



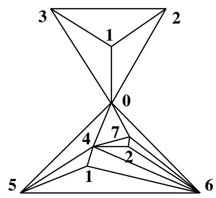
Q_9 $(6^2.5^2.4^2.3^2)$



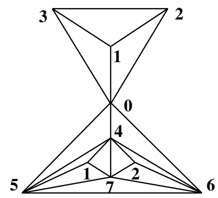
Q_{10} $(6^2.4^6)$



Q_{11} $(7^2.6.5^2.4^3)$

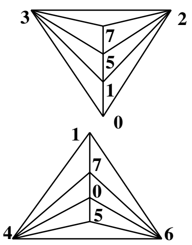


Q_{12} $(7.6^4.4^2.3)$

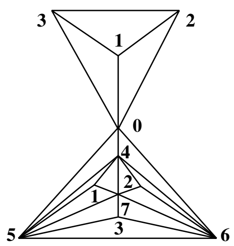


Q_{13} $(6^4.5^3.3)$

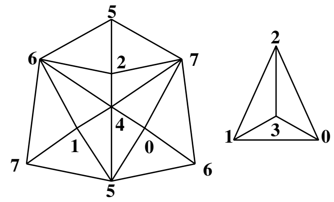
by $|\sigma|$. Since X is finite we can choose N so that $\sigma \cap \gamma = \emptyset$ implies $|\sigma| \cap |\gamma| = \emptyset$. The set $\mathcal{X} := \{|\sigma| \ : \ \sigma \in X, \sigma \cap \gamma = \emptyset \Rightarrow |\sigma| \cap |\gamma| = \emptyset\}$ is called a *geometric simplicial complex* corresponding to X or a *geometric realization* of X. The topological space $|X| := \cup_{\sigma \in X} |\sigma|$ is called a *geometric carrier* of X. Clearly, if two finite complexes have a common geometric realization, then they are isomorphic and isomorphic finite complexes have homeomorphic geometric carriers [8]. We identify a complex with its geometric realization.



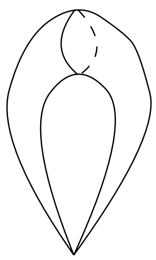
Q_{14} $(7^4.5^4)$



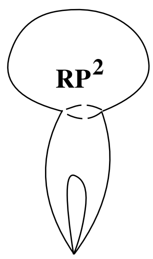
Q_{15} (6^8)



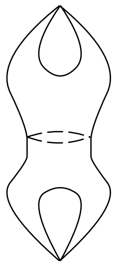
Q_{16} $(7^3.6^4.3)$



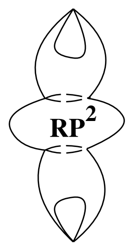
P



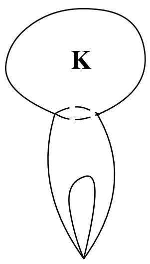
$\mathbf{RP^2 \# P}$



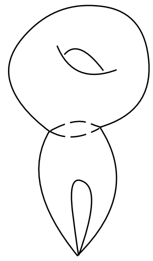
P # P



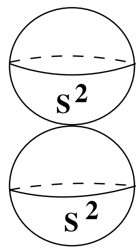
$\mathbf{RP^2 \# P \# P}$

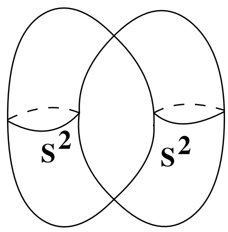


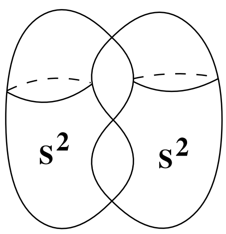
K # P

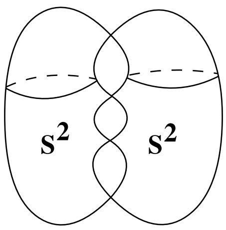


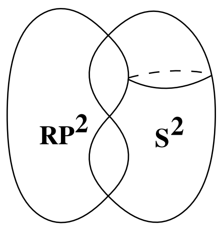
$\mathbf{(S^1 x S^1) \# P}$











If M is an n-vertex two-dimensional weak pseudomanifold with f_1 edges and f_2 triangles then $3f_2 = 2f_1$. Therefore,

$$\chi(M) = n - f_1 + f_2 = n - \frac{f_1}{3} = n - \frac{f_2}{2}. \tag{1}$$

In [5], we have seen the following:

PROPOSITION 3.1

There are exactly 13 *distinct two-dimensional weak pseudomanifolds on* 7 *vertices, namely,* T_1, R_2, R_3, R_4, $S_5, \ldots, S_9$, P_1, P_2, Q_1 *and* Q_2.

Altshuler and Steinberg [2] showed the following:

PROPOSITION 3.2

There are exactly 14 *distinct combinatorial* 2*-spheres on 8 vertices, namely,* $S_{10}, \ldots, S_{23}$.

If X_1, X_2 are two simplicial complexes with disjoint vertex sets, then their *join* $X_1 * X_2$ is the complex whose simplices are those of X_1 and X_2 and the unions of simplices of X_1 with simplices of X_2. If both X_1 and X_2 are pseudomanifolds then so is $X_1 * X_2$ [3]. Observe that $S^0_2(\{a,b\}) * S^0_2(\{c,d\}) = C_4(a,c,b,d)$.

If M is a two-dimensional simplicial complex and $\tau = abc$ is a 2-simplex then $(M\backslash\tau) \cup (\partial\tau * v)$ denotes the two-dimensional complex whose 2-simplices are abv, acv, bcv and those of M other than τ, where $v \notin V(M)$. This complex is said to be obtained from M by *starring* the vertex v in τ. Observe that R_2 is obtained from R_1 by starring the vertex 0 in 123. Similarly, if ab is an edge (contained in abc and abd) and $v \notin V(M)$ then $(M!\backslash\{ab\}) \cup (C_4(a,c,b,d) * v)$ denotes the two-dimensional complex whose 2-simplices are acv, adv, bcv, bdv and those of M other than abc and abd. This complex is said to be obtained from M by *starring* the vertex v in the edge ab. The complex R_3 is obtained from R_1 by starring the vertex 0 in the edge 12.

Let M, N be two simplicial complexes with $\sigma_1, \ldots, \sigma_m \in M$ and $\tau_1, \ldots, \tau_m \in N$. We say $(M, \sigma_1, \ldots, \sigma_m)$ and $(N, \tau_1, \ldots, \tau_m)$ are isomorphic (denoted by $(M, \sigma_1, \ldots, \sigma_m) \cong (N, \tau_1, \ldots, \tau_m)$) if there exists an isomorphism φ from M to N such that $\varphi(\sigma_i) = \tau_i$ for $1 \le i \le m$.

Let M be a d-dimensional weak pseudomanifold. Let u and v be two distinct vertices of M such that uv is not an edge. If $V(\mathrm{Lk}(u)) \cap V(\mathrm{Lk}(v)) = \emptyset$ then define the complex $\widetilde{M} = \{\tau \in M : u \notin \tau, v \notin \tau\} \cup \{(\tau\backslash\{u\}) \cup \{w\} : u \in \tau\} \cup \{(\tau\backslash\{v\}) \cup \{w\} : v \in \tau\}$. This $\widetilde{M}$ is called the simplicial complex obtained from M by *identifying* u and v (to a new vertex w). Observe that P_1 is obtained from S_{20} by identifying vertices 0 and 3 of S_{20} and P_2 is is obtained from S_{16} by identifying vertices 0 and 5 of S_{16}.

If $\tau_1 = abc$ and $\tau_2 = xyz$ are two disjoint 2-simplices of a two-dimensional simplicial complex M and $v \notin V(M)$ then $(M\backslash\{\tau_1, \tau_2\}) \cup ((\partial\tau_1 \cup \partial\tau_2) * v)$ denotes the complex whose 2-simplices are abv, acv, bcv, xyv, yzv, xzv and those of M other than τ_1 and τ_2. Observe that we get P_{33} from P_2 and P_{39} from T_1 by this process.

From these definitions one gets the following:

PROPOSITION 3.3

Let $\widetilde{M}$ *be obtained from* M *by identifying two vertices* u *and* v.

(a) *If* M *is a* (*weak*) *pseudomanifold, then so is* $\widetilde{M}$.
(b) *If* $\widetilde{N}$ *is obtained from* N *by identifying two vertices* u_1 *and* v_1 *and* $(N, \{u_1\}, \{v_1\}) \cong (M, \{u\}, \{v\})$, *then* $\widetilde{N} \cong \widetilde{M}$.
(c) *If* M *is a two-dimensional pseudomanifold and both* u *and* v *are non-singular then* $|\widetilde{M}|$ *is homeomorphic to the connected sum of* $|M|$ *and the pinched sphere.*

PROPOSITION 3.4

Let M be a two-dimensional simplicial complex.

(a) *Let τ be an i-simplex ($1 \leq i \leq 2$) and $\widetilde{M}$ be obtained from M by starring a vertex in τ. If M is a weak pseudomanifold, pseudomanifold or combinatorial 2-manifold then so is $\widetilde{M}$ with the same geometric carrier.*
(b) *Let τ_1, τ_2 are disjoint 2-simplices of M and $u, v, w \notin V(M)$. Let $\widehat{M} := (M \backslash \{\tau_1, \tau_2\}) \cup ((\partial\tau_1 \cup \partial\tau_2) * w)$. Let N be the complex obtained from M by starring u in τ_1 and v in τ_2. Let $\widetilde{N}$ be obtained from N by identifying u and v. If M is a (weak) pseudomanifold then so is $\widetilde{N}$ and $\widetilde{N} \cong \widehat{M}$ (and hence $|\widehat{M}|$ is homeomorphic to the connected sum of $|M|$ and the pinched sphere whenever M is a pseudomanifold).*

PROPOSITION 3.5

Let M_1 and M_2 be two-dimensional weak pseudomanifolds.

(a) *Let $\widetilde{M}_j$ be obtained from M_j by starring a vertex on an i-simplex ($1 \leq i \leq 2$) σ_j for $j = 1, 2$. If $(M_1, \sigma_1) \cong (M_2, \sigma_2)$, then $\widetilde{M}_1 \cong \widetilde{M}_2$.*
(b) *If $(M_1, \sigma_1, \tau_1) \cong (M_2, \sigma_2, \tau_2)$ and $u_1, u_2 \notin V(M_1) \cup V(M_2)$, where σ_j, τ_j are disjoint 2-simplices of M_j for $j = 1, 2$, then $(M_1 \backslash \{\sigma_1, \tau_1\}) \cup ((\partial\sigma_1 \cup \partial\tau_1) * u_1) \cong (M_2 \backslash \{\sigma_2, \tau_2\}) \cup ((\partial\sigma_2 \cup \partial\tau_2) * u_2)$.*

PROPOSITION 3.6

Let $\varphi\colon K \to L$ be a simplicial map of degree $d > 0$, where K and L are two-dimensional oriented pseudomanifolds. For a vertex v of L, let $S_v := \{\sigma \in K : \varphi(\sigma)$ is a 2-simplex containing $v\}$.

(a) *If σ is a 2-simplex of L then $\varphi^{-1}(\sigma)$ contains at least d simplices.*
(b) *If for some 2-simplex σ of L, $\varphi^{-1}(\sigma)$ contains d or $d+1$ 2-simplices, say $\sigma_1, \ldots, \sigma_d$ (or σ_{d+1}), then σ_i and σ_j have at most one vertex in common for $i \neq j$.*
(c) *If for some vertex v of degree c of L, S_v contains the 2-simplices $\tau_i \cup \{u_i\}$, $\tau_i \cup \{v_i\}$, where $\varphi(u_i) = \varphi(v_i) = v$ for $1 \leq i \leq p$, then $\#(S_v) \geq cd + 2p$.*

Proof. (a) follows from the definition of the degree of a simplicial map.

If possible let $\varphi(uvx) = \varphi(uvy) = \sigma$. If $\varphi_2\colon C_2(K) \to C_2(L)$ is the homomorphism induced by φ then, for any orientations of K and L, $\varphi_2(+uvx) = -\varphi_2(+uvy)$ in $C_2(L)$. So, if m is the coefficient of $+\sigma$ in $\varphi_2(C_2(K))$ then $|m| \leq d+1-2$ and hence $\deg(\varphi) \leq d-1$, a contradiction. This proves (b).

By the same argument as in (b), $\varphi_2(+\tau_i \cup \{u_i\}) = -\varphi_2(+\tau_i \cup \{v_i\})$, for $1 \leq i \leq p$. Therefore, for each 2-simplex σ containing v (as the degree of φ is d) $\#(\varphi_2^{-1}(\sigma) \backslash \{\tau_i \cup \{u_i\}, \tau_i \cup \{v_i\} : 1 \leq i \leq p\}) \geq d$. This proves (c). □

For a simplicial map $\varphi\colon K \to L$, a d-simplex σ is said to be *collapsing* if $\varphi(\sigma)$ is not a d-simplex. Let $\varphi\colon K \to S^2_4$ be a simplicial map, where K is an m-vertex two-dimensional weak pseudomanifold and S^2_4 is the standard 2-sphere (with 2-simplices $\sigma_1, \ldots, \sigma_4$). The map φ is said to be of *type* (n_1, n_2, n_3, n_4) if n_i is the number of triangles (in K) with image σ_i ($1 \leq i \leq 4$) and $n_1 \geq n_2 \geq n_3 \geq n_4$.

4. Proof of Theorem 1.1

Throughout this section M is an 8-vertex combinatorial 2-manifold.

The first lemma follows from the description of R_2, R_3 and R_4 in §2.

Lemma 4.1. *If R_2, R_3 and R_4 are as in* §2 *then we have the following:*

(i) $(R_2, v_0v_1v_2) \overset{\alpha_1}{\cong} (R_2, v_0v_1v_3) \overset{\alpha_2}{\cong} (R_2, v_0v_2v_3)$.

(ii) $(R_2, v_1v_3v_5) \overset{\alpha_2}{\cong} (R_2, v_2v_3v_6) \overset{\alpha_3}{\cong} (R_2, v_1v_2v_4)$.

(iii) $(R_2, \sigma) \cong (R_2, v_3v_4v_5)$, *for* $\sigma = v_3v_4v_6, v_2v_5v_6, v_2v_4v_5, v_1v_4v_6, v_1v_5v_6$.

(iv) $(R_3, v_0v_1v_3) \overset{\alpha_2}{\cong} (R_3, v_0v_2v_3) \overset{\alpha_4}{\cong} (R_3, v_0v_2v_4) \overset{\alpha_2}{\cong} (R_3, v_0v_1v_4)$.

(v) $(R_3, v_1v_3v_5) \overset{\alpha_2}{\cong} (R_3, v_2v_3v_6) \overset{\alpha_4}{\cong} (R_3, v_2v_4v_5) \overset{\alpha_2}{\cong} (R_3, v_1v_4v_6)$.

(vi) $(R_3, v_3v_4v_5) \overset{\alpha_2}{\cong} (R_3, v_3v_4v_6)$, $(R_3, v_1v_5v_6) \overset{\alpha_2}{\cong} (R_3, v_2v_5v_6)$.

(vii) $(R_4, \sigma_1) \cong (R_4, \sigma_2)$ *for any 2 triangles* σ_1, σ_2 *of* R_4,

where $\alpha_1, \dots, \alpha_8 \colon \{v_0, \dots, v_6\} \to \{v_0, \dots, v_6\}$ *are the permutations given by* $\alpha_1 = (v_2, v_3)(v_4, v_5)$, $\alpha_2 = (v_1, v_2)(v_5, v_6)$, $\alpha_3 = (v_1, v_3)(v_4, v_6)$, $\alpha_4 = (v_3, v_4)(v_5, v_6)$, $\alpha_5 = (v_2, v_3)(v_0, v_6)$, $\alpha_6 = (v_2, v_4)(v_0, v_5)$, $\alpha_7 = (v_1, v_4)(v_0, v_6)$ *and* $\alpha_8 = (v_1, v_3)(v_0, v_5)$. *Here* $(R_i, \tau) \overset{\alpha_l}{\cong} (R_j, \sigma)$ *means* (R_i, τ) *and* (R_j, σ) *are isomorphic via the map* α_l.

Lemma 4.2. *If* $\chi(M) = 1$ *and M has a vertex of degree* 3, *then M is isomorphic to* $R_5, \dots, R_{11}$ *or* R_{12} *defined above.*

Proof. Let v_7 be a vertex of degree 3 of M and let $\mathrm{Lk}(v_7) = C_3(a, b, c)$. Since $M \not\cong S^2_4$, abc is not a simplex. Let $\widetilde{M} = (M \setminus \{v_7\}) \cup \{\tau\}$. Then $\widetilde{M}$ is a 7-vertex combinatorial 2-manifold with $\chi(\widetilde{M}) = \chi(M) = 1$. Hence, by Lemma 4.1, $(\widetilde{M}, \tau)$ is isomorphic to $(R_2, v_0v_1v_2)$, $(R_2, v_2v_3v_6)$, $(R_2, v_1v_5v_6)$, $(R_3, v_0v_1v_3)$, $(R_3, v_2v_3v_6)$, $(R_3, v_3v_4v_5)$, $(R_3, v_1v_5v_6)$ or $(R_4, v_0v_1v_3)$.

If $(\widetilde{M}, \tau)$ is isomorphic to $(R_2, v_0v_1v_2)$ then, by Proposition 3.5(a), $M = (\widetilde{M} \setminus \{\tau\}) \cup (\partial\tau * v_7)$ is isomorphic to $R_{2,1}$, where $R_{2,1} = (R_2 \setminus \{v_0v_1v_2\}) \cup (\{v_0v_1, v_1v_2, v_0v_2\} * v_7)$.

Since $R_{2,1}$ is R_5, M is isomorphic to R_5.

Similarly in the other cases M is isomorphic to one of the following: $R_{2,2} := (R_2 \setminus \{v_2v_3v_6\}) \cup (\{v_2v_3, v_3v_6, v_2v_6\} * v_7)$, $R_{2,3} := (R_2 \setminus \{v_1v_5v_6\}) \cup (\{v_1v_5, v_5v_6, v_1v_6\} * v_7)$, $R_{3,1} := (R_3 \setminus \{v_0v_1v_3\}) \cup (\{v_0v_1, v_1v_3, v_0v_3\} * v_7)$, $R_{3,2} := (R_3 \setminus \{v_2v_3v_6\}) \cup (\{v_2v_3, v_3v_6, v_2v_6\} * v_7)$, $R_{3,3} := (R_3 \setminus \{v_3v_4v_5\}) \cup (\{v_3v_4, v_4v_5, v_3v_5\} * v_7)$, $R_{3,4} := (R_3 \setminus \{v_1v_5v_6\}) \cup (\{v_1v_5, v_5v_6, v_1v_6\} * v_7)$, $R_{4,1} := (R_4 \setminus \{v_0v_1v_3\}) \cup (\{v_0v_1, v_1v_3, v_0v_3\} * v_7)$.

Now observe that $R_{2,2} = R_6$, $R_{2,3} = R_7$, $R_{3,1} = R_8$, $R_{3,2} = R_9$, $R_{3,3} = R_{10}$, $R_{3,4} = R_{11}$ and $R_{4,1} = R_{12}$. This proves the lemma. □

Lemma 4.3. *If* $\chi(M) = 1$ *and there exists a vertex of degree* 7 *of M and no vertex of degree* 3, *then M is isomorphic to* $R_{13}, \dots, R_{16}$ *or* R_{17}.

Proof. By (1), the number of 2-simplices of M is 14. Let v_0 be a vertex of degree 7 and $\mathrm{Lk}(v_0) = C_7(v_1, \dots, v_7)$.

Claim. Any 2-simplex not containing v_0 contains exactly one edge from $C_7(v_1, \dots, v_7)$.

Since the degree of each vertex is more than 3, $v_i v_{i+1} v_{i+2}$ is not a simplex for $1 \le i \le 7$ (addition in the subscript is modulo 7). So, no 2-simplex contains more than one edge from $C_7(v_1, \dots, v_7)$. Let abc be a simplex not containing v_0. If none of ab, bc or ca is from $C_7(v_1, \dots, v_7)$ then the number of 2-simplices is more than 14 (7 through v_0, 7 through the edges in $C_7(v_1, \dots, v_7)$ and abc), a contradiction. These prove the claim.

First consider the case when there exists no triangle of the form $v_i v_{i+1} v_{i+3}$ or $v_i v_{i+1} v_{i+5}$. In this case, by the claim, the other triangles are $v_1v_2v_5$, $v_2v_3v_6$, $v_3v_4v_7$, $v_1v_4v_5$, $v_2v_5v_6$, $v_3v_6v_7$ and $v_1v_4v_7$. Then M is R_{17}.

Now, assume that there exists a simplex of the form $v_i v_{i+1} v_{i+3}$ or $v_i v_{i+1} v_{i+5}$. We may assume that $v_1v_2v_4$ is a simplex (the case when $v_1v_2v_6$ is a simplex gives isomorphic complexes). Then, by repeated use of the claim, the following are the possibilities for the remaining 6 triangles.

(1) $v_2v_4v_5,\ v_2v_3v_5,\ v_3v_5v_6,\ v_3v_4v_6,\ v_4v_6v_7,\ v_1v_4v_7,$
(2) $v_2v_4v_5,\ v_2v_3v_5,\ v_3v_5v_6,\ v_3v_6v_7,\ v_1v_3v_7,\ v_1v_3v_4,$
(3) $v_2v_4v_5,\ v_2v_3v_5,\ v_3v_5v_6,\ v_3v_6v_7,\ v_3v_4v_7,\ v_1v_4v_7,$
(4) $v_2v_4v_5,\ v_2v_5v_6,\ v_2v_3v_6,\ v_3v_4v_6,\ v_4v_6v_7,\ v_1v_4v_7,$
(5) $v_2v_4v_5,\ v_2v_5v_6,\ v_2v_3v_6,\ v_3v_6v_7,\ v_3v_4v_7,\ v_1v_4v_7,$
(6) $v_2v_4v_5,\ v_2v_5v_6,\ v_2v_3v_6,\ v_3v_6v_7,\ v_1v_3v_7,\ v_1v_3v_4,$
(7) $v_2v_4v_5,\ v_2v_5v_6,\ v_2v_6v_7,\ v_2v_3v_7,\ v_3v_4v_7,\ v_1v_4v_7,$
(8) $v_2v_4v_5,\ v_2v_5v_6,\ v_2v_6v_7,\ v_2v_3v_7,\ v_1v_3v_7,\ v_1v_3v_4.$

For $1 \le i \le 8$, let M_i be the combinatorial 2-manifold arising in case (i) above. Let $\alpha_1, \alpha_2, \alpha_3 \colon \{v_0, \dots, v_7\} \to \{v_0, \dots, v_7\}$ be the permutations given by $\alpha_1 = (v_1, v_6)(v_2, v_5)(v_3, v_4)$, $\alpha_2 = (v_1, v_5)(v_2, v_4)(v_6, v_7)$ and $\alpha_3 = (v_1, v_4)(v_2, v_3)(v_5, v_7)$. Then

$$M_8 \overset{\alpha_3}{\cong} M_2 \overset{\alpha_1}{\cong} M_1 = R_{13}, \quad M_6 \overset{\alpha_2}{\cong} M_3 = R_{14}, \quad M_7 \overset{\alpha_2}{\cong} M_4 = R_{15}, \quad M_5 = R_{16}.$$

So, in this case, M is isomorphic to R_{13}, R_{14}, R_{15} or R_{16}. This proves the lemma. □

Lemma 4.4. *If* $\chi(M) = 1$ *and* $3 < \deg(v) < 7$, *for each vertex* v, *then* M *is isomorphic to* R_{18}, R_{19} *or* R_{20}.

Proof. Since (by (1)) the number of edges of M is 21, there exists a vertex (say v_0) of degree 6. Let $\mathrm{Lk}(v_0) = C_6(v_1, \dots, v_6)$. Let v_7 be the remaining vertex. If $\mathrm{Lk}(v_0)$ and $\mathrm{Lk}(v_7)$ have no common edge then number of 2-simplices is more than 14 (6 through v_0, 6 more through the edges in $C_6(v_1, \dots, v_6)$ and at least 4 more through v_7), a contradiction. Without loss of generality, we can assume that $v_4v_5v_7$ is a simplex.

Let vv_1v_2 ($\neq v_0v_1v_2$) be the other simplex containing v_1v_2. Since the degree of each vertex is more than 3, $v \notin \{v_3, v_6\}$. Hence, without loss of generality, we can assume that $v = v_4$ or v_7.

Case I. $v_1v_2v_4$ is a simplex. As $v_2v_3v_4$ is not a simplex, $\mathrm{Lk}(v_4) = C_6(v_3, v_0, v_5, v_7, v_2, v_1)$. Since $v_1v_5v_6$ is not a simplex, $\mathrm{Lk}(v_1)$ is $C_5(v_6, v_0, v_2, v_4, v_3)$ or $C_6(v_6, v_0, v_2, v_4, v_3, v_7)$.

Subcase I.1. $\mathrm{Lk}(v_1) = C_5(v_6, v_0, v_2, v_4, v_3)$. Since $\deg(v_7) > 3$, $v_2v_5 \notin \mathrm{Lk}(v_7)$. Therefore v_2v_3 or v_2v_6 is in $\mathrm{Lk}(v_7)$.

In the first case, from the links of v_3 and v_7, $v_3v_6v_7$ and $v_5v_6v_7$ are simplices. Then M is isomorphic, via the map $(v_0, v_6, v_2, v_1, v_5, v_7)(v_3, v_4)$, to R_{20}.

In the second case, $v_2v_3v_6$ is a simplex through v_2 and hence $v_5v_6v_7$ is a simplex. Then M is isomorphic, via the map $(v_0, v_6, v_3, v_5, v_7)(v_1, v_4, v_2)$, to R_{19}.

Subcase I.2. $\mathrm{Lk}(v_1) = C_6(v_6, v_0, v_2, v_4, v_3, v_7)$. Then $\mathrm{Lk}(v_7)$ is $C_6(v_2, v_4, v_5, v_3, v_1, v_6)$ or $C_6(v_2, v_4, v_5, v_6, v_1, v_3)$. If $\mathrm{Lk}(v_7) = C_6(v_2, v_4, v_5, v_3, v_1, v_6)$ then $\mathrm{Lk}(v_2) = C_6(v_1, v_0, v_3, v_6, v_7, v_4)$. These give more than 14 triangles, which is not possible. So, $\mathrm{Lk}(v_7) = C_6(v_2, v_4, v_5, v_6, v_1, v_3)$. Then M is isomorphic, via the map $(v_0, v_2, v_5)(v_7, v_3, v_4, v_1, v_6)$, to R_{19}.

Case II. $v_1v_2v_7$ is a simplex. Since the degree of each vertex is more than 3, the 2-simplex $(\neq v_0v_3v_4)$ containing v_3v_4 is $v_1v_3v_4$, $v_3v_4v_6$ or $v_3v_4v_7$.

In the first case, by similar argument as above, M is isomorphic to R_{18} or R_{20}.

In each of the other two cases, M is isomorphic to R_{18}, R_{19} or R_{20}. This completes the proof of Lemma 4.4. □

Lemma 4.5. *If* $\chi(M) = 0$ *and there exists a vertex of degree* 3 *in* M, *then* M *is isomorphic to* T_3.

Proof. Let $\deg(v_0) = 3$ and let $\mathrm{Lk}(v_0) = C_3(v_1, v_2, v_3)$. Since $M \not\cong S^2_4$, $v_1v_2v_3$ is not a simplex. Let $\widetilde{M}$ be the complex on the vertex set $V(M)\setminus\{v_0\}$ and whose 2-simplices are $v_1v_2v_3$ and those of M which do not contain v_0. Then $\widetilde{M}$ is a 7 vertex 2-manifold which has 14 triangles and hence 21 edges. From Proposition 3.1, $\widetilde{M}$ is isomorphic to T_1. Then, by Proposition 3.5(a), M is isomorphic to T_3. □

Lemma 4.6. *If* $\chi(M) = 0$ *and* M *has no vertex of degree* 7, *then* M *is isomorphic to* T_2.

Proof. Since the degree of each vertex is less than 7, the degree of each vertex has to be 6. Let v_0 be a vertex whose link is $C_6(v_1, \dots, v_6)$. Let v_7 be the remaining vertex. Then $v_1, \dots, v_6 \in \mathrm{Lk}(v_7)$. Since the degree of each vertex is more than 3, $v_iv_{i+1}v_{i+2}$ is not a simplex in M for $1 \leq i \leq 6$ (addition in the subscripts are modulo 6). Since $|M|$ is not homeomorphic to S^2, $\mathrm{Lk}(v_7) \neq C_6(v_1, \dots, v_6)$. Without loss of generality, assume $v_1v_2 \notin \mathrm{Lk}(v_7)$. Then either $v_1v_2v_4$ or $v_1v_2v_5$ is the other 2-simplex containing v_1v_2. Assume, without loss of generality, that $v_1v_2v_4$ is a simplex. So, v_3v_0, v_0v_5 and v_1v_2 are edges in the link of v_4.

Claim. Neither v_1v_3 nor v_2v_5 is an edge in the link of v_4.

If $v_2v_5 \in \mathrm{Lk}(v_4)$ then $\mathrm{Lk}(v_4) = C_6(v_3, v_0, v_5, v_2, v_1, v_7)$. Since $v_1v_5v_6$ is not a simplex, $\mathrm{Lk}(v_1) = C_6(v_6, v_0, v_2, v_4, v_7, v_3)$ and hence $\mathrm{Lk}(v_3) = C_6(v_2, v_0, v_4, v_7, v_1, v_6)$. These imply $\deg(v_2) = 7$, a contradiction.

If $v_1v_3 \in \mathrm{Lk}(v_4)$ then $\mathrm{Lk}(v_1) = C_6(v_6, v_0, v_2, v_4, v_3, v_7)$, $\mathrm{Lk}(v_4) = C_6(v_3, v_0, v_5, v_7, v_2, v_1)$ and $\mathrm{Lk}(v_3) = C_6(v_2, v_0, v_4, v_1, v_7, v)$, where $v = v_5$ or v_6. If $v = v_5$ then, $\mathrm{Lk}(v_5) = C_6(v_4, v_0, v_6, v_2, v_3, v_7)$. These imply, $v_0v_2, v_1v_2, v_3v_2, \dots, v_7v_2$ are edges in M and hence, $\deg(v_2) = 7$, a contradiction. If $v = v_6$ then $C_3(v_1, v_3, v_6) \subseteq \mathrm{Lk}(v_7)$, a contradiction. This proves the claim.

Since $v_2v_3v_4$ is not a simplex, by the claim, $\mathrm{Lk}(v_4) = C_6(v_3, v_0, v_5, v_1, v_2, v_7)$ and $\mathrm{Lk}(v_1) = C_6(v_6, v_0, v_2, v_4, v_5, v_7)$. Then $\mathrm{Lk}(v_5) = C_6(v_6, v_0, v_4, v_1, v_7, y)$, where $y = v_2$ or v_3. If $y = v_2$ then $\deg(v_2) = 7$, a contradiction. So, $y = v_3$. These imply $\mathrm{Lk}(v_2) = C_6(v_1, v_0, v_3, v_6, v_7, v_4)$. Then M is isomorphic, via the map $(v_1, v_4, v_5, v_3, v_7)(v_2, v_6)$, to T_2. □

Lemma 4.7. *If* $\chi(M) = 0$ *and there exists a vertex of degree* 7 *in* M *and no vertex of degree* 3*, then* M *is isomorphic to* $T_4, \dots, T_8$*,* $K_1, \dots, K_5$ *or* K_6.

Proof. Let $\deg(v_0) = 7$ and let $\mathrm{Lk}(v_0) = C_7(v_1, \dots, v_7)$. Since the degree of each vertex is more than 3 and $M \neq R_{17}$, there exists $i \in \{1, \dots, 7\}$ such that $v_i v_{i+1} v_{i+3}$ or $v_i v_{i+1} v_{i+5}$ is a simplex (additions in the subscripts are modulo 7). We may assume that $v_1v_2v_4$ is a simplex (cases when $v_1v_2v_6$ is a simplex give isomorphic complexes). Then the 2-simplex ($\neq v_0v_1v_7$) containing v_1v_7 is $v_1v_7v_3$, $v_1v_7v_4$ or $v_1v_7v_5$.

Case I. $v_1v_4v_7$ is a simplex. If $v_4v_7v_6$ is a simplex then $\mathrm{Lk}(v_7) = C_4(v_1, v_0, v_6, v_4)$, $\mathrm{Lk}(v_4) = C_7(v_3, v_0, v_5, v_2, v_1, v_7, v_6)$ and $\mathrm{Lk}(v_1) = C_4(v_2, v_0, v_7, v_4)$. Now, $\mathrm{Lk}(v_2)$ is $C_5(v_3, v_0, v_1, v_4, v_5)$ or $C_6(v_3, v_0, v_1, v_4, v_5, v_6)$. In the first case, $\mathrm{Lk}(v_3) = C_5(v_2, v_0, v_4, v_6, v_5)$. Then the links of the remaining vertices are complete and the degree sequence of M is $7^2 \cdot 5^4 \cdot 4^2$ and hence $\chi(M) = -1$, a contradiction. In the second case, the links of the remaining vertices are complete and the degree sequence of M is $7^2 \cdot 6^2 \cdot 4^4$, a contradiction again. So, the 2-simplex ($\neq v_0v_4v_7$) containing v_4v_7 is $v_3v_4v_7$ or $v_4v_5v_7$.

Subcase I.1. $v_3v_4v_7$ is a simplex. Since $v_4v_5v_6$ is not a simplex, $\mathrm{Lk}(v_4) = C_6(v_3, v_0, v_5, v_2, v_1, v_7)$ and hence $\mathrm{Lk}(v_1) = C_4(v_7, v_0, v_2, v_4)$. Then the edge v_5v_6 (of $\mathrm{Lk}(v_0)$) is either in $\mathrm{Lk}(v_2)$ or in $\mathrm{Lk}(v_3)$.

If v_5v_6 is in $\mathrm{Lk}(v_2)$, then $\mathrm{Lk}(v_2)$ is $C_7(v_3, v_0, v_1, v_4, v_5, v_6, v_7)$ or $C_6(v_3, v_0, v_1, v_4, v_5, v_6)$. In the first case, the links of the remaining vertices are complete. These imply that $f_2(M) = 14$ and hence $\chi(M) = -1$, a contradiction. In the second case, $\mathrm{Lk}(v_6) = C_5(v_7, v_0, v_5, v_2, v_3)$. Then no more 2-simplices are possible. These imply that $f_2(M) = 14$, a contradiction again. So, v_5v_6 is in $\mathrm{Lk}(v_3)$.

If $v_2v_3v_5$ is a simplex then $\mathrm{Lk}(v_3) = C_6(v_2, v_0, v_4, v_7, v_6, v_5)$. These complete all the links of the remaining vertices. In this case, $f_2(M) = 14$, a contradiction. Hence, the 2-simplex ($\neq v_0v_2v_3$) containing v_2v_3 is $v_2v_3v_6$. Then $\mathrm{Lk}(v_3) = C_6(v_2, v_0, v_4, v_7, v_5, v_6)$. Now, $\mathrm{Lk}(v_2)$ and $\mathrm{Lk}(v_7)$ show that the remaining two simplices are $v_2v_6v_7$ and $v_2v_5v_7$. Then M is isomorphic, via the map $(v_0, v_4, v_2)(v_1, v_3, v_5)$, to K_3.

Subcase I.2. $v_4v_5v_7$ is a simplex. To complete $\mathrm{Lk}(v_4)$, $v_2v_4v_6$ and $v_3v_4v_6$ have to be simplices. Then the other triangle containing v_5v_6 is either $v_2v_5v_6$ $v_3v_5v_6$.

In the first case, using the similar method as above, M is isomorphic to T_4 or T_7.

In the second case, M is isomorphic to K_3.

Case II. $v_1v_3v_7$ is a simplex. Since v_6v_7 is an edge, one of $v_2v_6v_7$, $v_3v_6v_7$ or $v_4v_6v_7$ is a triangle.

In the first case, M is isomorphic to $K_1, \dots, K_4$, T_4 or T_5.

In the second case, M is isomorphic to T_5, T_6, T_7, $K_1, \dots, K_4$ or K_6.

In the third case, M is isomorphic to T_4 or K_3.

Case III. $v_1v_5v_7$ is a simplex. Then one of $v_1v_5v_6$, $v_2v_5v_6$ or $v_3v_5v_6$ is a triangle containing v_5v_6.

In the first case, M is isomorphic to $K_1, \dots, K_6$.

In the second case, M is isomorphic to $T_5, \dots, T_8$, K_2, K_3 or K_4,

In the third case, M is isomorphic to $K_1, \dots, K_5$, T_5 or T_8. This completes the proof of Lemma 4.7. □

Lemma 4.8. *The combinatorial* 2*-manifolds mentioned in Theorem* 1.1 *are pairwise non-isomorphic.*

Proof. For $2 \leq i < j \leq 8$, $T_i \cong T_j$ implies that the degree sequence of T_i is the same as the degree sequence of T_j. This implies that $(i, j) = (5, 8)$. If φ is an isomorphism from T_5 to T_8, then $\varphi(v_1v_2) = v_2v_5$ (since $\deg_{T_5}(v_j) = \deg_{T_8}(\varphi(v_j))$. This implies $\varphi(v_1v_2v_4) = v_2v_5v_0$ or $v_2v_5v_3$. Then $\varphi(v_4) = v_0$ or v_3. But $\deg_{T_5}(v_4) = 6$ and $\deg_{T_8}(v_0) = 5 = \deg_{T_8}(v_3)$, a contradiction. So, $T_i \ncong T_j$ for $2 \leq i \neq j \leq 8$.

Similarly, for $1 \leq i < j \leq 6$, $K_i \cong K_j$ implies that the degree sequence of K_i is the same as the degree sequence of K_j and hence $(i, j) = (2, 4)$. If α is an isomorphism from K_2 to K_4, then $\alpha(\{v_0, v_4, v_5\}) = \{v_1, v_4, v_5\}$ (since α takes vertices of degree 7 in K_2 to vertices of degree 7 of K_4). Now, $v_0v_4v_5 \in K_2$ whereas $v_1v_4v_5 \notin K_4$. Hence, $K_2 \ncong K_4$. So, $K_i \ncong K_j$ for $1 \leq i \neq j \leq 6$.

Repeating the above argument we find that for $5 \leq i < j \leq 20$, $R_i \cong R_j$ implies $(i, j) = (5, 10), (14, 16)$ or $(18, 19)$.

If possible, let $\beta_1 : R_5 \to R_{10}$ be an isomorphism. Since $\deg(\beta_1(v)) = \deg(v)$ for each vertex v in R_5, $\beta_1(v_7) = v_7$ and $\beta_1(v_0) = v_0$. Now, v_0v_7 is an edge in R_5 whereas v_0v_7 is not an edge in R_{10}. So, $R_5 \ncong R_{10}$.

If possible, let $\beta_2 : R_{14} \to R_{16}$ be an isomorphism. Since $\deg(\beta_2(v)) = \deg(v)$ for each vertex v in R_{14}, $\beta_2(v_0) = v_0$ and $\beta_1(\{v_3, v_4\}) = \{v_2, v_4\}$. Now, $v_0v_3v_4$ is a simplex in R_{14} whereas $v_0v_2v_4$ is not a simplex in R_{16}. Thus, $R_{14} \ncong R_{16}$.

In R_{19}, both the degree 4 vertices form an edge but that is not the case in R_{18}. Thus, $R_{18} \ncong R_{19}$. So, $R_i \ncong R_j$ for $5 \leq i \neq j \leq 20$.

If, for $10 \leq i < j \leq 23$, $S_i \cong S_j$ then the degree sequence of S_i is equal to the degree sequence of S_j. This implies that $i = 16$ and $j = 18$. Now, if γ is an isomorphism between S_{16} and S_{18}, then γ takes vertices of degree 3 in S_{16} to vertices of degree 3 of S_{18}. Observe that the links of the vertices of degree 3 of S_{16} have a vertex of degree 4 whereas those of S_{18} have no vertex of degree 4. This shows that such a γ does not exist. Hence, $S_i \ncong S_j$ for $10 \leq i \neq j \leq 23$.

The lemma now follows from the fact that S_i $(10 \leq i \leq 23)$ triangulates S^2, R_j $(5 \leq j \leq 20)$ triangulates $\mathbb{R}P^2$, T_k $(2 \leq k \leq 8)$ triangulates $S^1 \times S^1$, K_p $(1 \leq p \leq 6)$ triangulates the Klein bottle and $|D|$ is disconnected. □

Proof of Theorem 1.1. Let M be an 8-vertex combinatorial 2-manifold. Then $\frac{3\times 8}{2} \leq f_1(M) \leq \binom{8}{2}$, $\chi(M) = 8 - f_1(M) + f_2(M)$ and $3f_2(M) = 2f_1(M)$. This implies $12 \leq f_1(M) \leq 27$ and hence $-1 \leq \chi(M) \leq 4$. But it is known (e.g., see [7,9]) that there does not exist any 8-vertex combinatorial 2-manifold M with $\chi(M) = -1$. If $\chi(M) \geq 3$ then $|M|$ is disconnected and since the number of vertices is 8, M is isomorphic to D. So, if $M \neq D$ then $0 \leq \chi(M) \leq 2$. The theorem now follows from Lemmas 4.2, ..., 4.8 and Proposition 3.2. □

5. Proof of Theorem 1.2

Throughout this section M is an 8-vertex two-dimensional weak pseudomanifold which is not a combinatorial 2-manifold.

Lemma 5.1. *If M has a vertex, say v_0, whose link is of the form $C_3(v_1, v_2, v_3) \sqcup C_4(v_4, v_5, v_6, v_7)$, where $v_1v_2v_3$ is a simplex, then M is isomorphic to Q_3, Q_5, Q_6, Q_{12} or Q_{16}.*

Proof. Here we have two cases: (I) At least one of v_4v_6 or v_5v_7 is an edge of M and (II) Neither v_4v_6 nor v_5v_7 is an edge of M.

Case I. We can assume, without loss of generality, that v_4v_6 is an edge of M.

Subcase I.1. The triangles through v_4v_6 are $v_4v_5v_6$ and $v_4v_6v_7$. Then, M is Q_3.

Subcase I.2. Exactly one of $v_4v_5v_6$ or $v_4v_6v_7$ is a simplex. Assume, without loss of generality, that $v_4v_5v_6$ is a simplex. Then, the other triangle through v_4v_6 must be one of $v_1v_4v_6$, $v_2v_4v_6$ or $v_3v_4v_6$. Assume, without loss of generality, that $v_1v_4v_6$ is a simplex in M. Then $v_1v_6v_7$ and $v_1v_4v_7$ have to be simplices. Here M is Q_5.

Subcase I.3. Neither $v_4v_5v_6$ nor $v_4v_6v_7$ is a simplex of M. The triangles through v_4v_6 are of the form $v_1v_4v_6$, $v_2v_4v_6$ or $v_3v_4v_6$. Assume, without loss of generality, that $v_1v_4v_6$ and $v_2v_4v_6$ are simplices of M. The triangle ($\neq v_1v_4v_6$) having v_1v_4 as an edge is either $v_1v_4v_5$ or $v_1v_4v_7$. We can assume, without loss of generality, that $v_1v_4v_5$ is a simplex. Then, the other triangle through v_1v_5 is $v_1v_5v_6$ or $v_1v_5v_7$.

In the first case (by considering $\mathrm{Lk}(v_6)$ and $\mathrm{Lk}(v_4)$), $v_2v_6v_7$ and $v_2v_4v_7$ are simplices. Here M is Q_{12}.

In the second case, $\mathrm{Lk}(v_4) = C_6(v_5, v_0, v_7, v_2, v_6, v_1)$ and $\mathrm{Lk}(v_1) = C_3(v_0, v_2, v_3) \sqcup C_4(v_6, v_4, v_5, v_7)$ and hence $\mathrm{Lk}(v_6) = C_6(v_5, v_0, v_7, v_1, v_4, v_2)$. The link of v_7 shows that $v_2v_5v_7$ is a simplex. Then M is Q_{16}.

Case II. Assume, without loss of generality, that the second 2-simplex containing v_4v_5 is $v_1v_4v_5$. Then (since, v_4v_6 and v_5v_7 are non-edges), $\mathrm{Lk}(v_1) = C_3(v_0, v_2, v_3) \sqcup C_4(v_4, v_5, v_6, v_7)$. In this case, M is Q_6. □

Lemma 5.2. *If M has a vertex, say v_0, whose link is of the form $C_3(v_1, v_2, v_3) \sqcup C_4(v_4, v_5, v_6, v_7)$, where v_4v_6 and v_5v_7 are edges but $v_1v_2v_3$ is not a simplex, then M is isomorphic to $P_{29}, \ldots, P_{32}$, P_{35}, P_{37}, P_{38}, Q_{11} or Q_{14}.*

Proof. We have three cases: (I) Both $v_4v_5v_6$ and $v_4v_6v_7$ are simplices, (II) exactly one of $v_4v_5v_6$ and $v_4v_6v_7$ is a simplex and (III) neither $v_4v_5v_6$ nor $v_4v_6v_7$ is a simplex.

Case I. The triangles through v_5v_7 are of the form $v_1v_5v_7$, $v_2v_5v_7$ or $v_3v_5v_7$. We can assume, without loss of generality, that $v_1v_5v_7$ and $v_2v_5v_7$ are in M. Consider the triangle through v_1v_5 other than $v_1v_5v_7$. Clearly, it is either $v_1v_2v_5$ or $v_1v_3v_5$.

In the former case, to complete $\mathrm{Lk}(v_1)$ and $\mathrm{Lk}(v_3)$, $v_1v_3v_7$ and $v_2v_3v_7$ have to be simplices. Here M is Q_{11}.

In the latter case, $v_2v_3v_5$ has to be a simplex. This implies $v_1v_2v_7$ is also a simplex. Then M is isomorphic, via the map (v_5, v_7), to Q_{11}.

Case II. Assume, without loss of generality, that $v_4v_5v_6$ is a simplex. Then, the other triangle through v_4v_6 must be one of $v_1v_4v_6$, $v_2v_4v_6$ or $v_3v_4v_6$. Assume, without loss of generality, that $v_1v_4v_6$ is in M. The triangles through v_5v_7 are of the form $v_1v_5v_7$, $v_2v_5v_7$ or $v_3v_5v_7$. Without loss of generality we can assume that either $v_1v_5v_7$ and $v_2v_5v_7$ are in M or $v_2v_5v_7$ and $v_3v_5v_7$ are in M.

In the first case, M is P_{29}, P_{30} or P_{31}.

In the second case, M is isomorphic to P_{32}, P_{35} or Q_{14}.

Case III. Without loss of generality, assume that the triangles through v_4v_6 are $v_1v_4v_6$ and $v_2v_4v_6$. Then assume, without loss of generality, that the triangles through v_5v_7 are either $v_1v_5v_7$ and $v_2v_5v_7$ or $v_2v_5v_7$ and $v_3v_5v_7$.

In the first case, M is isomorphic to P_{37}.

In the second case, M is isomorphic to P_{38}. □

Lemma 5.3. *If there exists a vertex, say v_0, whose link in M is of the form $C_3(v_1, v_2, v_3) \sqcup C_4(v_4, v_5, v_6, v_7)$, where $v_1v_2v_3$ and v_4v_6 are not simplices, then M is isomorphic to P_3, P_4, P_6, P_9, P_{11}, P_{13}, P_{15}, P_{22}, P_{23}, P_{25}, P_{28}, P_{30}, P_{34}, Q_7, Q_8 or Q_{11}.*

Proof. The complex $\widetilde{M} = (M \setminus \{v_0\}) \cup \{v_1v_2v_3, v_4v_5v_6, v_4v_6v_7\})$ is a 7-vertex two-dimensional weak pseudomanifold. From the classification of 7-vertex two-dimensional weak pseudomanifolds, we observe that $(\widetilde{M}, v_1v_2v_3, v_4v_5v_6, v_4v_6v_7)$ is (isomorphic to) $(R_2, v_0v_1v_2, v_3v_4v_5, v_3v_4v_6)$, $(R_3, v_0v_1v_3, v_2v_4v_5, v_2v_5v_6)$, $(R_3, v_1v_5v_6, v_0v_2v_3, v_0v_2v_4)$, $(S_5, v_2v_3v_4, v_1v_6v_7, v_1v_5v_6)$, $(S_6, v_1v_6v_7, v_2v_3v_4, v_2v_4v_5)$, $(S_7, v_2v_4v_5, v_3v_6v_7, v_1v_3v_7)$, $(S_7, v_1v_3v_7, v_4v_5v_6, v_2v_5v_6)$, $(S_7, v_1v_3v_7, v_2v_4v_5, v_2v_5v_6)$, $(S_7, v_1v_3v_7, v_2v_4v_5, v_4v_5v_6)$, $(S_8, v_1v_2v_3, v_4v_5v_6, v_5v_6v_7)$, $(S_8, v_2v_5v_7, v_1v_3v_4, v_1v_4v_6)$, $(S_8, v_5v_6v_7, v_1v_2v_3, v_1v_3v_4)$, $(S_9, v_1v_2v_3, v_4v_5v_6, v_5v_6v_7)$, $(Q_1, v_1v_2v_3, v_4v_5v_7, v_4v_5v_6)$, $(Q_1, v_1v_2v_3, v_4v_6v_7, v_4v_5v_7)$, $(Q_2, v_1v_2v_3, v_4v_5v_7, v_5v_6v_7)$, $(P_1, v_1v_5v_6, v_2v_3v_7, v_2v_3v_4)$, $(P_2, v_2v_3v_7, v_1v_5v_6, v_1v_4v_5)$ or $(P_2, v_2v_3v_4, v_1v_5v_6, v_5v_6v_7)$.

First, assume that $(\widetilde{M}, v_1v_2v_3, v_4v_5v_6, v_4v_6v_7)$ is $(R_2, v_0v_1v_2, v_3v_4v_5, v_3v_4v_6)$. Let $\widehat{M}$ be the complex obtained from $\widetilde{M}$ by starring a vertex u_8 in $v_1v_2v_3$ and u_9 in the edge v_4v_6. Let $\widehat{R_2}$ be the complex obtained from R_2 by starring a vertex v_8 in $v_0v_1v_2$ and v_9 in the edge v_3v_4. Then, by Proposition 3.4(a), $\widehat{M} \cong \widehat{R_2}$.

The complex obtained from $\widehat{M}$ by identifying u_8 and u_9 (to a new vertex v_0) is M and the complex obtained from $\widehat{R_2}$ by identifying v_8 and v_9 (to a new vertex v_7) is P_{22}. Therefore, by Proposition 3.3(b), $M \cong P_{22}$.

Similarly, if $(\widetilde{M}, v_1v_2v_3, v_4v_5v_6, v_4v_6v_7)$ is one of $(R_3, v_0v_1v_3, v_2v_4v_5, v_2v_5v_6)$, $(R_3, v_1v_5v_6, v_0v_2v_3, v_0v_2v_4)$, $(S_5, v_2v_3v_4, v_1v_6v_7, v_1v_5v_6)$, $(S_6, v_1v_6v_7, v_2v_3v_4, v_2v_4v_5)$, $(S_7, v_1v_3v_7, v_4v_5v_6, v_2v_5v_6)$, $(S_7, v_1v_3v_7, v_2v_4v_5, v_2v_5v_6)$, $(S_7, v_1v_3v_7, v_2v_4v_5, v_4v_5v_6)$, $(S_8, v_2v_5v_7, v_1v_3v_4, v_1v_4v_6)$, $(S_8, v_5v_6v_7, v_1v_2v_3, v_1v_3v_4)$, $(Q_1, v_1v_2v_3, v_4v_5v_7, v_4v_5v_6)$, $(Q_1, v_1v_2v_3, v_4v_6v_7, v_4v_5v_7)$, $(P_1, v_1v_5v_6, v_2v_3v_7, v_2v_3v_4)$ or $(P_2, v_2v_3v_4, v_1v_5v_6, v_5v_6v_7)$, then M is P_{23}, P_{25}, P_6, P_{11}, P_4, P_9, P_3, P_{15}, P_{13}, Q_7, Q_8, P_{34} or P_{28} respectively.

If $(\widetilde{M}, v_1v_2v_3, v_4v_5v_6, v_4v_6v_7)$ is $(S_7, v_2v_4v_5, v_3v_6v_7, v_1v_3v_7)$, $(S_8, v_1v_2v_3, v_4v_5v_6, v_5v_6v_7)$, $(S_9, v_1v_2v_3, v_4v_5v_6, v_5v_6v_7)$, $(Q_2, v_1v_2v_3, v_4v_5v_7, v_5v_6v_7)$ or $(P_2, v_2v_3v_7, v_1v_5v_6, v_1v_4v_5)$ then M is isomorphic to P_6, P_{11}, P_{15}, Q_{11} or P_{30} respectively. □

Lemma 5.4. *If M has no singular vertex of degree* 7*, then M is isomorphic to P_5, P_7, P_8, P_{10}, P_{12}, P_{14}, $P_{16}, \ldots, P_{21}$, P_{24}, P_{26}, P_{27}, P_{33}, P_{36}, P_{39}, Q_4, Q_9, Q_{10}, Q_{13} or Q_{15}.*

Proof. Since M is not a combinatorial 2-manifold, there exists a vertex, say v_0, whose link is of the form $C_3(v_1, v_2, v_3) \sqcup C_3(v_4, v_5, v_6)$. Since $M \ncong Q_1$, at most one of $v_1v_2v_3$ or $v_4v_5v_6$ can be a simplex. Let v_7 be the remaining vertex of M.

Case I. Exactly one of $v_1v_2v_3$ or $v_4v_5v_6$ is a simplex. Assume, without loss of generality, that $v_1v_2v_3$ is a simplex. Then the triangle ($\neq v_0v_4v_5$) through v_4v_5 is $v_1v_4v_5$, $v_2v_4v_5$, $v_3v_4v_5$ or $v_4v_5v_7$. Without loss of generality, we can assume that either $v_1v_4v_5$ or $v_4v_5v_7$ is a simplex.

In the first case, M is isomorphic to Q_9, Q_{13} or Q_{15}.

In the second case, M is isomorphic to Q_4, Q_9 or Q_{13}.

Case II. Neither $v_1v_2v_3$ nor $v_4v_5v_6$ is a simplex. Then $\widetilde{M} = (M \setminus \{v_0\}) \cup \{v_1v_2v_3, v_4v_5v_6\}$ is a 7-vertex two-dimensional weak pseudomanifold. Since M has no singular vertex of degree

7, none of $v_1, \dots, v_6$ is singular in $\widetilde{M}$. From the classification of 7-vertex two-dimensional weak pseudomanifolds we observe that $(\widetilde{M}, v_1v_2v_3, v_4v_5v_6)$ is (isomorphic to) $(T_1, v_1v_2v_4, v_3v_5v_6)$, $(R_2, v_0v_1v_3, v_2v_5v_6)$, $(R_3, v_0v_1v_3, v_2v_4v_5)$, $(R_3, v_0v_1v_3, v_2v_5v_6)$, $(R_4, v_0v_1v_3, v_2v_4v_5)$, $(S_5, v_2v_3v_4, v_1v_6v_7)$, $(S_5, v_2v_3v_4, v_1v_5v_6)$, $(S_6, v_2v_3v_4, v_1v_6v_7)$, $(S_7, v_1v_2v_3, v_4v_5v_6)$, $(S_7, v_1v_2v_7, v_3v_4v_6)$, $(S_7, v_1v_2v_7, v_4v_5v_6)$, $(S_7, v_1v_3v_7, v_4v_5v_6)$, $(S_8, v_1v_2v_3, v_4v_5v_6)$, $(S_8, v_1v_3v_4, v_5v_6v_7)$, $(S_8, v_1v_2v_7, v_4v_5v_6)$, $(S_9, v_1v_2v_3, v_4v_5v_6)$, $(Q_1, v_1v_2v_3, v_4v_5v_6)$, $(P_1, v_1v_2v_6, v_3v_4v_5)$ or $(P_2, v_1v_5v_6, v_2v_3v_4)$.

If $(\widetilde{M}, v_1v_2v_3, v_4v_5v_6)$ is $(T_1, v_1v_2v_4, v_3v_5v_6)$ then, by Proposition 3.5(b), M is P_{39}.

Similarly, if $(\widetilde{M}, v_1v_2v_3, v_4v_5v_6)$ is $(R_2, v_0v_1v_3, v_2v_5v_6), \dots, (S_9, v_1v_2v_3, v_4v_5v_6)$, $(Q_1, v_1v_2v_3, v_4v_5v_6)$, $(P_1, v_1v_2v_6, v_3v_4v_5)$ or $(P_2, v_1v_5v_6, v_2v_3v_4)$, then M is P_{24}, P_{26}, P_{27}, P_{21}, P_7, P_5, P_8, P_{12}, P_{10}, P_{14}, P_{17}, P_{18}, P_{19}, P_{16}, P_{20}, Q_{10}, P_{36} or P_{33} respectively. □

Lemma 5.5. *The weak pseudomanifolds mentioned in Theorem* 1.2 *are pairwise non-isomorphic.*

Proof. First, we observe that all the P_i's $(1 \leq i \leq 39)$ are pseudomanifolds, while all the Q_i's $(1 \leq i \leq 16)$ are not pseudomanifolds.

For $3 \leq i < j \leq 39$, $P_i \cong P_j$ implies that the degree sequences of P_i and P_j are the same and the singular degree sequences of P_i and P_j are the same and hence, from the description of P_i's in §2, $(i, j) \in \{(17, 18), (19, 20), (22, 25)\}$.

If there exists an isomorphism $\alpha\colon P_{17} \to P_{18}$, then $\alpha(v_0) = v_0$ (since, these are the only singular vertices). In P_{17}, each of the C_3's in the link of v_0 has a vertex of degree 4, while only one C_3 in $\mathrm{Lk}(v_0)$ in P_{18} has a vertex of degree 4. Therefore, α is not an isomorphism.

If $\beta\colon P_{19} \to P_{20}$ is an isomorphism, then $\beta(v_0) = v_0$ (since, these are the only singular vertices). We see that $\mathrm{Lk}(v_0)$ in P_{19} has a vertex (namely, v_3) of degree 4 whereas $\mathrm{Lk}(v_0)$ in P_{20} has no vertex of degree 4. So, $P_{19} \ncong P_{20}$.

If there exists an isomorphism $\psi\colon P_{22} \to P_{25}$, then $\psi(v_7) = v_7$ (since, these are the only singular vertices of degree 7). In P_{25} the link of v_7 has a C_3 all of whose vertices have degree 6 but the link of v_7 in P_{22} has no such C_3. So, $P_{22} \ncong P_{25}$.

For $3 \leq i < j \leq 16$, $Q_i \cong Q_j$ implies that the degree sequences of Q_i and Q_j are the same and hence, from the description of Q_i's in §2, $(i, j) = (6, 7)$.

Two degree 3 vertices in Q_6 form an edge but that is not the case in Q_7. Thus, $Q_6 \ncong Q_7$. This completes the proof of the lemma. □

Proof of Theorem 1.2. Let M be a two-dimensional 8-vertex weak pseudomanifold which is not a combinatorial 2-manifold. Then M has a singular vertex.

First consider the case when M has a singular vertex, say v_0, of degree 7. Then the link of v_0 is of the form $C_3 \sqcup C_4$. Let $\mathrm{Lk}(v_0)= C_3(v_1, v_2, v_3) \sqcup C_4(v_4, v_5, v_6, v_7)$.

If $v_1v_2v_3 \in M$ then, by Lemma 5.1, M is isomorphic to Q_3, Q_5, Q_6, Q_{12} or Q_{16}.

If $v_1v_2v_3$ is not a simplex then we have two cases, namely, (i) either both v_4v_6 and v_5v_7 are edges of M or (ii) at least one of v_4v_6 or v_5v_7 is a non-edge of M, say (without loss of generality) v_4v_6 is not an edge. Then, by Lemmas 5.2 and 5.3, M is isomorphic to P_3, P_4, P_6, P_9, P_{11}, P_{13}, P_{15}, P_{22}, P_{23}, P_{25}, $P_{28}, \dots, P_{32}$, P_{34}, P_{35}, P_{37}, P_{38}, Q_7, Q_8, Q_{11} or Q_{14}.

Finally, consider the case when M has no singular vertex of degree 7. In this case, by Lemma 5.4, M is isomorphic to P_5, P_7, P_8, P_{10}, P_{12}, P_{14}, $P_{16}, \dots, P_{21}$, P_{24}, P_{26}, P_{27}, P_{33}, P_{36}, P_{39}, Q_4, Q_9, Q_{10}, Q_{13} or Q_{15}. This completes the proof. □

6. Applications

Proof of Corollary 1.3. Observe that $S_1, \ldots, S_{23}$ triangulate S^2, $R_1, \ldots, R_{20}$ triangulate $\mathbb{R}P^2$, $T_1, \ldots, T_8$ triangulate $S^1 \times S^1$, $K_1, \ldots, K_6$ triangulate the Klein bottle (K) and D triangulates $S^2 \sqcup S^2$. These, Proposition 3.1 and Theorem 1.1 imply Corollary 1.3(i).

By Proposition 3.3(c), $P_1, \ldots, P_{20}$ triangulate the pinched sphere (P), $P_{21}, \ldots, P_{27}$ triangulate $\mathbb{R}P^2\#P$, $P_{28}, \ldots, P_{36}$ triangulate $P\#P$, P_{37} triangulates $\mathbb{R}P^2\#P\#P$, P_{38} triangulates $K\#P$ and P_{39} triangulates $(S^1 \times S^1)\#P$.

Also (from the pictures in §2) Q_1, Q_3 and Q_4 triangulate the union of two S^2's having one point in common, $Q_2, Q_5, \ldots, Q_{10}$ triangulate the union of two S^2's having two points in common, Q_{11}, Q_{12} and Q_{13} triangulate the union of two 2-spheres having three points in common, Q_{14} and Q_{15} triangulate the union of two 2-spheres having four points in common, Q_{16} triangulates the union of S^2 and $\mathbb{R}P^2$ having three points in common. These, Proposition 3.1 and Theorem 1.2 imply Corollary 1.3(ii). □

Proof of Theorem 1.4. Let $\varphi\colon K_n^2 \to S_4^2$ be a simplicial map, where K_n^2 is an oriented n-vertex two-dimensional pseudomanifold and S_4^2 is the 4-vertex 2-sphere with an orientation.

If $n \leq 7$, then there exists a vertex, say a, of S_4^2 whose inverse image contains less than 2 vertices and hence there exists a triangle through a whose inverse image contains less than 3 triangles. Thus, $\deg(\varphi) < 3$.

If $n = 8$, then $f_2(K_8^2) \leq 18$ and hence $\deg(\varphi) \leq 18/4$. Let φ be of type (n_1, n_2, n_3, n_4). Assume, $\deg(\varphi) \geq 3$. By the same argument as above, each vertex of S_4^2 has two inverse images. Let $\varphi^{-1}(a) = \{a_1, a_2\}$, $\varphi^{-1}(b) = \{b_1, b_2\}$, $\varphi^{-1}(c) = \{c_1, c_2\}$ and $\varphi^{-1}(d) = \{d_1, d_2\}$, where a, b, c and d are the vertices of S_4^2. So, there does not exist any 2-simplex σ such that $\varphi(\sigma)$ is a vertex and hence we have:

Claim 6.1. Each collapsing triangle contains exactly one collapsing edge. On the other hand, both the triangles through a collapsing edge are collapsing.

It is also easy to see the following:

Claim 6.2. If $S_a := \{\sigma \in K_8^2 : \varphi(\sigma) \text{ is a 2-simplex containing } a\}$, then $\#(S_a) \leq 12$. Further, if a_1a_2 is an edge then $\#(S_a) \leq (\deg(a_1) - 2) + (\deg(a_2) - 2) \leq 10$.

If $\deg(\varphi) = 4$, then $f_2(K_8^2) \geq 16$ and hence $\chi(K_8^2) \leq 0$. If $\chi(K_8^2) = -1$ then by Theorems 1.1 and 1.2, $K_8^2 = P_{37}, P_{38}$ or P_{39}. Since P_{37} and P_{38} are non-orientable, $K_8^2 = P_{39}$. Clearly, from the degree sequence of P_{39}, there exists an edge xy such that $\varphi(x) = \varphi(y)$. Then (by Claim 6.2) $\deg(\varphi) \leq 10/3$, a contradiction. So, $\chi(K_8^2) = 0$ and hence $f_2(K_8^2) = 16$. Then φ is of type $(4, 4, 4, 4)$ and hence there is no collapsing 2-simplex and hence, by Claim 6.1, no collapsing edge. So, if $\varphi(u) = \varphi(v)$, then uv is a non-edge and conversely (since the number of non-edges is $\binom{8}{2} - 24 = 4$). These imply that the 4 non-edges are disjoint and hence the degree sequence of K_8^2 is 6^8. Then, by Theorems 1.1 and 1.2, $K_8^2 = T_2$ or P_{36}. In both the cases v_3v_6 is a non-edge and hence $\varphi(v_3) = \varphi(v_6)$. If $K_8^2 = T_2$ then $\varphi(v_2v_3v_7)$ and $\varphi(v_2v_6v_7)$ are the same 2-simplex, a contradiction to Proposition 3.6(b). If $K_8^2 = P_{36}$ then $\varphi(v_2v_3v_4)$ and $\varphi(v_2v_4v_6)$ are the same 2-simplex, a contradiction to Proposition 3.6(b). Thus $\deg(\varphi) \leq 3$. This proves the first part of the theorem.

Now assume $\deg(\varphi) = 3$. In this case $\#(S_a) \geq 9$ for each vertex a of S_4^2. Thus, $f_2(K_8^2) \geq 12$ and hence, by Theorems 1.1 and 1.2, $-1 \leq \chi(K_8^2) \leq 2$.

Case I. $\chi(K_8^2) = -1$. By Theorem 1.2 (as K_8^2 is orientable), $K_8^2 = P_{39}$. Let $\varphi(v_0) = a$. If $\varphi(v_7) = a$ then (since $\#(S_a) \geq 9$) $\varphi(v_1)$, $\varphi(v_2)$, $\varphi(v_4)$ are distinct and $\varphi(v_3)$, $\varphi(v_5)$, $\varphi(v_6)$ are distinct. Again, $\varphi(v_1) = \varphi(v_3)$ or $\varphi(v_5)$ implies that there are 4 collapsing 2-simplices through v_7. This implies, $\#(S_a) \leq 8$, a contradiction. So, $\varphi(v_1) = \varphi(v_6)$. Similarly, $\varphi(v_2) = \varphi(v_5)$ and $\varphi(v_3) = \varphi(v_4)$. Then $\varphi^{-1}(bcd)$ contains no 2-simplices, a contradiction. So, $v_7 \notin \varphi^{-1}(a)$. If $v_1 \in \varphi^{-1}(a)$ then $\varphi(v_0v_5v_6) = \varphi(v_1v_5v_6)$ and hence, by Proposition 3.6(c), $\#(S_a) \geq 9 + 2 = 11$. On the other hand $\#(S_a) \leq (\deg(v_0) - 2) + (\deg(v_1) - 2) = 9$, a contradiction. Similarly, for v_2, v_3, v_4, v_5 or $v_6 \in \varphi^{-1}(a)$ we get contradictions.

Case II. $\chi(K_8^2) = 0$. By Claim 6.1, the inverse image of each triangle of S_4^2 contains 3, 5 or 7 triangles. Also, by Claim 6.2, $(n_1, n_2, n_3, n_4) = (7, 3, 3, 3)$ or $(5, 5, 3, 3)$ is not possible. So, (n_1, n_2, n_3, n_4) is $(5, 3, 3, 3)$ or $(3, 3, 3, 3)$.

Subcase II.1. If $(n_1, n_2, n_3, n_4) = (5, 3, 3, 3)$, then we have two collapsing triangles and hence, by Claim 6.1, exactly one collapsing edge. Assume (if necessary, by taking a composition with an automorphism of S_4^2) that the number of triangles of $\varphi^{-1}(abc)$ is 5. Then $\#(S_a) \geq 5 + 3 + 3 = 11$ and hence, by Claim 6.2, a_1a_2 is not an edge. So, $11 \leq \deg(a_1) + \deg(a_2) \leq 12$. Similarly, b_1b_2 and c_1c_2 are not edges, $11 \leq \deg(b_1) + \deg(b_2)$, $\deg(c_1) + \deg(c_2) \leq 12$ and $\#(S_b), \#(S_c) \geq 11$. So, d_1d_2 is the collapsing edge. Without loss of generality, let $\deg(a_1) = \deg(b_1) = \deg(c_1) = 6$.

Since the sum of the degrees of all the vertices is 48, we get $12 \leq \deg(d_1)+\deg(d_2) \leq 14$. If $\deg(d_1) + \deg(d_2) = 12$ then (by Claim 6.2) $\#(S_d) \leq 12 - 4$, a contradiction. If $\deg(d_1) + \deg(d_2) = 14$ then, $\deg(d_1) = \deg(d_2) = 7$. Without loss, we can assume that $\deg(a_2) = \deg(b_2) = 5$ and $\deg(c_2) = 6$. Let x, y be the vertices of K_8^2 such that d_1d_2x and d_1d_2y are 2-simplices. If $x = a_1$, then $\#(S_a) \leq 6 + 5 - 1 = 10$, a contradiction. By a similar argument we see that $x, y \notin \{a_1, a_2, b_1, b_2\}$. Hence $x, y \in \{c_1, c_2\}$. Then $\#(S_c) \leq 12 - 2 = 10$, a contradiction. Thus, $\deg(d_1) + \deg(d_2) = 13$.

We can assume (if necessary, by taking composition with automorphisms of S_4^2 and K_8^2, i.e., up to an equivalence) that $\deg(d_1) = 7$, $\deg(d_2) = 6$, $\deg(b_2) = \deg(c_2) = 6$ and $\deg(a_2) = 5$. Hence there is no collapsing 2-simplex through a_1 or a_2. So, the degree sequence of K_8^2 is $7 \cdot 6^6 \cdot 5$ and hence, by Theorems 1.1 and 1.2, $K_8^2 = P_{35}$ and $d_1 = v_0$, $a_2 = v_1$. We can also assume that $\{b_1, b_2\} = \{v_2, v_6\}$ and $\{c_1, c_2\} = \{v_3, v_6\}$. Then, $\{a_1, d_2\} = \{v_5, v_7\}$.

If $(a_1, d_2) = (v_7, v_5)$ then $\{v_1v_2v_4, v_7v_2v_4, v_1v_3v_6, v_7v_3v_6\} \subseteq S_a$. This implies, by Proposition 3.6(c), $\#(S_a) \geq 13$, a contradiction. So, $(a_1, d_2) = (v_5, v_7)$. Then, φ is equivalent to h.

Subcase II.2. If $(n_1, n_2, n_3, n_4) = (3, 3, 3, 3)$, then we have 4 collapsing triangles and hence, by Claim 6.1, exactly 2 collapsing edges.

Without loss of generality, we can assume that a_1a_2 and b_1b_2 are edges whereas c_1c_2 and d_1d_2 are not. Then $9 \leq \deg(c_1) + \deg(c_2)$, $\deg(d_1) + \deg(d_2) \leq 12$. We see that $(\deg(a_1) - 2) + (\deg(a_2) - 2) = \#(S_a) = 9$. Hence $\deg(a_1) + \deg(a_2) = 13$. Similarly, $\deg(b_1) + \deg(b_2) = 13$. We can assume that $\deg(a_1) = \deg(b_1) = 7$ and $\deg(a_2) = \deg(b_2) = 6$. Thus, $\deg(c_1) + \deg(c_2) + \deg(d_1) + \deg(d_2) = 22$. This shows that $10 \leq \deg(c_1) + \deg(c_2)$, $\deg(d_1) + \deg(d_2) \leq 12$. Since $\deg(c_1), \deg(c_2) \leq 6$, it is clear that $\deg(c_1), \deg(c_2) \geq 4$. Thus, there exists no vertex of degree 3. We may assume that $\deg(c_1) + \deg(c_2) \geq \deg(d_1) + \deg(d_2)$, $\deg(c_1) \geq \deg(c_2)$ and $\deg(d_1) \geq \deg(d_2)$. Then we have the following three possibilities for the degrees of the remaining four vertices.

(II.2.1) $\deg(c_1) = \deg(c_2) = \deg(d_1) = 6$ and $\deg(d_2) = 4$.
(II.2.2) $\deg(c_1) = \deg(c_2) = 6$ and $\deg(d_1) = \deg(d_2) = 5$.
(II.2.3) $\deg(c_1) = \deg(d_1) = 6$ and $\deg(c_2) = \deg(d_2) = 5$.

(II.2.1) In this case, from Theorems 1.1 and 1.2, we see that K_8^2 is T_4. The vertex of degree 4 of T_4 is v_0. Hence $d_2 = v_0$. We can assume, without loss of generality, that $a_1 = v_7$ and $b_1 = v_2$. Since v_4 and v_5 are the only vertices of degree 6 which do not form an edge, we can assume $c_1 = v_4$ and $c_2 = v_5$. Then $\varphi^{-1}(acd)$ contains $v_0v_4v_7$ and $v_0v_5v_7$, a contradiction to Proposition 3.6(b) (since, $\#(\varphi^{-1}(\sigma)) = 3 = \deg(\varphi)$, for each 2-simplex σ in S_4^2).

(II.2.2) In this case, the degree sequence of K_8^2 is $7^2 \cdot 6^4 \cdot 5^2$ and hence, by Theorems 1.1 and 1.2, K_8^2 is T_5, T_8, K_1, P_{26}, P_{32}, P_{33} or P_{34}. As K_1 and P_{26} are non-orientable, K_8^2 is not K_1 or P_{26}. Since the two degree 5 vertices of T_5 form an edge, $K_8^2 \neq T_5$.

If K_8^2 is T_8 then $\{d_1, d_2\} = \{v_0, v_3\}$ and $\{a_1, b_1\} = \{v_2, v_5\}$. Then $\varphi^{-1}(abd)$ contains $v_0v_2v_5$ and $v_2v_3v_5$, a contradiction to Proposition 3.6(b).

If $K_8^2 = P_{32}$ then $\{d_1, d_2\} = \{v_3, v_6\}$ and we can assume $a_1 = v_0$, $b_1 = v_7$. Then $\{c_1, c_2\} = \{v_1, v_5\}$ and $\{a_2, b_2\} = \{v_2, v_4\}$. If $(a_2, b_2) = (v_2, v_4)$ then $v_0v_1v_2$, $v_0v_2v_3$, $v_0v_4v_7$, $v_2v_4v_7$ are collapsing and hence $\#(S_a) \leq 7$, a contradiction. If $(a_2, b_2) = (v_4, v_2)$ then $v_0v_4v_5$, $v_0v_4v_7$, $v_2v_4v_7$, $v_2v_5v_7$ are collapsing and hence $\#(S_a) \leq 8$, a contradiction.

If $K_8^2 = P_{33}$ then $\{d_1, d_2\} = \{v_3, v_5\}$ and $\{a_1, b_1\} = \{v_1, v_4\}$. Then $\varphi^{-1}(abd)$ contains $v_1v_3v_4$ and $v_1v_4v_5$, a contradiction to Proposition 3.6(b).

If $K_8^2 = P_{34}$ then $\{d_1, d_2\} = \{v_2, v_3\}$ and $\{a_1, b_1\} = \{v_0, v_7\}$. Then $\varphi^{-1}(abd)$ contains $v_0v_2v_7$ and $v_0v_3v_7$, a contradiction to Proposition 3.6(b).

(II.2.3) In this case also, the degree sequence of K_8^2 is $7^2 \cdot 6^4 \cdot 5^2$ and hence (since K_8^2 is orientable), by Theorems 1.1 and 1.2, K_8^2 is T_5, T_8, P_{32}, P_{33} or P_{34}.

If $K_8^2 = T_5$ then assume (up to an equivalence) that, $a_1 = v_1$, $b_1 = v_2$, $c_2 = v_0$, $d_2 = v_5$. Then, $v_4 = d_1$. These imply, $\varphi(v_0v_2v_4) = \varphi(v_0v_2v_5) = bcd$, a contradiction to Proposition 3.6(b).

If $K_8^2 = T_8$ then assume, without loss of generality, that $a_1 = v_2$, $b_1 = v_5$, $c_2 = v_0$ and $d_2 = v_3$. Then $c_1 = v_1$ and $d_1 = v_6$. So, $\{a_2, b_2\} = \{v_4, v_7\}$. If $(a_2, b_2) = (v_4, v_7)$ the $\varphi(v_2v_3v_7) = \varphi(v_2v_6v_7) = abd$, a contradiction to Proposition 3.6(b). If $(a_2, b_2) = (v_7, v_4)$ the $\varphi(v_0v_5v_7) = \varphi(v_1v_5v_7) = abc$, a contradiction to Proposition 3.6(b).

If $K_8^2 = P_{32}$ then we can assume (up to an equivalence) $a_1 = v_0$, $b_1 = v_7$, $c_2 = v_3$, $d_2 = v_6$ and hence $c_1 = v_4$ and $d_1 = v_2$. Then $\varphi(v_1v_2v_4) = \varphi(v_1v_4v_6)$, a contradiction to Proposition 3.6(b).

If $K_8^2 = P_{33}$ then assume, without loss of generality, that $a_1 = v_1$, $b_1 = v_4$, $c_2 = v_3$ and $d_2 = v_5$. Then $c_1 = v_6$ and $d_1 = v_2$. Clearly, $(a_2, b_2) = (v_0, v_7)$ or (v_7, v_0). In the first case $\varphi(v_1v_3v_4) = \varphi(v_0v_3v_4) = abc$, a contradiction to Proposition 3.6(b). In the second case, $\varphi(v_1v_4v_5) = \varphi(v_4v_5v_7) = abd$, a contradiction to Proposition 3.6(b) again.

If $K_8^2 = P_{34}$ then assume (up to an equivalence) that, $a_1 = v_0$, $b_1 = v_7$, $c_2 = v_2$ and $d_2 = v_3$. Then $c_1 = v_5$, $d_1 = v_6$ and $\{a_2, b_2\} = \{v_1, v_4\}$. If $(a_2, b_2) = (v_1, v_4)$ then $\varphi(v_0v_3v_7) = \varphi(v_1v_3v_7) = abd$, a contradiction to Proposition 3.6(b). So, $(a_2, b_2) = (v_4, v_1)$. In this case, φ is equivalent to g.

Case III. $\chi(K_8^2) = 2$. In this case $f_2(K_8^2) = 12$ and hence there is no collapsing 2-simplex and $(n_1, n_2, n_3, n_4) = (3, 3, 3, 3)$. So, $\deg(a_1) + \deg(a_2) = \deg(b_1) + \deg(b_2) = \deg(c_1) + \deg(c_2) = \deg(d_1) + \deg(d_2) = 9$. Further, by Claim 6.1, a_1a_2, b_1b_2, c_1c_2 and d_1d_2 are not edges.

Observe that K_8^2 cannot have a vertex of degree 4. (If possible let there exist a vertex, say a_1, of degree 4. Let b_1c_1, c_1d_1, d_1x and xb_1 be the edges in $\mathrm{Lk}(a_1)$. Clearly, $x \neq a_2$. If $x = b_2$ or d_2 then there is a collapsing 2-simplex, a contradiction. If $x = c_2$ then Proposition 3.6(b) is contradicted.) Hence, by Theorems 1.1 and 1.2, K_8^2 is S_{15} or S_{20}. But, S_{20} has no pair of vertices, the sum of whose degrees is 9. So, K_8^2 is S_{15}.

If $\deg(a_1) \geq \deg(a_2)$, then $(a_1, a_2) \in \{(v_1, v_5), (v_2, v_0), (v_4, v_7), (v_6, v_3)\}$. We observe that $(v_1, v_2)(v_5, v_0)$, $(v_1, v_4)(v_5, v_7)$, $(v_1, v_6)(v_3, v_5)$, $(v_2, v_4)(v_0, v_7)$, $(v_2, v_6)(v_0, v_3)$ and $(v_4, v_6)(v_0, v_3)$ are all automorphisms of S_{15}. So, we may assume that $(a_1, a_2) = (v_1, v_5)$, $(b_1, b_2) = (v_2, v_0)$, $(c_1, c_2) = (v_4, v_7)$ and $(d_1, d_2) = (v_6, v_3)$. Then (up to an equivalence), $\varphi = f$. The theorem now follows from Example 2.1. □

Remark 6.3. Some of the steps in the proofs of the lemmas in §4 and 5 are similar to the others. Hence we have omitted these details for the sake of brevity. Complete proofs are available with the authors.

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