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Smoothness of density of states for random decaying interaction

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Abstract. In this paper we consider one dimensional random Jacobi operators with decaying independent randomness and show that under some condition on the decay vis-a-vis the distribution of randomness, that the distribution function of the average spectral measures of the associated operators are smooth.

Keywords. Random Jacobi operators; density of states; smoothness.

1. Introduction

One of the questions that is of interest in the theory of Anderson localization is the smoothness of the density of states. The Anderson model is the perturbation of the discrete Laplacian on a lattice perturbed by a potential coming from independent and identically distributed random variables indexed by the lattice. The precise model is given below. Continuity results on the density of states in such models are widely known and we refer to the book of Carmona and Lacroix [1], Figotin–Pastur [3] for the results and references.

There were several results on the question of the smoothness of the density of states relating it to the smoothness of the density of the probability distribution according to which the random variables are distributed. For a complete list of results on the smoothness question of the density of states we refer to the paper of Companino–Klein [2].

Suppose the random variables are distributed according to an absolutely continuous probability measure μ with density f (that is $(1 + |t|)^\alpha \widehat{f}(t)$ is integrable for some $\alpha > 0$). Simon–Taylor [8] showed that even if f has some fractional smoothness, but has compact support, then the density of states is infinitely smooth. After this result, Companino–Klein [2] gave a proof that related the degree of decay of f together with the fractional smoothness to the smoothness of the density of states.

We address the following question in this paper. If instead of taking i.i.d random variables we took random variables which are independent but have decaying coupling constants will these results hold? We will give the model below.

Our motivation for this question is to look at models that exhibit a mobility edge and see if the smoothness results are valid across the mobility edge, since in higher dimensional Anderson model with small disorder it is expected that there are mobility edges and the density of states is expected to be smooth across them.

We have to first look at the correct object which is the analog of the density of states in such models, since there is no single candidate for it. Several definitions on the lines of those for the stationary case can be considered. There is a definition of the density of states, which happens to be non-random, for the case of decaying randomness, given in Gordon–Jacksic–Molchanov–Simon [4].

We however take a definition of the integrated density of states, based on the spectral measures in this paper.

2. The model

We consider the space $\ell^2(\mathbb{Z})$ and the discrete Laplacian

$$(\Delta u)(n) = u(n+1) + u(n-1), \quad u \in \ell^2(\mathbb{Z}).$$

We take independent and identically distributed random variables $q^{\omega}(n)$which are distributed according to the probability measure μ that satisfies the conditions given below.

We then consider the operators

$$(H^{\omega}u)(n) = (\Delta u)(n) + a_n q^{\omega}(n)u(n), \quad u \in \ell^2(\mathbb{Z}), \tag{1}$$

where a_n is a sequence of positive numbers. We shall henceforth denote by V^{ω} the operator of multiplication by $a_n q^{\omega}(n)$ on $\ell^2(\mathbb{Z})$.

Hypothesis 2.1. Let μ be absolutely continuous probability measure and let its characteristic function $h(t) = \int \mathrm{e}^{itx}\mathrm{d}\mu(x)$ satisfy

(1) $(1+|t|)^{\alpha}h(t)$ is bounded for some $\alpha > 0$.

(2) There is a positive integer n, such that $(1+|t|)^{\alpha}h^{(j)}(t)$ is bounded for each $0 \le j \le n$ and some $\alpha > 0$.

We note that the Fourier transform h of the measure μ satisfying the above hypothesis is in $L^p(\mathbb{R})$ for large enough p, since $h(t) \le C/(1+|t|)^{\alpha}$, with $\alpha > 0$. Further Cauchy–Schwarz inequality implies that $h(t) < 1, \quad t \ne 0$. Hence $\int |h(t)|^m \mathrm{d}t$ goes to zero as $m \to \infty$. We shall define for any fixed positive integers k, j and N, the set

$$X_{k,\,j,\,N} = \{-k-N, -k-N-1, \ldots, -1, 0, 1, \ldots, j+N-1, j+N\}.$$

We define the numbers

$$\beta_{k,\,j,\,N,\,N'} = \sup_{S(N') \subset X_{k,\,j,\,N}} \prod_{i \in S(N')} |a_i^{-\alpha/2}|,$$

for any fixed positive integer N' where $S(N')$ is a subset of $X_{k,\,j,\,N}$ of cardinality N'. Let

$$P_{k,\,j,\,N}(S) = \{i \in X_{k,\,j,\,N} \backslash S(N') : i+1 \text{ or } i-1 \text{ is in } X_{k,\,j,\,N} \backslash S(N')\},$$

so that P defines the set of consecutive integers in $X_{k,\,j,\,N}$ that are not in $S(N')$.

In this paper, unless otherwise explicitly stated, we set $\|h\|_p^p = \int_0^{\infty} |h(r^2)|^p \mathrm{d}r$ for any positive number p. The reason for this non-standard notation is that it is in this form that the L^p norms occurs in the estimates (essentially in Lemmas 3.2 and 3.3).

Hypothesis 2.2. Let μ and h be as in the above hypothesis. Let $\{a_m\}$ be a sequence of positive numbers satisfying the following assumptions. There is an N_0 such that for all $|k| \ge N_0$,

(1) $a_k^{-1/2}\|h\|_{|k|}^{|k|} < 1$,

(2) Let N and $N'(< N)$ be arbitrary but fixed positive integers, then the condition

$$\sum_{k,j \geq N_0} \beta_{k,j,N,N'}(k+j+N)^{N'} \left(\sup_{S(N') \subset X_{k,j,N}} \prod_{(i,i+1) \in P_{k,j,N}(S)} \sigma_i \sigma_{i+1} \right) < \infty, \tag{2}$$

is valid, where in the product the pairs $(i, i+1)$ are not repeated and where notationally we have set $\sigma_i = |a_i|^{-(1/2|i|)}\|h\|_{|i|}$.

Remark.

1. We note that the sequence a_k of positive numbers could go to zero, or be bounded below as $k \to \infty$. In the case when a_k is a constant or goes to ∞, the hypothesis is trivially satisfied. Only in the case when a_k goes to zero is it non-trivial and the allowed sequences depend on the function h.
2. We note that since $\|h\|_p$ goes to 1 as $p \to \infty$, the condition (1) on a_k shows that the sequence cannot decay faster than an exponential, and certainly it cannot vanish on infinite subsets of the lattice. This shows that our proof of theorem 2.3 is not applicable for example for finite rank perturbations of Δ, for which the conclusions of the theorem are not valid.
3. The point of defining the set $P_{k,\ j,\ N}$ is that the estimate in Lemma 3.1 uses a pair of operators to get an L^p estimate, so the condition is given in that form.

We consider the standard orthonormal basis $\{e_n\}$ for $\ell^2(\mathbb{Z})$ $(e_n(m) = \delta_{n,m})$. Given the operators H^ω defined in eq. (1), we consider the spectral measures ν_n^ω associated with the standard basis $\{e_n\}$ and the operators H^ω, so that for any bounded continuous function ϕ we have

$$\langle e_n, \phi(H^\omega)e_n \rangle = \int_{\mathbb{R}} \phi(x)\, \mathrm{d}\nu_n^\omega(x).$$

The measure class of the operator H^ω is given by the total spectral measure

$$\sum_{k \in \mathbb{Z}} \alpha_k \nu_k^\omega, \quad \alpha_k > 0, \sum \alpha_k < \infty,$$

for example.

Under the above assumptions on V^ω and μ, it is a standard calculation (see Carmona–Lacroix [1], §V.1) to verify that the (probability measure valued maps) $\omega \to \nu_k^\omega$ are measurable for each k. Hence the map $\omega \to \sum_{k \in \mathbb{Z}} \alpha_k \nu_k^\omega$ is also measurable (as a finite measure valued map). Therefore the following measures are defined as

$$\sigma_k = \int \mathrm{d}\mathbb{P}(\omega)\nu_k^\omega \quad \sigma = \int \mathrm{d}\mathbb{P}(\omega) \sum_{k \in \mathbb{Z}} \alpha_k \nu_k^\omega, \quad \alpha_k > 0 \quad \sum_k \alpha_k < \infty$$

and we have $\sigma = \sum_{k \in \mathbb{Z}} \alpha_k \sigma_k$, by an application of Fubini.

Further if each of the σ_k is absolutely continuous, then so is σ and if the densities of σ_k are n times continuously differentiable with all the derivatives bounded, then the density of σ is also n times continuously differentiable.

While presenting a talk on this paper, Simon remarked that, it is enough to take two instead of the infinitely many ν_k above, since any two ν_k in the above are enough to get a total spectral measure and also that it is possible to use Simon–Taylor [8] to obtain the results similar to ours.

We shall henceforth fix a single summable sequence α_k of positive numbers below when we consider the measure σ.

We consider the measure σ defined above associated with the operators given in eq. (1). We state our main theorems below.

Theorem 2.3. *Consider the operators given in equation* (1)*, with the measure μ satisfying hypothesis* 2.1 *and the sequence $\{a_n\}$ satisfying the hypothesis* 2.2*. Suppose the Fourier transform h of μ satisfies the condition* $(1 + |t|)^\alpha h^{(j)}(t)$ *is bounded for* $\alpha > 0$ *and* $j = 0, 1, \ldots, n$*, then the measure σ is absolutely continuous with its density n/2* (*or respectively* $(n-1)/2$) *times continuously differentiable for even n* (*respectively odd n*).

Remark.

(1) We note that we need the condition on the sequence a_n in relation to h, otherwise the theorem is not true. Take for example the case when $a_0 = 1$ and $a_k = 0$ for all other k. In this case σ is not differentiable at the band edges.
(2) In the case when a_k grows with k, the hypothesis 2.2 is trivially satisfied.
(3) When $a_k \equiv 1$, which is the Anderson model, our definition of σ is a constant multiple of the density of states, since σ_k is independent of k and agrees with the density of states thus recovering the results of Companino–Klein [2].

3. The supersymmetric trick

We follow essentially the ideas of Companino–Klein [2] of using the supersymmetric replica trick to prove the above theorem. We make the necessary changes in their proof to cover our assumptions.

The idea behind the supersymmetric trick is the following: First note that if A is an invertible matrix of size N, then the matrix elements $A^{-1}(x, y)$ of its inverse can be written as the ratio of two determinants, namely $\det(A_{xy})/\det(A)$, where A_{xy} is the matrix obtained from A by dropping the xth row and the yth column. The denominator in this expression can be written in terms of a Gaussian integral, while the numerator is written using the definition of the determinant using antisymmetric tensor products. These two are combined together as a supersymmetric Gaussian integral.

Let us recall, from Companino–Klein [2], the basic steps involved. Let $z \in \mathbb{C}$, then define $H_L^\omega = P_L H^\omega P_L$, where P_L is the orthogonal projection onto $\ell^2([-L, \ldots, L])$. Then using the supersymmetric formalism, one can write

$$\begin{aligned} G_L^\omega(z, x_1, x_2) &= \left\langle e_{x_1}, (H_L^\omega - z)^{-1} e_{x_2} \right\rangle \\ &= i \int \psi(x_1)\overline{\psi}(x_2) \exp\left\{ -i \sum_{x=-L}^{L} \Phi(x) \cdot (H_L^\omega - z)\Phi(x) \right\} D_L\Phi. \end{aligned} \tag{3}$$

In the above $\Phi(x) = (\phi(x), \psi(x), \overline{\psi}(x))$, with $\phi(x) \in \mathbb{R}^2$ and $\psi(x), \overline{\psi(x)}$ are in a Grassman algebra. Notationally

$$\Phi(x) \cdot \Phi(y) = \phi(x) \cdot \phi(y) + \frac{1}{2}(\overline{\psi(x)}\psi(y) + \overline{\psi(y)}\psi(x))$$

and

$$D_L\Phi = \prod_{x=-L}^{L} \mathrm{d}^2\phi(x)\mathrm{d}\psi(x)\mathrm{d}\overline{\psi(x)}.$$

The 'integration' with respect to $\psi, \overline{\psi}$ is a functional given as $\int \mathrm{d}\psi\mathrm{d}\overline{\psi}(a + b\psi + c\overline{\psi} + d\psi\overline{\psi}) = -d$. Since any power series in the symbols $\psi, \overline{\psi}$ reduces to an expression as in the above integrand, the definition of 'integral' could be extended to any such power series.

Given these rules one finds that $\int \exp(-\Phi(x) \cdot \Phi(x))\mathrm{d}\Phi(x) = 1$, as can be checked by expanding the expression $\exp(-\psi(x)\overline{\psi(x)}\mathrm{d}\psi\mathrm{d}\overline{\psi}) = 1$.

Using these rules, taking averages over ω the relation below follows.

$$G_L(z, x_1, x_2) = \mathbb{E}(G_L^{\omega}(z, x_1, x_2)) = i \int \psi(x_1)\overline{\psi}(x_2) \prod_{x=-L}^{L} \beta_x(\Phi(x)^2; z) \times \exp\left\{-i \sum_{x=-L}^{L-1} \Phi(x) \cdot \Phi(x+1)\right\} D_L\Phi, \quad (4)$$

where we have defined $\beta_x(r; z) = h(a_x r)\exp(-izr), x \in \mathbb{Z}$ with h being the Fourier transform of the measure μ. We recall that $h(\Phi^2)$ is defined to be $h(\phi^2) + h'(\phi^2)\overline{\psi}\psi$, the prime denoting the derivative of h as a function of a real variable.

The above equation reduces, after using the rules of supersymmetric integration to reduce the integral and writing it in polar coordinates,

$$G_L(z, 0, 0) = 2i \int_0^{\infty} r\mathrm{d}r \left\{ \left(\prod_{k=1}^{L} TB_k(z)1 \right)(r^2) \right\} \beta_0(r^2; z) \times \left\{ \left(\prod_{k=-1}^{-L} TB_k(z)1 \right)(r^2) \right\}. \quad (5)$$

In the above equation we have used the notation

$$(Tf)(r^2) = -2 \int_0^{\infty} J_0(rs) f'(s^2)\, s\mathrm{d}s \text{ and } (B_k f)(r^2) = \beta_k(r^2; z) f(r^2),$$

with J_0 denoting the Bessel function of order 0, given by

$$J_0(r) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\theta\, \mathrm{e}^{-ir\cos(\theta)},$$

and denoted by 1 the constant function with value 1. The integral in (5) converges absolutely for $\mathrm{Im}(z) > 0$.

We note that an expression similar to eq. (5), results for $G_L(z,k,k)$ for any $k \in \mathbb{Z}$ (if we take the interval of length $2L+1$ around k) and the analysis is similar to the one that will be done for the case of $k=0$, so while we give the proofs for $k=0$, they are also valid for any k. We will henceforth denote $G_L(z,0,0)$ as simply $G_L(z)$ and work with it.

Before proceeding further we explain the idea involved in the proofs. In eq. (5), the right hand side is estimated for any L large enough and it is shown that the derivatives of the right hand side exist as a function of E in $z = E + i\epsilon$, for any E in $\mathbb{R}$ and any $\epsilon \geq 0$. This requires identifying the function spaces between which the operators $TB_k(E+i0)$, $(\mathrm{d}^{(j)}/\mathrm{d}E^{(j)})TB_k(E+i0)$ are bounded. This problem translates itself into the boundedness problem of a different collection of operators. The reason is that the only place where z dependence comes in is in the function β_k and that too as e^{izr^2}, so the derivatives (of β_k in E) become operators of multiplication by r^2. Therefore if T were replaced by the identity operator, in the above expression, it would amount to deciding in the product $\prod_{k=1}^{L} h(a_k r^2)\mathrm{e}^{-iEr^2}$, whether for fixed n, $r^{2n}\prod_{k=1}^{L} h(a_k r^2)$ is a bounded function or not for a given L to conclude if $G_L(E)$ is n times differentiable. This is of course a over simplification and the operator T is necessary for the convergence of the product as L goes to infinity, to some non-zero quantity.

We begin proving a series of lemmas to prove the main theorem. Let us first define the (Hilbert) spaces

$$\mathcal{H}_n = \left\{ f \in \mathcal{C}^n :: \sum_{m=0}^{n}\sum_{k=0}^{m} 2^k \|r^{m-1/2} f^{(k)}(r^2)\|_2^2 < \infty \right\}$$

for any non-negative integer n, where

$$\mathcal{C}_0 = \{f : [0,\infty) \to \mathbb{C},\ f \text{ measurable}\}$$

and

$$\mathcal{C}^n = \{f : [0,\infty) \to \mathbb{C},\ f \text{ is } n\text{–}1 \text{ times differentiable with } f^{(n-1)} \text{ absolutely continuous}\}.$$

We also need the associated spaces

$$\mathcal{H}_0^0 = \mathcal{H}_0 \text{ and } \mathcal{H}_n^0 = \{f \in \mathcal{H}_n : f(0) = 0\}.$$

Lemma 3.1. *The operator T can be written as*

$$(Tf)(r^2) = f(0) + (Rf)(r^2), \ \textit{where}\ (Rf)(r^2) = r\int_0^\infty J_{-1}(rs) f(s^2)\,\mathrm{d}s. \tag{6}$$

The operators T and R are respectively unitary on $\mathcal{H}_n$ and $\mathcal{H}_n^0$ and T leaves each $\mathcal{H}_n^0$ invariant and there $T = R$.

The proof of the above lemma is direct from using Hankel transforms, see equation in (3.5) [2] for example.

Lemma 3.2. *Let β be continuous with $(1+r^2)^{\gamma/2}\beta(r^2)$ be bounded, for some $\gamma > 0$. Then, RB_kRB_l maps $L^\infty(\mathbb{R}^+)$ to $\mathcal{H}_0$, for any k and l.*

Proof. Since $(1+r^2)^{\alpha/2}\beta(r^2)$ is bounded, it follows that $r^{-1/2}\beta(r^2) \in L^q(\mathbb{R}^+)$ for all $2(1-\alpha) < q < 2$ and $\beta \in L^p(\mathbb{R}^+)$ for all $1/\alpha < p \leq \infty$. The map R is related to the Hankel transform via $r^{-1/2}(Rf)(r^2) = H_{-1}(s^{-1/2}f(s^2))(r)$ and we have the Hölder inequality $\|H_n(f)\|_p \leq \|f\|_q,\ \ 1/p + 1/q = 1,\ \ 1 \leq p \leq 2$, for the Hankel transforms. This shows that $|r^{-1/2}RB_l f|^2$ is in $L^p(\mathbb{R}^+)$ for each $1 < p < p_0$. Therefore we pick such p, based on q for which $|\beta_k|^2 \in L^q$, so that $1/q + 1/p = 1$ and we get the bounds

$$\begin{aligned}\|RB_kRB_l f\|^2_{\mathcal{H}_0} &= \|r^{-1/2}B_kRB_l f\|_2^2 \leq \||\beta_k|^2\|_q \||r^{-1/2}RB_l f|^2\|_p \\ &\leq \|\beta_k\|^2_{2q}\|r^{-1/2}\beta_l\|^2_{2q/(q+1)}\|f\|^2_\infty. \qquad (7)\end{aligned}$$

This estimate shows the lemma and in fact it also gives an explicit bound for the norm of the operator RB_kRB_l as a map from L^∞ to $\mathcal{H}_0$.

The proof of the next lemma is identical to that of Lemma 3.1.

Lemma 3.3. *Suppose $\beta \in L^p(\mathbb{R}^+),\ 2 < p \leq \infty$, then RB_kRB_l is bounded as a map from $\mathcal{H}_0$ to itself and the operator norm of this operator has the bound*

$$\|RB_kRB_l\|_{\mathcal{H}_0} \leq \|\beta_k(r^2)\|_p\|\beta_l(r^2)\|_p \leq (a_ka_l)^{-1/2p}\|\beta(r^2)\|_p^2$$

is valid for any k and l.

Remark. In the above the last inequality is from the definition of β_k.

The proof of the lemma below is clear from the definitions of the spaces $\mathcal{H}_n^0$ and the assumptions on β_k which implies that all the derivatives $\beta_k^{(j)}$ are bounded.

Lemma 3.4. *Suppose β satisfies $(1+r^2)^{\alpha/2}\beta^{(j)}(r^2)$ is bounded for each $j = 0, 1, \ldots, n$. Then the operator B_k of multiplication by $\beta_k(r^2; z) = h(a_kr^2)\mathrm{e}^{izr^2}$ is bounded from $\mathcal{H}_n^0$ to itself and the bound is given by*

$$\|B_kf\|_{\mathcal{H}_n^0} \leq C(n,z)\|f\|_{\mathcal{H}_n^0}$$

is valid for any k, with the constant being given by

$$C(n,z) = n \sup_{j=1,\ldots,n}\{\sup_{r\in[0,\infty)} |(1+r^2)^{\alpha/2}\beta^{(j)}(r^2;z)|\}.$$

The proof of the following lemma is given at the end of the paper. The idea behind the proof is the following. If in the product $RB_{i_1}\ldots RB_{i_{l_n}}$, we replace Rs by the identity, then the resulting operator maps the space $\mathcal{H}_0$ into functions in $\mathcal{H}_0$ which have a decay rate of roughly r^{-n} at infinity for suitable l_n, since each B is an operator of multiplication by a function β having a decay rate of $r^{-\alpha}$ at ∞. On the other hand the relation

$$(-2)^m r^{m+k-(1/2)}(Rf)^{(m)}(r) = (-2)^k H_{m+k-1}(s^{m+k-(1/2)}f^{(k)}(s^2))(r), \quad (8)$$

for $k = 0, 1, \ldots, m = 0, 1, \ldots$, valid between the operator R and the Hankel transform H_l shows that the Rs convert a bit of decay into a bit of smoothness. Thus in combination Rs and Bs should map $\mathcal{H}_0$ into $\mathcal{H}_n$ for suitably high powers. Also note that one cannot do better than $\mathcal{H}_n$ since the function β itself is only n times differentiable, by assumption. These heuristic expectations are proved by using interpolation theorems.

Let β denote the function $\beta(r^2; z) = h(r^2)\mathrm{e}^{-izr^2}$, $z \in \overline{\mathbb{C}^+}$. Here we denote by M the operator of multiplication given by $Mf(r^2) = rf(r^2)$. The transpose A^t of a bounded operator A is defined after eq. (21).

Lemma 3.5. *Let μ be a measure satisfying hypothesis* 2.1 (1), (2) *with the number α and the integer n given as in the hypothesis. Then the following are valid.*

1. *The operator M^k is bounded as a map from $\mathcal{H}_j$ to $\mathcal{H}_{j-k}$ provided, $j-k \geq 0$.*
2. *Given $0 \leq k \leq n$, there is an integer l_k depending upon k and α such that $RB_{i_1}RB_{i_2}\dots RB_{i_{l_k}}$ is a bounded operator from $\mathcal{H}_0$ to $\mathcal{H}_k^0$. These operators are bounded uniformly in z in compact subsets of $\mathbb{C}$. The numbers l_k satisfy $0 \leq l_0 \leq l_1 \leq \cdots \leq l_n$.*
3. *The above boundedness statement is also valid if we replace RB by $(RB)^t$ in the above.*

Lemma 3.6. *Let μ be a measure satisfying the hypothesis* 2.1 (1), (2) *with the number α and the integer n given as in the hypothesis. Then for each absolutely continuous bounded function f on $\mathbb{R}^+$, the limits*

$$\begin{aligned} T^+(z)f &= \lim_{L\to\infty}(TB_1TB_2\dots TB_L)f \quad \textit{and} \\ T^-(z)f &= \lim_{L\to\infty}(TB_{-1}TB_{-2}\dots TB_{-L})f \end{aligned} \tag{9}$$

exist in $\mathcal{H}_n$. The convergence is uniform on sets of bounded real z in $\overline{\mathbb{C}^+}$. Further they are given by the power series expansion

$$T^{\pm}(z)f = \sum_{k=0}^{\infty}\left(\prod_{i=\pm 1}^{\pm l_n \pm 1}(TB_i)(z)\right)\prod_{j=\pm l_n \pm 1}^{\pm k}(RB_j)(z)f, \tag{10}$$

valid for all $z \in \overline{\mathbb{C}^+}$, with the convergence compact uniform in $\overline{\mathbb{C}^+}$.

Proof. We will prove the lemma for T^+, the proof for T^- is similar. First consider the sequence $K_L = TB_1\dots TB_Lf$. This is in $\mathcal{H}_n$ for $L \geq l_n$, so it is enough to show that this sequence is Cauchy in $\mathcal{H}_0$ – the reason being that we can write $K_L = (TB_1\cdots TB_{l_n})\tilde{K_L}$, so that whenever $\tilde{K_L}$ is Cauchy in $\mathcal{H}_0$, K_L will be one in $\mathcal{H}_n$ by Lemma 3.5. We consider K_L itself and show that it is Cauchy in $\mathcal{H}_0$, since it differs from $\tilde{K_L}$ by a finite product. Let f be absolutely continuous, then β_k being n times differentiable, $\beta_k f$ is also absolutely continuous and the function TB_kf is well-defined and is given (see eq. (6)) as $(TB_kf)(r^2) = f(0) + (RB_kf)(r^2)$. Therefore using this relation repeatedly we have, using the fact that $\beta_k(0) = 1$ for any k,

$$\begin{aligned} K_L(r^2) &= (TB_1TB_2\dots TB_Lf)(r^2) \\ &= f(0)\left[1 + (RB_1 1) + (RB_1RB_2 1) + \cdots + (RB_1\dots RB_{L-1}1)\right](r^2) \\ &\quad + (RB_1\dots R_{B_L}f)(r^2). \end{aligned} \tag{11}$$

Consider $M > L$ and look at

$$\begin{aligned} \|K_L - K_M\|_{\mathcal{H}_0} &\leq |f(0)|\sum_{k=L+1}^{M}\left\|\left(\prod_{j=1}^{k}RB_k\right)1\right\|_{\mathcal{H}_0} \\ &\quad + \|RB_1\dots R_{B_L}f\|_{\mathcal{H}_0} + \|RB_1\dots R_{B_M}f\|_{\mathcal{H}_0}. \end{aligned} \tag{12}$$

We estimate the right hand side in an index dependent way. For this notice that the function β_k is in $L^p(\mathbb{R}^+)$ for $p > 1/\alpha$, where α is the rate of decay of the function β to zero at ∞.

Therefore for every integer n larger than $1/\alpha$, the function β_k is in L^n. We use this fact to estimate the sum above. We consider for each $k > l_n$, a number $l_n + 1 \le m_k \le l_n + 2$, such that $k - m_k - 1$ is even and estimate the right hand side of the above inequality as

$$\|K_L - K_M\|_{\mathcal{H}_0} \le |f(0)| \sum_{k=L+1}^{M} C(l_n) \left\| \left(\prod_{j=0}^{(k-m_k-1)/2} RB_{m_k+2j} RB_{m_k+2j+1} \right) 1 \right\|_{\mathcal{H}_0} + C(l_n)\|RB_{m_L} \dots R_{B_L} f\|_{\mathcal{H}_0} + C(l_n)\|RB_{m_M} \dots R_{B_M} f\|_{\mathcal{H}_0}. \tag{13}$$

Using the estimates of Lemma 3.5, we get that

$$\begin{aligned}\|K_L - K_M\|_{\mathcal{H}_0} \le |f(0)| &\sum_{k=L+1}^{M} C(l_n) \\ &\times \left(\prod_{j=0}^{\frac{k-m_k-1}{2}-1} (a_{m_k+2j} a_{m_k+2j+1})^{-1/2(m_k+2j+1)} \|h\|^2_{m_k+2j+1} \right) \\ &\times \|RB_{k-1} RB_k 1\|_{\mathcal{H}_0} + C(l_0)\|RB_{m_L} \dots R_{B_L} f\|_{\mathcal{H}_0} \\ &+ C(l_0)\|RB_{m_M} \dots R_{B_M} f\|_{\mathcal{H}_0}. \end{aligned} \tag{14}$$

We note that by hypothesis 2.2(1), and the fact that $|h(t)| < 1$ for any t non-zero, we get for any integer n large enough, $a_n^{-1}\|h\|^{n+1}_{n+1} < 1$, since $\|h^{n+1}_{n+1}\| \le \|h\|^n_n$. To ease writing we set $\alpha_n = a_n^{-1/2}\|h\|^n_n$, then using Lemma 3.6 to estimate the last parts, we get

$$\begin{aligned}\|K_L - K_M\|_{\mathcal{H}_0} \le\ &|f(0)| \sum_{k=L+1}^{M} C(l_n) \\ &\times \left(\prod_{j=0}^{\frac{k-m_k-1}{2}-1} (\alpha_{m_k+2j} \alpha_{m_k+2j+1})^{1/(m_k+2j+1)} \right) \|RB_{k-1} RB_k 1\|_{\mathcal{H}_0} \\ &+ C(l_n) \left(\prod_{j=0}^{\frac{L-m_L-1}{2}-1} (\alpha_{m_L+2j} \alpha_{m_L+2j+1})^{1/(m_L+2j+1)} \right) \|RB_{L-1} R_{B_L} f\|_{\mathcal{H}_0} \\ &+ C(l_n) \left(\prod_{j=0}^{\frac{M-m_M-1}{2}-1} (\alpha_{m_M+2j} \alpha_{m_M+2j+1})^{1/(m_M+2j+1)} \right) \|RB_{M-1} R_{B_M} f\|_{\mathcal{H}_0}. \end{aligned} \tag{15}$$

Using the estimate in Lemma 3.6, we get

$$\begin{aligned}\|K_L - K_M\|_{\mathcal{H}_0} \le |f(0)| &\sum_{k=L+1}^{M} C(l_n) \mathrm{e}^{\left(\sum_{j=0}^{\frac{k-m_k-1}{2}-1} \frac{1}{m_k+2j+1} \ln(\alpha_{m_k+2j+1}) \right)} \\ &\times (\alpha_{k-1} a_k^{\frac{-1}{4(k-1)}} \|r^{-1/2} h(r^2)\|_{(2-\frac{2}{k})}) \end{aligned}$$

$$+ C(l_n)\mathrm{e}^{\left(\sum_{j=0}^{\frac{L-m_L-1}{2}-1} \frac{1}{m_L+2j+1} \ln(\alpha_{m_L+2j+1})\right)}$$

$$\times (\alpha_{L-1} a_k^{\frac{-1}{4(L-1)}} \|r^{-1/2} h(r^2)\|_{(2-\frac{2}{L})}) \|f\|_\infty$$

$$+ C(l_n)\mathrm{e}^{\left(\sum_{j=0}^{\frac{M-l_M-1}{2}-1} \frac{1}{l_M+2j+1} \ln(\alpha_{l_M+2j+1})\right)}$$

$$\times (\alpha_{M-1} a_k^{\frac{-1}{4(M-1)}} |r^{-1/2} h(r^2)\|_{(2-\frac{2}{M})}) \|f\|_\infty. \tag{16}$$

Now using the hypothesis 2.2(1), (2), we get the bound

$$\|K_L - K_M\|_{\mathcal{H}_0} \leq \sum_{k=L+1}^{M} \frac{C_1(l_0, f)}{|k|^{1+\epsilon}} + \frac{C_1(l_0, f)}{|L|^{1+\epsilon}} + \frac{C_1(l_n, f)}{|M|^{1+\epsilon}}, \tag{17}$$

for some $\epsilon > 0$ and some constants $C_1(l_n, f)$, showing that the right hand side goes to zero as L and M go to infinity.

The expression in eq. (11) alongwith the above estimates also show the last assertion of the lemma, since K_L can be written as

$$K_L = TB_1 \ldots TB_{l_{0+1}} \tilde{K_L},$$

with the expression and the estimates are valid as well for $\tilde{K_L}$ in the place of K_L in eq. (11). The number n in the proposition below is as in Theorem 2.3.

PROPOSITION 3.7

The boundary values $\mathbb{E}\{G^\omega(E + i0, 0, 0)\}$, *which we call* $G(E)$, *exist for each* E *in* $\mathbb{R}$ *and* $G(E)$ *is given by the following power series, which converges absolutely and uniformly for* E *in compacts of* $\mathbb{R}$.

$$G(E) = \sum_{k,j=0}^{\infty} K_{k,j}(E),$$

$$K_{k,j}(E) = 2i \int_0^\infty \left(\left(\prod_{i=-1}^{-nl_n-1} TB_k \right) \left(\prod_{i=0}^{j} (RB_{-l_n-2-i} \right) 1 \right) (r^2; E)$$

$$\times \beta_0(r^2; E) \left(\left(\prod_{i=1}^{nl_n+1} TB_k \right) \left(\prod_{i=0}^{k} (RB_{l_n+2+i} \right) 1 \right) (r^2; E)\, r\mathrm{d}r. \tag{18}$$

Proof. We note that the family of operators H_L^ω converge in the strong resolvent sense to H^ω, pointwise in ω, therefore for each $z \in \mathbb{C}^+$, the quantities $G_L^\omega(z, 0, 0)$ converge to $G^\omega(z, 0, 0)$ and since the limits are bounded for each ω, so do their averages. Therefore we have the expression, using the previous lemma,

$$G(z) = 2i \int_0^\infty (T^-(z)1)(r^2)\beta_0(r^2)(T^+(z)1)(r^2) r\mathrm{d}r = \lim_{L,M\to\infty} \sum_{k=0}^{L} \sum_{j=0}^{M} K_{k,j}(z), \tag{19}$$

where $K_{k,\,j}$ is the function defined in equation (18).

Since the above limit exists and the summands are uniformly bounded, as in the estimates in eqs. (13)–(17), in $\mathrm{Im}(z)$ for $\mathrm{Re}(z)$ in compacts, the right hand side converges to the sum stated in eq. (18) for z in reals also. Using these estimates, the proposition follows.

Lemma 3.8. *The function $K_{k,j}(E)$ defined in eq.* (18), $j, k = 0, 1, 2, \ldots$ *is in* $C^{([n/2]-1)}(\mathbb{R})$. *Further we have the estimate*

$$|\mathrm{d}^l/\mathrm{d}E^l K_{k,j}| \leq C(n, h, \alpha, z) j^{nl_n} \beta_{k,j,nl_n} \times \prod_{(i,i+1)\in X_{k,j,nl_n}\setminus S} |a_i^{-(1/2i)} a_{i+1}^{-(1/2(i+1))}| \|h\|_i \|h\|_{i+1}. \tag{20}$$

Proof. We note first that the largest integer k such that $0 \leq 2k \leq n$ is precisely $[n/2]$, which equals $n/2$ if n is even and $(n-1)/2$ if n is odd. We shall work with even n, set $N = n/2$, the proof for the case of odd n is similar.

We first make a few observations before proceeding with the proof. Let us denote the operator of multiplication by r as M, then, it is clear that the operator valued functions $TB_i(E)$ and $RB_i(E)$ are both differentiable in E, with the derivatives agreeing with $TiM^2B_i(E)$ and $RiM^2B_i(E)$ respectively for each i which are bounded as maps from $\mathcal{H}_n$ to $\mathcal{H}_{n-2}$. Next note that if $f \in \mathcal{H}_k^0$ for any k, then we have the equality $\prod_{i=m_1}^{m_2}(TB_i)f = \prod_{i=m_1}^{m_2}(RB_i)f$ is valid for any m_1, m_2.

Therefore we consider a smooth partition of the identity $1 = \chi_1 + \chi_2$ with $\mathrm{supp}(\chi_1) \subset [0, 1]$ and $\mathrm{supp}(\chi_2) \subset [1/2, \infty)$, and write

$$K_{k,j} = K_{k,j,1} + K_{k,j,2} + K_{k,j,3},$$

where the right hand side elements are defined by

$$\begin{aligned}
K_{k,j,1}(E) &= 2i \left\langle\!\!\left\langle 1, \left(\prod_{i=0}^{j}(RB_{-l_n-2-i})\right)^t \left(\prod_{i=-1}^{-nl_n-1} TB_i\right)^t B_0 \left(\prod_{i=1}^{nl_n+1} TB_k\right)\right.\right.\\
&\quad \times \left.\left.\left(\prod_{i=0}^{k}(RB_{l_n+2+i})\right)\chi_1 1\right\rangle\!\!\right\rangle,\\
K_{k,j,2}(E) &= 2i \left\langle\!\!\left\langle \chi_1 1, \left(\prod_{i=0}^{j}(RB_{-nl_n-2-i}\right)^t \left(\prod_{i=-1}^{-nl_n-1} TB_i\right)^t B_0 \left(\prod_{i=1}^{nl_n+1} TB_k\right)\right.\right.\\
&\quad \times \left.\left.\left(\prod_{i=0}^{k}(RB_{l_n+2+i}\right)\chi_2 1\right\rangle\!\!\right\rangle,\\
K_{k,j,3}(E) &= 2i \left\langle\!\!\left\langle \chi_2 1, \left(\prod_{i=0}^{j}(RB_{-nl_n-2-i}\right)^t \left(\prod_{i=-1}^{-nl_n-1} TB_i\right)^t B_0 \left(\prod_{i=1}^{nl_n+1} TB_k\right)\right.\right.\\
&\quad \times \left.\left.\left(\prod_{i=0}^{k}(RB_{l_n+2+i}\right)\chi_2 1\right\rangle\!\!\right\rangle,
\end{aligned} \tag{21}$$

using the notation $\langle\langle u, v\rangle\rangle$ for the bilinear form $\int_0^\infty u(r^2)v(r^2)r^{-1}\mathrm{d}r$, which is continuous on $\mathcal{H}_0$. This bilinear form is related to the inner product on $\mathcal{H}_0$ by $\langle u, v\rangle = \langle\langle \overline{u}, v\rangle\rangle$. We have also used in the above equation the notation A^t for the transpose of a bounded operator on $\mathcal{H}_0$, defined by $A^t u = \overline{A^*\overline{u}}$ for any vector u in $\mathcal{H}_0$, where A^* denotes the adjoint of the operator A. In this notation we have the relation $\langle\langle A^t u, v\rangle\rangle = \langle\langle u, Av\rangle\rangle$. It is then an obvious fact, following from the equivalence of the boundedness of A and A^*, that A is bounded if and only if A^t is bounded. Therefore in the above bilinear form we can shift bounded operators from right to the left at will by transposing the operators. We do this without further explanation in the expressions occurring below.

We will now prove the required estimate for the last term in the above inequality, the proof of the other two terms is similar. The idea is the following, if we take ℓ derivatives of the above as a function of E, then the derivatives of the operator valued functions $B_i(E)$ has to be taken in the product over the index i. When multiple derivatives are taken, then the derivative operation acts on different factors of the product as per the product rule of differentiation. This means that if the product has L factors and we take ℓ-fold derivative, the result will be a sum of l^l distinct terms. Therefore in the expression below we consider a typical term in such a sum of $(k + j + 2nl_n)^l$ terms coming out of taking the l-fold derivative of $K_{k,j,3}(E)$ in the above equation as a function of E. So a typical term in the expansion of $\mathrm{d}^l K_{k,j,3}(E)/\mathrm{d}E^l$ looks like

$$2i \left\langle \phi, (iM)^{2k_1} R^t \left(\prod_{i=x_0}^{x_1} (RB_i)^t\right) (iM)^{2k_2} \left(\prod_{i=x_1+1}^{x_2} (RB_i)^t\right) \dots (iM)^{k_{r-1}} \right.$$
$$\left(\prod_{i=x_r-1}^{x_r} (RB_i)^t\right) B_0 (iM)^{2k_r} \left(\prod_{i=x_r}^{x_{r+1}} (RB_i) \dots (iM)^{2k_{s-1}} \left(\prod_{i=x_{s-1}}^{x_s} (RB_i)\right)\right)$$
$$\left. R(iM)^{k_s} \psi \right\rangle, \quad (22)$$

where $-nl_n - j \le x_0 \le x_1 \le \dots \le x_{r-1} \le 0 \le x_r \le \dots \le x_{s-1} \le x_s = k + nl_n$ and $\sum k_i = l$. We have also set

$$\phi = B_{-nl_n-1-j}\chi_2 1, \quad \psi = B_{nl_n+1-k}\chi_2 1,$$

and notice that both ϕ and ψ are in $\mathcal{H}_0$ in view of the assumptions 2.1. We shall denote the operator product in eq. (22) as Ξ so that it can be written as $\langle\langle \phi, \ \Xi\psi\rangle\rangle$, for easy reference.

The above expression is because each differentiation with respect to E gives rise to an iM^2 factor. Inspecting the above expression, we see that the factor $(iM)^{k_m}(\prod_{i=x_m}^{x_{m+1}}(RB_i))$, or its transpose, is a bounded operator, in view of Lemma 3.5, provided $x_{m+1} - x_m \ge l_{k_m}$, where l_{k_m} is the number given in Lemma 3.5(2), for k_m. Otherwise it is not bounded, and we need to look at the next factor until we find a block of RB_i (or its transpose) such that there are 'enough of them' to make the previous $M^{k_m} M^{k_{m+1}} \dots$ bounded. Therefore suppose r_1 is the index such that

$$\begin{aligned} x_1 - x_0 &\le l_{2k_1} \\ &\dots\dots \\ x_{r_1-1} - x_{r_1-2} &\le l_{2k_{r_1-1}} \\ x_{r_1} - x_{r_1-1} &\ge l_{2k_1+2k_2+\dots+2k_{r_1}}, \end{aligned} \quad (23)$$

then the block of operators up to x_{r_1} is bounded and we inspect the next block of operators. Since the number of factors is finite this operation can be done finitely many times to exhaust the product in the above expression. The reason the above inequalities (especially the lower bound in the above) is valid is that since there are $nl_n + k + j$ factors in the expression for $K_{k,\ j,\ 3}$, taking $l(< N = n/2)$ derivatives will affect at most l of those factors, therefore there will be a block of at least l_n consecutive factors somewhere in the product for which the derivative is not taken (or equivalently where the factor M does not appear).

Suppose we obtain a collection of indices $r_1, \ldots, r_m$ such that

$$\begin{aligned} x_{r_{i-1}+1} - x_{r_{i-1}} &\leq l_{2k_{r_{i-1}+1}} \\ &\ldots\ldots \\ x_{r_i-1} - x_{r_i-2} &\leq l_{2k_{r_i-1}} \\ x_{r_i} - x_{r_i-1} &\geq l_{2k_{r_{i-1}}+2k_{r_{i-1}+1}+\cdots+2k_{r_i}}, \quad i = 1, \ldots, m-1, \end{aligned} \tag{24}$$

and

$$\begin{aligned} x_{r_{m-1}+1} - x_{r_{m-1}} &\leq l_{2k_{r_{m-1}+1}} \\ &\ldots\ldots \\ x_{r_m-1} - x_{r_m-2} &\leq l_{2k_{r_m-1}} \\ x_{r_m} - x_{r_m-1} &\geq l_{2k_{r_{m-1}}+2k_{r_{m-1}+1}+\cdots+2k_{r_m}} + l_{2k_{r_m+1}+2k_{r_m+2}+\cdots+2k_{r_s}}, \\ x_{r_m+1} - x_{r_m} &\leq l_{2k_{r_m+1}} \\ &\ldots\ldots \\ x_{r_s} - x_{r_s-1} &\leq l_{2k_{r_s}}. \end{aligned} \tag{25}$$

The above condition on the indices implies that, for any $l \leq N$, we have

$$\begin{aligned} \sum_{i=1}^{m} (x_{r_i} - x_{r_i-1}) &\geq k + j + nl_n - \left(\sum_{i=1}^{m} \sum_{i'=1}^{r_i-1} l_{2k_{i'}} \right) \\ &\geq k + j + nl_n - (Nl_n) \geq k + j + Nl_n. \end{aligned} \tag{26}$$

Therefore this inequality shows that in the set

$$\{-k - nl_n, \ldots, j + nl_n\}$$

there is a subset S whose complement has at least $k + j + N(l_n - 2)$ consecutive integers.

Now looking at the eq. (22) and the subsequent definition of the operator Ξ, we see that ϕ and ψ are in $\mathcal{H}_0$, so if Ξ is bounded from $\mathcal{H}_0$ to itself, then we can estimate its operator norm. Lemmas 3.5 and 3.1 imply that this is precisely the case and using these lemmas together with the above inequalities for the indices x_i, we obtain the estimate

$$|\langle\langle \phi, \Xi\psi \rangle\rangle| \leq \|\phi\|_{\mathcal{H}_0} \|\Xi\|_{\mathcal{H}_0,\mathcal{H}_0} \|\psi\|_{\mathcal{H}_0}, \tag{27}$$

while the operator norm of Ξ has the bound

$$\|\Xi\|_{\mathcal{H}_0,\mathcal{H}_0} \leq \beta_{k,j,nl_n,l} \prod_{(i,i+1)\in X_{k,j,nl_n}\setminus S(l)} |a_i^{-(1/2i)} a_{i+1}^{-(1/2(i+1))}| \|h\|_i \|h\|_{i+1}. \tag{28}$$

The above estimates imply that, since the expansion of $\mathrm{d}^l/\mathrm{d}E^l K_{k,j,3}$ has $(k+j+nl_n)^l$ such terms, we get the following bound, which we make independent of ℓ, by taking a cruder bound than necessary,

$$|\mathrm{d}^l/\mathrm{d}E^l K_{k,j,3}| \leq (k+j+nl_n)^{nl_n} C(n,h,\alpha,z)\beta_{k,j,nl_n,N} \times \prod_{(i,i+1)\in X_{k,j,nl_n}\setminus S(N)} |a_i^{-(1/2i)} a_{i+1}^{-(1/2(i+1))}| \|h\|_i \|h\|_{i+1}. \tag{29}$$

This proves the lemma.

Proof of Theorem 2.3. Using Proposition 3.7, Lemma 3.8 and Assumption 2.2, the theorem follows since $G(E+i0)$ is seen to be differentiable $n/2$ or $(n-1)/2$ times, depending upon whether n is even or odd.

Proof of Lemma 3.5. We follow the proof of Theorem 5.1 of Companino–Klein [2] (we follow their notation also), but we need explicit bounds on the operator norms which we also obtain. Part (1) of the Lemma is a direct use of the definition of M and the spaces involved.

We turn to part (2), where the statement is obvious for the case $k=0$. We prove the case $k=n$ by induction, the proof for any $0<k<n$ is identical, with n replaced by k.

The following spaces are defined first:

$$\begin{aligned} &X_0 = Y_0 = Z_0 = \mathcal{H}_0, \quad W_0 = \mathcal{H}^0_{n-1}, \quad W_1 = Z_1 = \mathcal{H}^0_n, \\ &X_1 = \{f : \mathbb{R}^+ \to \mathbb{C} : \|(1+r^2)^{n/2} r^{-(1/2)} f(r^2)\|_2 < \infty\}, \\ &Y_1 = \{f : \mathbb{R}^+ \to \mathbb{C} : \sum_{k=0}^{n} \|r^{k-(1/2)} f^{(k)}(r^2)\|_2 < \infty\}. \end{aligned} \tag{30}$$

Denote by $X_t, Y_t, Z_t, t \in [0,1]$, the interpolating spaces between X_0, X_1, Y_0, Y_1 and Z_0, Z_1 pairs respectively. X_t is given explicitly by

$$X_t = \{f : \mathbb{R}^+ \to \mathbb{C} : \|(1+r^2)^{nt/2} r^{-(1/2)} f(r^2)\|_2 < \infty\}, \quad t \in [0,1].$$

The further collection of spaces

$$V_t = \{f : \mathbb{R}^+ \to \mathbb{C} : \|r^{m-1+t-(1/2)} f(r^2)\|_2 < \infty\}, \quad t \in [0,1],$$

are defined. The interpolation spaces between V_t and V_1 are denoted by $V_t^{(2)}$ and so on (taking $V_t^{(1)} = V_t$) and at the mth stage the space is written as $V_t^{(m)}$. For any other pair W_0, W_1 also the $W_t^{(m)}$ is understood in the same way. In the following all the interpolations and the estimates use the Calderon–Lions interpolation theorem (see Theorem IX.20, [7]) and the estimates in that theorem for the norms of the interpolating operators.

We note from eq. (8) that the operator R is bounded between the spaces X_0 to Y_0 and also Y_0 to X_0 and also from X_1 to Y_1 and Y_1 to X_1, with the operator norms bounded by 1. Therefore R is also bounded as

$$\|Rf\|_{X_t} \leq \|f\|_{Y_t}, \quad \|Rf\|_{Y_t} \leq \|f\|_{X_t}.$$

For $\sigma = \alpha/n$, we have an explicit estimate, using the bound $\|(1+r^2)^{\alpha/2}h(r^2)\|_\infty \leq C$, valid by the assumptions 2.1 on h,

$$\|B_k f\|_{X_\sigma} \leq \left(\int |(1+r^2)^\alpha r^{-1}|h(a_k r^2)||\mathrm{e}^{izr^2}|^2|f(r^2)|\mathrm{d}r\right)^{1/2} \leq C a_k^{-\alpha/2}\|f\|_{X_0}.$$

The above two estimates show that

$$\|RB_k f\|_{Y_\sigma} \leq C a_k^{-\alpha/2}\|f\|_{X_0}. \tag{31}$$

Next step is to interpolate between the spaces Ys and Zs. For this consider the operator valued function $S_k(\zeta)$ given by the operator of multiplication by the function $S_k(\zeta, r^2) = \mathrm{e}^{\zeta^2}\beta_k(r^2; z)(1+r^2)^{(\sigma-\zeta)n/2}$, with $\mathrm{Re}(\zeta)$ in [0,1]. Then the hypothesis 2.1 on the function h yields

$$\|S_k(0)f\|_{Z_0} \leq C\mathrm{e}^{\sigma^2} a_k^{-\alpha/2}\|f\|_{Y_0}$$

and

$$\begin{aligned}\|S(1)f\|_{Z_1} &\leq \sum_{m=0}^{n}\sum_{l=0}^{n}\|r^{m-1/2}(eB_k(r^2; z)(1+r^2)^{(\sigma-1)n/2}f)^{(l)}\|_2 \\ &\leq \sum_{m=0}^{n}\sum_{l=0}^{n}\sum_{i\leq l}^{n}\tbinom{l}{i}\|(eB_k(r^2; z)(1+r^2)^{(\sigma-1)n/2})^{(l-i)}r^{m-k}\|_\infty \\ &\quad\times \|r^{k-1/2}f^{(k)}\|_2 \leq C(n, h, z, k)\|f\|_{Y_1}.\end{aligned} \tag{32}$$

In the above estimate, the k dependence in the constant C is such that $a_k^\alpha C(n, h, z, k)$ has a uniform bound in k. Also since the derivatives of the function h are bounded by assumption 2.1 and any finite number of derivatives of the functions e^{izr} are bounded polynomially in z, the stated uniform boundedness in compacts in z is valid for the constant C. Combining the above estimates and using the interpolation theorem for obtaining the bounds between Y_σ and Z_σ, we get

$$\|RB_k RCB_l f\|_{Z_\sigma} \leq D(\sigma, n, h)a_k^{-\alpha/2}a_l^{-\alpha/2}\|f\|_{X_0}, \tag{33}$$

where we have used a cruder bound than is given by the interpolation theorem (using which one would get $a_k^{-\alpha/2}a_l^{-\alpha/2+\sigma/2}$). Again using the interpolation theorem we get, again taking a crude bound,

$$\left\|\prod_{i=1}^{2m} RB_{k_i} f\right\|_{Z_\sigma^{(m)}} \leq D(\sigma, n, h)^{2m}\left(\prod_{i=1}^{2m} a_{m_i}^{-\alpha/2}\right)\|f\|_{X_0}. \tag{34}$$

We use induction now to assume that the theorem is valid for $(n-1)$ and prove it for n. Therefore we assume the estimate

$$\left\|\prod_{i=1}^{l_{n-1}} RB_{m_i} f\right\|_{\mathcal{H}_{n-1}^0} \leq D^{l_{n-1}}\left(\prod_{i=1}^{l_{n-1}} a_{m_i}^{-\alpha/2}\right)\|f\|_{\mathcal{H}_0}, \tag{35}$$

for $(n-1)$ for some l_{n-1} and we have to show that a similar estimate is valid for n with some l_n replacing the l_{n-1}.

In view of the above estimate, we have that $\prod_{i=1}^{l_{n-1}} RB_{m_i}$ is a bounded map from Z_0 to W_0 and since RB_i maps $\mathcal{H}_n i^0$ to itself it is also bounded from Z_1 to W_1 (both of which are just $\mathcal{H}_n^0$). Therefore using interpolation m times one gets that $\prod_{i=1}^{l_{n-1}} RB_{m_i}$ is bounded from X_0 to $W_\sigma^{(m)}$, with the explicit bound, coming from combining the estimates of eqs (34) and (35),

$$\left\| \prod_{i=1}^{l_{n-1}+2m} RB_{m_i} f \right\|_{W_\sigma^{(m)}} \leq D^{l_{n-1}+2m} \left(\prod_{i=1}^{l_{n-1}+2m} a_{m_i}^{-\alpha/2} \right) \|f\|_{\mathcal{H}_0}, \tag{36}$$

where m is chosen so that $(1-\sigma)^m < \alpha$.

We need one final set of estimates to pass from $W_\sigma^{(m)}$ to $\mathcal{H}_n^0$. For this first note that the operator of differentiation $Df = f'$, has the bounds

$$\|D^k f\|_{X_0} \leq \|f\|_{W_0}, \quad \|D^k f\|_{V_1} \leq C(n)\|f\|_{W_1}, \tag{37}$$

for $k = 0, 1, \ldots, n-1$, where the spaces V_t are as defined earlier. Therefore identifying the interpolation spaces at the mth stage

$$V_\sigma^{(m)} = \{f : \mathbb{R}^+ \to \infty : \|r^{n-(1-\sigma)^m-(1/2)} f\|_2 < \infty,$$

we find that if $f \in W_\sigma^{(m)}$, then $f^k \in V_\sigma^{(m)}$, for $k = 0, 1, \ldots, n-1$. This implies that $(B_i f)^k \in V_1$ for $k = 0, 1, \ldots, n-1$, since $(1 + a_i r^2)^{\alpha/2} B_i$ is n times differentiable and all the derivatives are bounded. Since $(1-\sigma)^m < \alpha$, we find that

$$\|r^{n-1/2} B_i f\|_2 \leq C_1'(n, z, h) a_i^{-\alpha} \|f\|_{V_\sigma^m a}.$$

Therefore if $f \in X_0$, then $\prod_{i=1}^{l_{n-1}+2m} RB_{m_i} \in W_\sigma^m i$. Then, for f in $\mathcal{H}_0$,

$$B_{m_{l_{n-1}+2m+2}} \prod_{i=1}^{l_{n-1}+2m+1} RB_{m_i} \in V_1,$$

$$\left(\prod_{i=1}^{l_{n-1}+2m} RB_{m_i} f \right)^{(k)} \in V_1, \quad k = 0, 1, \ldots, n. \tag{38}$$

Together the above estimates imply that $\prod_{i=1}^{l_{n-1}+2m+2} RB_{m_i}$ is a bounded map from $\mathcal{H}_0$ to $\mathcal{H}_n^0$ and setting $l_n = l_{n-1} + 2m + 2$, and collecting the above estimates together we get

$$\left\| \left(\prod_{i=1}^{l_{n-1}+2m+2} RB_{m_i} \right) f \right\|_{\mathcal{H}_n^0} \leq D(n, h, z)^{l_n} \left(\prod_{i=1}^{l_n} a_{m_i}^{-\alpha/2} \right) \|f\|_{\mathcal{H}_0}. \tag{39}$$

Finally the statement of boundedness for the transposed operators is clear from the above proofs, since R^t is also a unitary map between X_σ and Y_σ and B_i^t is also a multiplication operator with the same properties as B_i for each i.

4. Examples and discussion

In this section we present examples of functions h and sequences a_k that satisfy the assumptions 2.1 and 2.2 and discuss the applicability of the results to various operators for which the spectral properties are already known.

1. Our first example is

$$h(t) = \frac{1}{(1+t^2)^{\alpha}}, \quad \alpha \in \mathbb{Z}^+,$$

and

$$a_k = \begin{cases} |k|^{-\beta}, & k \neq 0, \\ 1, & k = 0, \end{cases} \qquad 0 < \beta < 1/2.$$

Clearly h is infinitely differentiable and all the derivatives satisfy $(1+t^2)^{\alpha} h^{(j)}(t)$ which is bounded as a function of t. We compute the L^i norm of h to verify the next condition

$$\left| \int_{\mathsf{R}^+} |h(r^2)|^i \,\mathrm{d}r \right|^{1/i} \leq \left| \int_{\mathsf{R}^+} |(1+r^4)|^{-i\alpha} \,\mathrm{d}r \right|^{1/i} \leq \left(\frac{\pi}{2[i\alpha]^{1/4}} \right)^{1/i},$$

by making use of the bound $(1+a)^b \geq (1+[b]a)$, for any positive b (where $[b]$ denotes its integer part) and a alongwith a change of variables to get the bound. The constant C is independent of i. This bound shows that

$$\|h\|_i \leq \left(\frac{\pi}{2[i\alpha]^{1/4}} \right)^{1/i}.$$

In the above we have taken i to be a positive integer, but the same bound is valid if we replace i by $|i|$ for any non-zero integer, a fact we use below. Therefore for any N and M we have

$$\begin{aligned} &(k+j+N)^{N+M} \left(\prod_{i=-1}^{-k+N} |a_{|i|}|^{-1/2|i|} \|h\|_{|i|} \right) \left(\prod_{i=1}^{j-N} |a_i|^{-1/2i} \|h\|_i \right) \\ &\leq (k+j+N)^{N+M} \left(\prod_{i=1}^{k-N} i^{\beta/2i} \frac{\pi^4}{[i\alpha]^{1/4i}} \right) \left(\prod_{i=1}^{j-N} i^{\beta/2i} \frac{\pi^4}{[i\alpha]^{1/4i}} \right) \\ &\leq \exp\left((N+M)\ln(k+j+N) + \sum_{i=1}^{k-N} \frac{1}{4i} \ln\left(\frac{\pi^4}{[i^{(1-2\beta)}\alpha]} \right) \right. \\ &\qquad \left. + \sum_{i=1}^{j-N} \frac{1}{4i} \ln\left(\frac{C}{[i^{(1-2\beta)}\alpha]} \right) \right). \end{aligned} \tag{40}$$

Since α, N, M are fixed quantities, when $(1-2\beta) > 0$, it is clear that the right hand side has a bound $Dk^{-2}j^{-2}$, which is summable as a function of k and j thus satisfying the assumption 2.2. The requirement $(1-2\beta) > 0$ is satisfied by the assumption on β

we made in this example. We note here however that this assumption on β means that spectrally the operators H^{ω} associated with these sequences a_n and the measure μ given in this example have only pure point spectrum and do not exhibit any mobility edge as can be seen for example from Kiselev–Last–Simon [6].

2. Our next example comes from the Levy stable laws. Let μ be a probability measure on $\mathbb{R}$ such that its Fourier transform is given by $h(t) = \mathrm{e}^{-|t|^{\alpha}}$, $1 < \alpha < 2$. Since, $\alpha > 1$, h is differentiable. We take $a_n = |n|^{-\beta}$, $0 < \beta < 1/\alpha$. We note that when α is close to 1, the values of β can be chosen to be bigger than $1/2$ so that we can cover operators that have mobility edges in the spectrum.
Computing the i norm of h we see that

$$\|h(t^2)\|_i^i = \int_0^{\infty} \mathrm{e}^{-i|t^2|^{\alpha}}\, \mathrm{d}t = \frac{1}{i^{1/2\alpha}} \int_0^{\infty} \mathrm{e}^{-|t|^{2\alpha}}\, \mathrm{d}t = \frac{C}{i^{1/2\alpha}}.$$

Therefore computing the quantity

$$\begin{aligned}
&(k+j+N)^{N+M} \left(\prod_{i=-1}^{-k+N} |a_{|i|}|^{-1/2|i|} \|h\|_{|i|} \right) \left(\prod_{i=1}^{j-N} |a_i|^{-1/2i} \|h\|_i \right) \\
&\leq (k+j+N)^{N+M} \left(\prod_{i=1}^{k-N} i^{\beta/2i} \frac{C^{1/i}}{[i^{\frac{1}{2\alpha}}]^{1/i}} \right) \left(\prod_{i=1}^{j-N} i^{\beta/2i} \frac{C^{1/i}}{[i^{\frac{1}{2\alpha}}]^{1/i}} \right) \\
&\leq \exp\left((N+M)\ln(k+j+N) + \sum_{i=1}^{k-N} \frac{1}{i} \ln\left(\frac{C}{[i^{(1/2\alpha)-(\beta/2)}]} \right) \right. \\
&\quad \left. + \sum_{i=1}^{j-N} \frac{1}{i} \ln\left(\frac{C}{[i^{(1/2\alpha)-(\beta/2)}]} \right) \right). \qquad (41)
\end{aligned}$$

Since we chose $\beta < 1/\alpha$, the above sum can be shown to be bounded by $Dk^{-2}j^{-2}$ for large j and k and hence summable in them.
Let us estimate the i norm of the derivative of h, which is

$$h'(x) = -\alpha|x|^{(\alpha-1)}\mathrm{e}^{-|x|^{\alpha}},$$

so

$$\begin{aligned}
\|h'(t^2)\|_i^i &= \int_0^{\infty} \alpha^i t^{2i(\alpha-1)} \mathrm{e}^{-it^{2\alpha}} \quad \mathrm{d}t \\
&= \frac{1}{i^{1/2\alpha}} \frac{\alpha^i}{i^{(1-(1/2\alpha))i}} \int_0^{\infty} t^{2(\alpha-1)} \mathrm{e}^{-t^{2\alpha}}\, \mathrm{d}t \leq \frac{C}{i^{1/2\alpha}}. \qquad (42)
\end{aligned}$$

This crude bound is similar to that of h itself and we can now verify Hypothesis 2.2 as in the first example. This verification shows that the Theorem 2.3 is valid with $n = 1$, showing that the associated density of states is continuous.

Similar estimates are valid when we take a Gaussian and the associated h, and we can show that the Assumption 2.2 is satisfied, but even in this case only the spectral type of pure point spectrum is covered by the examples.

The above examples satisfy the assumptions (i)–(iv) of Theorem 8.9 of Kiselev–Last–Simon [6], which has examples of operators H_{ω} with purely absolutely continuous, purely singular continuous or pure point spectrum in the interval $(-2,\ 2)$. These examples also can be extended to include the case when there is pure point spectrum outside $(-2,\ 2)$ by an application of the theorem of Kirsch–Krishna–Obermeit [5]. These provide examples of operators with 'continuous density of states' even when the spectrum has a transition from continuous to the pure point (or through a 'mobility edge'). However we are unable to provide examples at the moment, though we believe they exist, of μ and a_n with a high degree of differentiability for the density of states in the regime where there is continuous and pure point spectrum and mobility edges.

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