Proc. Indian Acad. Sci. (Math. Sci.) Vol. 112, No. 1, February 2002, pp. 131–146.

Wegner estimate for sparse and other generalized alloy type potentials

WERNER KIRSCH and IVAN VESELIĆ

Fakultät für Mathematik, Ruhr-Universität, Bochum, Germany*

Abstract. We prove a Wegner estimate for generalized alloy type models at negative energies (Theorems 8 and 13). The single site potential is assumed to be non-positive. The random potential does not need to be stationary with respect to translations from a lattice. Actually, the set of points to which the individual single site potentials are attached, needs only to satisfy a certain density condition. The distribution of the coupling constants is assumed to have a bounded density only in the energy region where we prove the Wegner estimate.

Keywords. Wegner estimate; density of states; random Schrödinger operators; non-stationary potentials; singular distribution of coupling constants.

1. Introduction

It is well-known that random Schrödinger operators show spectral behaviour which is unusual for earlier well studied atomic or N-body Hamiltonians. In dimension $d = 1$ such random operators typically have pure point spectrum. The same phenomenon is believed to occur in dimension $d = 2$. Physical arguments suggest that in higher dimensions and for low disorder random Schrödinger operators have pure point spectrum near the boundary of the spectrum, while absolutely continuous spectrum should occur well inside the spectrum.

Mathematically only the pure point spectrum at the spectral boundary is understood, especially at the infimum of the spectrum. We refer to [15,36,39,1,9,28] and especially to the recent book [37] and the extensive literature given there.

Most models treated so far by mathematicians are either alloy type models of the form

$$V_\omega(x) = \sum_{k\in\mathbb{Z}} q_k(\omega)\, u(x-k) \tag{1}$$

or close relatives. Moreover, all known proofs of localization in multidimensional configuration space require that the distribution of the random variables q_i is absolutely continuous (or at least Hölder continuous). While there seems to be no physical reason for this assumption it turns out to be crucial for the mathematical techniques to work, both for the Aizenman–Molchanov method and for the multiscale analysis. One of the major ingredients of the multiscale analysis is the Wegner estimate. It is exactly this step that requires regularity of the distribution of the random variables.

For one-dimensional models there are proofs of localization which do not require the continuous distribution of the coupling constants, cf. [5,12,4]. These proofs do not rely on

*SFB 237 'Unordnung und große Fluktuationen'; www.ruhr-uni-bochum.de/mathphys
Dedicated to Jean-Michel Combes on the occasion of his sixtieth birthday.

the multiscale analysis and do not imply regularity properties of the integrated densities of states.

In this paper we will among other things prove a Wegner estimate at low energies for distributions of q_k which may have a point mass *away* from the relevant extremal value of their support. As we work with non-positive single site potentials, this means that the distribution of the q_k is assumed to have a density near the supremum of its support but may be arbitrary singular elsewhere. Moreover, we will treat models with non stationary random variables as well as various deviations from the alloy-type model (including surface and sparse potentials). This estimate allows us to prove corresponding localization results (see [2]).

The Wegner estimate is also related to the integrated density of states (IDS). This quantity encodes mathematically the average number of electron states per unit volume up to a given energy. The application of our results to the usual alloy type model imply the Hölder continuity of the IDS at the infimum of the spectrum of the considered random operator.

2. Generalized alloy type models

We denote the unit cube at the lattice site $j \in \mathbb{Z}^d$ by $\Lambda(j) := \Lambda_1(j) := [-1/2, 1/2]^d + j$. Cubes of sidelength $l \in \mathbb{N}$ are denoted by $\Lambda_l(j) = [-l/2, l/2]^d + j$, while the abbreviation Λ (or Λ_l) stands for $\Lambda_1(0)$ (respectively $\Lambda_l(0)$). We let $L^p_{\text{loc,unif}}(\mathbb{R}^d)$ denote the space of measurable functions f for which there exists a finite constant $A > 0$ so that $\|f\|_{L^p(C)} \le A$, for any unit cube $C \in \mathbb{R}^d$. The class of Schrödinger operators which we consider is given by the following:

Assumption 1.

(i) $H_0 := -\Delta + V_0$ is a Schrödinger operator with a potential V_0 in $L^p_{\text{loc,unif}}(\mathbb{R}^d)$ where $p = 2$ for $d \le 3$ and $p > d/2$ for $d \ge 4$.

(ii) $\Xi := \{\xi_k\}_{k\in\mathbb{N}}$ is a countable set of points in $\mathbb{R}^d$. There is a function $\theta\colon \mathbb{R} \to \mathbb{R}$ such that the quantities $\mathcal{L}(j) := \#\{k \in \mathbb{N} |\, \xi_k \in \Lambda(j)\}$ satisfy

$$\mathcal{L}(j) \le C_1\theta(\|j\|_\infty) + C_2 \;\forall j \in \mathbb{Z}^d. \tag{2}$$

Typically θ will be a quadratic function of $\|j\|_\infty$.

(iii) The sequence $u_k \in L^\infty(\Lambda_{l_\infty})$, $k \in \mathbb{N}$ consists of non-positive, measurable functions called *single site potentials*. Here l_∞ is a length scale independent of $k \in \mathbb{N}$. There exists a $u_\infty < \infty$ such that

$$\|u_k\|_\infty \le u_\infty \;\forall k \in \mathbb{N}. \tag{3}$$

(iv) $(\Omega, \mathcal{A}, \mathbb{P})$ is a probability space and $q_k : \Omega \to \mathbb{R}$, $k \in \mathbb{N}$ is a family of random variables called *coupling constants*, each taking values in a subset of the interval $[0, q_+]$ for some fixed, positive $q_+ > 0$. Denote the conditional probability measure of q_k with respect to the other random variables $q_k^\perp := \{q_l\}_{l\in\mathbb{N}\setminus k}$ by μ_k. There exists a constant $q_c \in [0, q_+[$ such that the restriction $\mu_k|_{]q_c,q_+]}$ has a density f_k obeying

$$\sup_{k\in\mathbb{N}} \|f_k\|_\infty \le f_\infty < \infty. \tag{4}$$

DEFINITION 2

The set of points $\xi_k \in \mathbb{R}^d$, $k \in \mathbb{N}$ and the two sequences q_k, $k \in \mathbb{N}$ and u_k, $k \in \mathbb{N}$ define a *random potential* V_ω by

$$V_\omega(x) := \sum_{k\in\mathbb{N}} q_k(\omega) u_k(x - \xi_k). \tag{5}$$

Finally we have a family of random Schrödinger operators:

$$H = H_\omega := H_0 + V_\omega, \ \omega \in \Omega \tag{6}$$

which we call generalized alloy type models.

Remark 3.

1. It would be enough to assume that V_0 lies in an appropriate Kato-class.
2. The clustering condition (2) on the points in Ξ is needed on the one hand to ensure the essential self-adjointness (see § 3), on the other hand to obtain a Wegner estimate with reasonable volume dependence. For the self-adjointness result θ may grow at most quadratically, while interestingly for the Wegner estimate only a subexponential bound

$$\theta(x) \le \exp(x^\beta) \ \text{ for all } x > 0 \text{ and some } \beta < 1 \tag{7}$$

is necessary to give a useful estimate for the multiscale analysis, cf. e.g. [37].
3. In the above assumptions on the random operator H_ω the points $\xi_k \in \mathbb{R}^d$, $k \in \mathbb{N}$ form a deterministic set. In the case $\xi_k = k \in \mathbb{Z}^d$ the operator H_ω describes an Anderson or alloy type model. However, the set Ξ may be much more general, having nothing in common with a lattice structure. It is also interesting to consider the case where it is itself random, i.e. the points $\xi_k \in \mathbb{R}^d$, $k \in \mathbb{N}$ are the support of a random point process $\Xi(\omega)$.

We list several examples which are covered by Assumption 1.

Example 4.

(1) *'Standard' alloy type model.* The set Ξ is simply the lattice $\mathbb{Z}^d$. In this case the Wegner estimate is well understood even under much more general assumptions. We refer to [28,18,10] and the references therein. Particularly, there exist results also in the case where the coupling constants are unbounded [18], where the single site potentials are of long range type [27] or are allowed to change sign [29,38,16]. Moreover, there are results [9,18,38] valid not only at spectral band edges but for any bounded energy intervals, which imply the existence of a density of the IDS.

(2) *Alloy type model with random displacements.* The Wegner estimate has been proven also for some generalizations of the alloy type potential. Already in the paper [9] it was shown that one can incorporate random displacements of the lattice points, at which the single site potentials are attached. The random potential has the form

$$V_\omega(x) := \sum_{k\in\mathbb{Z}^d} q_k(\omega) u(x - k - \xi_k(\omega)),$$

where the ξ_k, $k \in \mathbb{Z}^d$ are independent, identically distributed (i.i.d.) random variables taking values in the unit cube at zero. The single site potential u has a fixed shape independent of k. In [40] it was shown that the random displacements are compatible with long range single site potentials in the sense that one can prove a Wegner estimate and thereby localization combining the techniques of [9] and [27].

(3) *Sparse potentials of alloy type.* In [25,17] several types of sparse alloy type potentials are considered and results about the properties of the spectrum of the corresponding Schrödinger operators are derived. Particularly the existence of absolutely continuous spectrum on the positive energy half axis is established. Our Wegner estimate applies to their models II and III. This is also the case for the analog models on $l^2(\mathbb{Z}^d)$. See [26,32] and the references therein for a discussion of sparse Schrödinger operators of discrete type.

(4) *Surface potentials of alloy type.* If the random potential V_ω is concentrated near a lower dimensional surface in $\mathbb{R}^d$, e.g.

$$V_\omega(x) := \sum_{k \in \mathbb{Z}^{d_1} \times 0} q_k(\omega) u_k(x-k), \tag{8}$$

we call V_ω a *surface potential*. Here $d = d_1 + d_2$, i.e. the lattice $\mathbb{Z}^d$ is directly decomposed into $\mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}$ and the 0 in (8) denotes the zero vector in $\mathbb{Z}^{d_2}$. Results on this type of potentials can be found amongst others in [13,21,22,6–8,19,20] for the discrete case and in [14,30,3] in the case of continuous configuration space.

(5) *Subset of $\mathbb{Z}^d$ containing no nearest neighbours.* Let $\Gamma \subset \mathbb{Z}^d$ be the maximal set containing 0 but no nearest neighbours, i.e. no two points $x, y \in \mathbb{Z}^d$ with $\|x-y\|_1 = 1$, and consider

$$V_\omega(x) := \sum_{k \in \Gamma} q_k(\omega) u_k(x-k).$$

The discrete analog of the corresponding Schrödinger operator is analysed in [33]. There a Wegner estimate for the discrete model can be found, as well as a proof of localization.

(6) *Ergodic Poissonian random potential with coupling constants. Ergodicity with respect to $\mathbb{R}^d$ or $\mathbb{Z}^d$ possible.* Let $p : \mathbb{R}^d \to [0, \infty[$ be a bounded $\mathbb{Z}^d$-periodic function and $\nu_\omega, \omega \in \Omega$ a Poisson point process with intensity measure $\mathrm{d}\lambda(x) = p(x)\mathrm{d}x$ where $\mathrm{d}x$ denotes the Lebesgue measure on $\mathbb{R}^d$, i.e. ν_ω is defined by

$$\mathbb{P}\{\omega | \#\{\operatorname{supp} \nu_\omega \cap A\} = n\} = \mathrm{e}^{-\lambda(A)} \frac{(\lambda(A))^n}{n!}$$

and the condition that for any two disjoint, measurable sets $A, B \subset \mathbb{R}^d$ the random variables $\#\{\operatorname{supp} \nu_\omega \cap A\}$ and $\#\{\operatorname{supp} \nu_\omega \cap B\}$ are independent. The process is assumed to be independent of the random variables $q_k, k \in \mathbb{N}$. We denote the (countable) set of points in $\operatorname{supp} \nu_\omega$ by $\Xi(\omega) := \{\xi_k(\omega)\}_{k \in \mathbb{N}}$. In Lemma 7 we prove that the collection of Poissonian points $\Xi(\omega) = \operatorname{supp} \nu_\omega$ satisfies almost surely condition (2) in Assumption 1 with $\theta(x) = x^2$.

The Lifschitz-Poisson model is given by the Schrödinger operator with the random potential

$$V_\omega(x) := \sum_{k \in \mathbb{N}} q_k(\omega) u_k(x - \xi_k(\omega)). \tag{9}$$

If we assume that all the single site potentials $u = u_k$ have the same shape and moreover the coupling constants $\omega_k, k \in \mathbb{N}$ are i.i.d., the stochastic process $V_\omega(x)$ is ergodic with respect to translations from $\mathbb{Z}^d$ since the intensity measure $p(x)dx$ is periodic. Particulary, if p is a constant, the potential $V_\omega(x)$ is even ergodic with respect to $\mathbb{R}^d$, see e.g. [23,9].

Actually we do not need to assume that the intensity measure λ has a density. Under the assumption that λ is a Radon measure the Poisson process ν_ω is well-defined as well as the potential in (9) (as a multiplication operator).

(7) *Decaying* (*sparse*) *Poissonian random potential with coupling constants.* Similarly as in the example before we can consider a function $p\colon \mathbb{R}^d \to [0, \infty[$ decaying at infinity. The induced Poissonian cloud of points defines again a random potential by the formula (9). However, now the potential is not any more ergodic since p decays at infinity.

(8) *Growing Poissonian random potential with coupling constants.* Now consider an intensity measure growing at infinity at a rate

$$\lambda(\Lambda(j)) \ \leq \text{const}\ \|j\|_\infty^\beta + \ \text{const}$$

for some $\beta < 2$ and define the Poisson point process and the Lifschitz–Poisson model as before. In Lemma 7 we prove that the points $\Xi(\omega) = \operatorname{supp} \nu_\omega$ satisfy condition (2) in Assumption 1 with $\theta(x) = x^2$ almost surely.

(9) *Compound diluted alloy potential.* In the Example (6) consider the intensity measure

$$\lambda(x) = \sum_{k\in\mathbb{Z}^d} \delta_k(x) \tag{10}$$

and moreover the case that all $u_n = u, n \in \mathbb{N}$ are equal and the $q_n, n \in \mathbb{N}$ are i.i.d. with single site measure μ. The potential has the form

$$V_\omega(x) = \sum_{k\in\mathbb{N}} q_k(\omega)u(x - \xi_k(\omega)) = \int_{\mathbb{R}^d} q_k(\omega)u(x - y)\ \mathrm{d}\,\nu_\omega(y). \tag{11}$$

However, it can also be written as an alloy type potential

$$V_\omega(x) = \sum_{k\in\mathbb{Z}^d} Q_k(\omega)u(x - k). \tag{12}$$

The new coupling constants $Q_k, k \in \mathbb{N}$ are i.i.d., too, and have the distribution measure

$$\mu_Q = \ \mathrm{e}^{-1} \left(\delta_0 + \sum_{n\in\mathbb{N}} \frac{1}{n!}(\mu)^{*n}\right). \tag{13}$$

Here $(\mu)^{*n}$ denotes the n-fold convolution of μ with itself.

(10) *Discrete analoga.* Our proofs remain valid if we replace everywhere in the Assumption 1 the continuous configuration space $\mathbb{R}^d$ by the lattice $\mathbb{Z}^d$ and consider the discrete analoga of continuous Schrödinger operators.

Remark 5.

1. Note that we require the single site distributions μ_k to be absolutely continuous near their extremal value q_+ since we want to prove a Wegner estimate only near the infimum of the spectrum. The new technique presented in § 5 requires the absolute continuity of the coupling constants only in the relevant energy region.

2. In [10] an abstract Wegner estimate applicable to quite general operators which need not be of Schrödinger type can be found.

Let $\Lambda^+ = \{k \in \mathbb{N} | \operatorname{supp} u_k \cap \Lambda \neq \emptyset\}$ be the set of indices whose coupling constants influence the value of the potential in the cube Λ and $L := \#\{\Lambda^+\}$ their cardinality. The expectation with respect to $q := \{q_k\}_{k\in\mathbb{Z}^d}$ is denoted by $\mathbb{E}$. The set of lattice points $\Lambda \cap \mathbb{Z}^d$ is denoted by $\tilde{\Lambda}$.

3. Self-adjointness

In this section we discuss the self-adjointness property of random Schrödinger operators obeying Assumption 1. Regarding the clustering property (2) we assume $\theta(x) = x^2$.

PROPOSITION 6

Under the Assumption 1 *with* $\theta(x) = x^2$ *each Schrödinger operator of the family* H_ω, $\omega \in \Omega$ *is essentially self-adjoint on* $C_0^\infty(\mathbb{R}^d)$. *For any cube* $\Lambda_l(j)$ *we can restrict* H_ω *to* $L^2(\Lambda_l(j))$ *with Dirichlet boundary conditions (b.c.), resulting in a self-adjoint operator* $H_\omega^{l,j}$.

Let $W_0^{2,2}(\Lambda_l(j))$ denote the domain of the Dirichlet Laplacian on $L^2(\Lambda_l(j))$. For any self-adjoint operator A on $L^2(\mathbb{R}^d)$ we will use the notation $A^{l,j}$ for its restriction to $W_0^{2,2}(\Lambda_l(j))$.

Proof. For an $x \in \mathbb{R}^d$ let $n \in \mathbb{Z}^d$ denote any lattice point with $\|x - n\| \leq 1/2$. By the uniform bound $u_k \geq -u_\infty$ for all $k \in \mathbb{N}$ we infer

$$\begin{aligned} -(q_+u_\infty)^{-1} V_\omega(x) &\leq \sum_{k\in\mathbb{N}} \chi_{\Lambda_{l_\infty}(x)}(\xi_k) \leq \sum_{k\in\mathbb{N}} \sum_{j\in\tilde{\Lambda}_{l_\infty+1}(n)} \chi_{\Lambda(j)}(\xi_k) \\ &\leq \sum_{j\in\tilde{\Lambda}_{l_\infty+1}(n)} (C_1\|j\|_\infty^2 + C_2) \\ &\leq (l_\infty + 2)^d (C_1(\|n\|_\infty + (l_\infty + 1)/2)^2 + C_2) \\ &\leq \text{ const } \|x\|_2^2 + \text{ const.} \end{aligned}$$

This implies that the potential in H_ω can be written as $V_1 + V_2$ satisfying $0 \geq V_1(x) \geq -\text{ const }\|x\|_2^2 - \text{ const}$ and $V_2 \in L^p_{\text{loc,unif}}(\mathbb{R}^d)$ with $p = p(d)$ as in Assumption 1. Now essential self-adjointness follows by the Farris–Lavine theorem, cf. Corollary of Theorem X.38 in [34].

On a finite cube we have $V_1 + V_2 \in L^p(\Lambda_l(j))$ which suffices for the self-adjointness. q.e.d.

In the case where the set of points $\Xi = \{\xi_k\}_{k\in\mathbb{N}}$ is generated by a random Poisson process we give a condition on the intensity measure $\lambda(x)$ of the process which ensures that the resulting family of random Schrödinger operators (9) is essentially self-adjoint almost surely.

Lemma 7. *Let* λ *be a Radon measure satisfying the growth condition*

$$\lambda(\Lambda(j)) \leq c_1 \|j\|^\beta + c_2 \tag{14}$$

for some $\beta < 2$ and all $j \in \mathbb{Z}^d$. Define the Poisson point process ν_ω with intensity measure λ as in Example 4 (6)*. Then the set of points $\Xi(\omega) = \operatorname{supp} \nu_\omega$ satisfies condition* (2) *of Assumption* 1 *with finite constants $C_1 = 1$, $C_2 = C_2(\omega)$ and $\theta(x) = x^2$ almost surely.*

Proof. It is sufficient to prove that

$$\mathbb{P}\{\omega| \text{ there are infinitely many } j \in \mathbb{Z}^d \text{ with } \mathcal{L}(j) > \|j\|^2\} = 0. \tag{15}$$

Namely, if (15) is satisfied, there exists for all ω in a set $\Omega' \subset \Omega$ of full measure an exceptional, finite set $\Gamma(\omega) \subset \mathbb{Z}^d$ such that

$$\mathcal{L}(j) = \mathcal{L}_\omega(j) = \#\{\operatorname{supp} \nu_\omega \cap \Lambda(j)\} \le \|j\|_\infty^2 \quad \forall j \in \mathbb{Z}^d \setminus \Gamma(\omega).$$

On the other hand, for almost all $\omega \in \Omega'$,

$$\#\{\operatorname{supp} \nu_\omega \cap \bigcup_{j \in \Gamma(\omega)} \Lambda(j)\}$$

is bounded by a finite real $C_2(\omega)$. Thus

$$\mathcal{L}_\omega(j) \le \|j\|_\infty^2 + C_2(\omega)$$

for almost all ω.

Equation (15) is proven by the Borel–Cantelli lemma if

$$\sum_{j \in \mathbb{Z}^d} \mathbb{P}\{\omega| \, \mathcal{L}_\omega(j) > \|j\|_\infty^2\} < \infty. \tag{16}$$

To prove (16) we apply the Čebyšev inequality and consider

$$\sum_{j \in \mathbb{Z}^d} \mathrm{e}^{-\|j\|_\infty^2} \, \mathbb{E}(\mathrm{e}^{\mathcal{L}_\omega(j)}). \tag{17}$$

Now for the Poisson process $\mathbb{E}(\mathrm{e}^{\mathcal{L}_\omega(j)}) = \mathrm{e}^{(e-1)\lambda(\Lambda(j))} \le \mathrm{e}^{(e-1)(c_1\|j\|_\infty^\beta + c_2)}$ by condition (14). Thus

$$(17) \le \sum_{j \in \mathbb{Z}^d} \exp(-\|j\|_\infty^2 + (\mathrm{e}-1)c_1\|j\|_\infty^\beta + (\mathrm{e}-1)c_2) < \infty.$$

q.e.d.

4. Wegner estimate for generalized alloy type potentials

One of the two crucial ingredients for the localization proof via multiscale analysis is the so-called Wegner estimate. It controls the probability of the appearance of quantum mechanical resonances, i.e. tunneling events, between disjoint regions of configuration space.

In this section we consider the case that the conditional single site measure μ_k has a density f_k on the whole of its support, i.e. in the language of Assumption 1 we have $q_c < 0$. The changes needed for the general case will be treated in the next section.

The Wegner estimate is an assertion about the spectral properties of a random Schrödinger operator restricted to a cube $\Lambda_l(j)$. We assume without loss of generality

$\inf \sigma(H_0^{l,j}) = 0$ which may be ensured by adding a constant to the potential. We denote the spectral projection of $H_\omega^{l,j}$ on the energy interval $I =]E_1, E_2[$ by $P_\omega^{l,j}(I)$. Denote $\Theta := \Theta(l, j) = \sum_{k \in \Lambda_l^+(j)} \theta(k)$, $E_+ := E + 2\epsilon$ for $E \in \mathbb{R}$ and some $\epsilon > 0$, and

$$\tilde{\Theta} := \tilde{\Theta}(l, j) := \max \left\{ \int_{\Lambda_l(j)} (|V_0 - E_+| + 1)^{\tilde{p}} \, dx, \Theta(l, j)^{\tilde{p}} \right\},$$

where $\tilde{p} = d/2$ for dimensions $d \geq 3$ and $\tilde{p} = 2$ in the one and two dimensional case. Note that $\int_\Lambda |V_0 - E_+|^{\tilde{p}} \, dx \leq l^d \sup_{m \in \mathbb{Z}^d} \int_{\Lambda(m)} |V_0 - E_+|^{\tilde{p}} \, dx < \infty$ by our Assumption 1.

Theorem 8. *Let $E < 0$ and $\epsilon > 0$ be such that $-\delta := E + 2\epsilon < 0$. Then there exists a constant $C(\delta)$ such that for all $l \in \mathbb{N}$*

$$\mathbb{E}\left[\mathrm{Tr}\, P_\omega^{l,j}(]E - \epsilon, E + \epsilon[)\right] \leq C(\delta)\, \epsilon\, \Theta\, \tilde{\Theta}. \tag{18}$$

Let $h \in]0, 1[$. Then there exists a constant $C(h, \delta)$ such that

$$\mathbb{E}\left[\mathrm{Tr}\, P_\omega^{l,j}(]E - \epsilon, E + \epsilon[)\right] \leq C(h, \delta)\, \epsilon^h\, \Theta. \tag{19}$$

COROLLARY 9

In the case $\Xi = \mathbb{Z}^d$ we have under the assumptions of Theorem 8

$$\mathbb{E}\left[\mathrm{Tr}\, P_\omega^{l,j}(]E - \epsilon, E + \epsilon[)\right] \leq \min[C(\delta)\epsilon\, l^{2d}, C(h, \delta)\epsilon^h\, l^d]. \tag{20}$$

This follows if we replace the estimates (29) and (30) in the proof of Theorem 8 by standard estimates as found in [24].

Remark 10.

1. By the Čebyšev inequality (18) and (19) imply

$$\mathbb{P}\{\omega |\, \sigma(H_\omega^{l,j}) \cap]E - \epsilon, E + \epsilon[\neq \emptyset\} \leq \mathbb{E}\{\mathrm{Tr}\, P_\omega^{l,j}(]E - \epsilon, E + \epsilon[)\}.$$
$$\leq \min\left\{C(\delta)\epsilon\, \Theta\, \tilde{\Theta}, C(h, \delta)\epsilon^h\, \Theta\right\}$$

   Actually for the application in the multiscale analysis only this weaker form of the Wegner estimate is needed. It tells us that the probability that an eigenvalue of the restricted operator hits a given energy interval becomes small if we shrink the interval.
2. The second bound (19) in Theorem 8 implies that the IDS is Hölder continuous below the spectrum of the unperturbed operator H_0, if we know *a priori* (by ergodicity) that the IDS exists. This is the case if $\Xi = \mathbb{Z}^d$, all single site potentials $u_k = u$ have the same shape and the coupling constants q_k are i.i.d. Then one can define the IDS as the thermodynamic limit $\Lambda_l(j) \to \mathbb{R}^d$ of the normalized eigenvalue counting functions

$$N_\omega^l(E) = l^{-d} \#\{i |\, \lambda_i(H_\omega^l) < E\} = l^{-d}\, \mathrm{Tr}\, P_\omega^l(]-\infty, E[) \tag{21}$$

   of H_ω^l, which converge for almost all ω to a limit $N := \lim_{l \to \infty} N_\omega^l$ which is ω-independent. Now Corollary 9 implies

$$N(E_2) - N(E_1) \leq C(h, \delta)(E_2 - E_1)^h \quad \forall E_1 \leq E_2 \leq -\delta < 0.$$

3. We state two versions of the upper bound in Theorem 8. In the ergodic case in the limit $l \to \infty$ the first one diverges while the second one yields the Hölder continuity of the IDS. Still, as one often has to consider operators restricted to *finite* cubes for technical reasons, the linear dependence on the length of the energy interval, as given in the fist bound (18), may be easier to work with.

Proof. Let $\rho \colon \mathbb{R} \to [0,1]$ denote a smooth monotone function taking the value 0 on $]-\infty, -\epsilon]$ and the value 1 on $[\epsilon, \infty[$. Denote with $E_n(q) = E_n^{l,j}(\omega)$ the nth eigenvalue of $H_\omega^{l,j}$ counted from below. We estimate the expectation of the trace of the spectral projector as in the proof of Proposition 1 in [24]:

$$\mathbb{E}\Big(\mathrm{Tr}\, P_\omega^{l,j}([E-\epsilon, E+\epsilon])\Big) \le \int_{\mathbb{R}^L} \prod_{k\in\Lambda^+} f_k(q_k)\, \mathrm{d}q_k \times \sum_{n\in\mathbb{N}} \int_{-2\epsilon}^{2\epsilon} \mathrm{d}t\, \rho'(E_n(q) - E + t). \tag{22}$$

One crucial step in Wegner's original paper (and in the proof of Proposition 1 in [24]) is the replacement of the derivative ρ' by derivatives with respect to the coupling constants $q_k, k \in \mathbb{Z}^d$. We write $\Lambda^+ = \Lambda_l^+$ and suppress the dependence on l. The chain rule gives

$$\sum_{k\in\Lambda^+} \frac{\partial \rho(E_n(q) - E + t)}{\partial q_k} = \rho'(E_n(q) - E + t) \sum_{k\in\Lambda^+} \frac{\partial E_n(q)}{\partial q_k},$$

so it remains to find a bound on $\sum_{k\in\Lambda^+}(\partial E_n(q)/\partial q_k)$ which is independent of n and Λ. To achieve this aim we modify a trick from the recent paper ([10], p. 16) and prove the following.

Lemma 11. *Assume that the n-th eigenvalue of the operator $H_\omega^{l,j}$ satisfies $E_n(q) \le -\delta < 0$. Then*

$$\rho'(E_n(q) - E + t) \le \frac{q_+}{\delta}\left[-\sum_{k\in\Lambda^+} \frac{\partial \rho(E_n(q) - E + t)}{\partial q_k}\right].$$

Proof of the lemma. Let ψ_n be the normalized eigenfunction corresponding to $E_n(q)$. Then ψ_n satisfies by definition $\langle \psi_n, (H_0^{l,j} - E_n(q))\psi_n\rangle = -\langle \psi_n, V_\omega^{l,j}\psi_n\rangle$. We have

$$\begin{aligned} -\sum_{k\in\Lambda^+} q_k\, \langle \psi_n, u_k(\cdot - \xi_k)\psi_n\rangle &= -\langle \psi_n, V^{l,j}\psi_n\rangle \\ &= \langle \psi_n, (H_0^{l,j} - E_n(q))\psi_n\rangle \ge \delta. \end{aligned}$$

Now we have by the Hellman–Feynman theorem

$$\begin{aligned} -\sum_{k\in\Lambda^+} \frac{\partial E_n(q)}{\partial q_k} &= -\sum_{k\in\Lambda^+} \langle \psi_n, u_k(\cdot - \xi_k)\psi_n\rangle \\ &\ge -q_+^{-1} \sum_{k\in\Lambda^+} q_k\, \langle \psi_n, u_k(\cdot - \xi_k)\psi_n\rangle \ge \frac{\delta}{q_+} \end{aligned} \tag{23}$$

uniformly for all eigenvalues not exceeding $\delta < 0$. This gives

$$\rho'(E_n(q) - E + t) = -\left[-\sum_{k\in\Lambda^+} \frac{\partial E_n(q)}{\partial q_k}\right]^{-1} \sum_{k\in\Lambda^+} \frac{\partial \rho(E_n(q) - E + t)}{\partial q_k}$$
$$\leq \frac{q_+}{\delta}\left[-\sum_{k\in\Lambda^+} \frac{\partial \rho(E_n(q) - E + t)}{\partial q_k}\right].$$

Note that since ρ is monotone increasing and u is non-positive,

$$-\sum_{k\in\Lambda^+} \frac{\partial \rho(E_n(q) - E + t)}{\partial q_k} \tag{24}$$

is a non-negative real.

Remark 12.

1. Our modification in comparison to ([10], p. 16) lies in the fact that we consider the quadratic form $\langle f, V_\omega^{l,j} f\rangle$ rather than the square of the norm $\|V_\omega^{l,j} f\|^2$ and thus replace quadratic dependence on the coupling constants by a linear one. The quadratic dependence leads to some restrictions of the single site potentials u to which the estimate can be applied, e.g. the single site potentials may not overlap.
   On the other hand, the estimate in [10] is more general, in that it can be used for Wegner estimates for energy intervals in spectral gaps.
2. The lemma presented above relies heavily — as is often the case for certain steps of the proof of the Wegner estimate — on the fact that the considered eigenvalues lie *outside* the unperturbed spectrum $\sigma(H_0)$. In a sense the proof of the result is a perturbation argument. Under additional assumptions an analog of the lemma can be proven also for energy eigenvalues inside the unperturbed spectrum, cf. [24].

Continuation of the proof of the theorem. Denote with $H_\omega^{l,j}(q_j = q_+) = H_\omega^{l,j} + (q_+ - q_k)u_k(\cdot - \xi_k)$ the operator $H_\omega^{l,j}$, but with the jth coupling constant set to its maximal value, and with $E_n(q, q_j = q_+)$ its nth eigenvalue. $H_\omega^{l,j}(q_j = 0)$ and $E_n(q, q_j = 0)$ are defined analogously.

We consider the term in (24) with the derivative with respect to q_j and integrate it over the same random variable:

$$-\int_0^{q_+} f_k(q_j)\,\mathrm{d}\,q_j\, \frac{\partial \rho(E_n(q) - E + t)}{\partial q_j} \leq f_\infty\big[\rho(E_n(q, q_j = 0) - E + t) - \rho(E_n(q, q_j = q_+) - E + t)\big], \tag{25}$$

where we used monotonicity of the functions $E \mapsto \rho(E)$ and $q_j \mapsto E_n(q_j)$. This gives

$$\mathbb{E}\big(\operatorname{Tr} P_\omega^{l,j}([E-\epsilon, E+\epsilon])\big) \leq \frac{q_+ f_\infty}{\delta} \sum_{j\in\Lambda^+} \int_{\mathbb{R}^{L-1}} \prod_{k\in\Lambda^+\setminus j} f_k(q_k)\,\mathrm{d}q_k \times \int_{-2\epsilon}^{2\epsilon} \mathrm{d}t \sum_{n\in\mathbb{N}} \big[\rho(E_n(q, q_j = 0) - E + t) - \rho(E_n(q, q_j = q_+) - E + t)\big]. \tag{26}$$

There are now two ways to estimate

$$\sum_{n\in\mathbb{N}}[\rho(E_n(q,q_j=0)-E+t)-\rho(E_n(q,q_j=q_+)-E+t)]$$
$$=\mathrm{Tr}[\rho(H_\omega^{l,j}(q_j=0)-E+t)-\rho(H_\omega^{l,j}(q_j=q_+)-E+t)]. \quad (27)$$

1. Either one simply bounds (27) by

$$\#\{n\in\mathbb{N}|E_n(q,q_j=0)>E-t-\epsilon,\ E_n(q,q_j=q_+)<E-t+\epsilon\}$$
$$\le \#\{n\in\mathbb{N}|E_n(q,q_j=q_+)<E+2\epsilon\} \quad (28)$$

   and then uses the Cwikel–Lieb–Rosenbljum or the Lieb–Thirring bound, depending on the dimension d, cf. [35]. In the following we suppress the argument $(q,q_j=q_+)$ of the eigenvalue, write $E_+:=E+2\epsilon$ and denote with

$$V=V_0+q_+u_j(\cdot-\xi_j)+\sum_{k\in\mathbb{N}\setminus j}q_k(\omega)u_k(x-\xi_k)$$

   the full potential energy. In the case $d\ge 3$ we have

$$\begin{aligned}\#\{n\in\mathbb{N}|E_n<E_+\}&\le C_d\int_{\Lambda\cap\{V\le E_+\}}|V-E_+|^{d/2}\,\mathrm{d}x\\&\le C_d\left[\||V_0-E_+|^{d/2}\|_{L^{d/2}(\Lambda)}+\sum_{k\in\Lambda^+}q_k\|u_k\|_{L^{d/2}(\Lambda)}\right]^{d/2}\\&\le 2^{d/2}C_d\max\{1,(q_+u_\infty)^{d/2}l_\infty^d\}\\&\quad\times\max\left\{\int_\Lambda|V_0-E_+|^{d/2}\,\mathrm{d}x,\ \Theta(l,j)^{d/2}\right\},\end{aligned} \quad (29)$$

   which leaves us with the case $d=2$ and $d=1$. Here we estimate similarly

$$\begin{aligned}\#\{n\in\mathbb{N}|E_n<E_+\}&\le C_d\int_{\Lambda\cap\{V\le E_++1\}}|V-E_+-1|^2\mathrm{d}x\\&\le C_d\left[\||V_0-E_+-1|^2\|_{L^2(\Lambda)}+\sum_{k\in\Lambda^+}q_k\|u_k\|_{L^2(\Lambda)}\right]^2\\&\le 4C_d\max\{1,(q_+u_\infty)^2l_\infty^d\}\max\left\{\int_\Lambda|V_0-E_+-1|^2\mathrm{d}x,\ \Theta(l,j)^2\right\}\end{aligned} \quad (30)$$

   finishing the proof of (18).
2. Or one proceeds using estimates on the spectral shift function, as done in [11], cf. eqs (3.29)–(3.32) there. For any $h\in]0,1[$ there is a $C(h)$ such that

$$\mathrm{Tr}\left[\rho(H_\omega^{l,j}(q_j=0)-E-t)-\rho(H_\omega^{l,j}(q_j=q_+)-E-t)\right]\le C(h)\,\epsilon^{h-1}.$$

   In this case we can bound (26) by $4C(h)\delta^{-1}q_+f_\infty\,\epsilon^h\,\Theta$.

q.e.d

5. Wegner estimate for single site measures with singular components

Consider a generalized alloy type potential V_ω where only the restrictions of the single site measures μ_k to the interval $]q_c, q_+]$ have a density f_k, i.e. the extremal configurations of the potential are continuously distributed. We prove a Wegner estimate at the bottom of the spectrum of H_ω.

In a sense q_c is a critical value for the random variables q_k: for $q_k > q_c$ we know that q_k is continuously distributed, for smaller values we know nothing. There is a corresponding decomposition of the 'probability' space $\times_{k\in\Lambda^+}\mathbb{R} = \mathbb{R}^L$ which will turn out to be useful. For a given configuration of coupling constants $\{q_k\}_{k\in\Lambda^+}$ set $A^{ac}(\omega) = \{k \in \Lambda^+ | \, q_k > q_c\}$, then

$$\sum_{A\subset\Lambda^+} \int_{\mathbb{R}^L} \prod_{k\in\Lambda^+} \mathrm{d}\,\mu_k(q_k)\, \chi_{\{A^{ac}(\omega)=A\}}\,(\omega) = 1. \tag{31}$$

Define now an auxiliary background potential $V_1 = \sum_{k\in\Lambda^+} q_c u_k(\cdot - \xi_k)$ and by adding a constant assume $\inf \sigma((H_0 + V_1)^{l,j}) = 0$. Furthermore set

$$V_\omega^1 := \sum_{k\in\Lambda^+, q_k\le q_c} q_k u_k(\cdot - \xi_k) + \sum_{k\in\Lambda^+, q_k> q_c} q_c u_k(\cdot - \xi_k) \ge V_1, \tag{32}$$

$$V_\omega^2 := \sum_{k\in\Lambda^+, q_k> q_c} r_k u_k(\cdot - \xi_k) = \sum_{k\in A^{ac}} r_k u_k(\cdot - \xi_k),$$

$$\text{where } r_k = q_k - q_c > 0. \tag{33}$$

Theorem 13. *Let $E < 0$ and $\epsilon > 0$ be such that $-\delta := E + \epsilon < 0$. Then there exists a constant $C(\delta)$ such that for all $l \in \mathbb{N}$*

$$\mathbb{E}\Big[\mathrm{Tr}\, P_\omega^{l,j}(]E-\epsilon, E+\epsilon[)\Big] \le C(\delta)\epsilon\; \Theta\, \tilde{\Theta}. \tag{34}$$

Let $h \in]0,1[$. Then there exists a constant $C(h,\delta)$ such that

$$\mathbb{E}\Big[\mathrm{Tr}\, P_\omega^{l,j}(]E-\epsilon, E+\epsilon[)\Big] \le C(h,\delta)\epsilon^h\; \Theta. \tag{35}$$

COROLLARY 14

In the case $\Xi = \mathbb{Z}^d$ we have under the assumptions of Theorem 13

$$\mathbb{E}\Big[\mathrm{Tr}\, P_\omega^{l,j}(]E-\epsilon, E+\epsilon[)\Big] \le \min[C(\delta)\epsilon\, l^{2d}, C(h,\delta)\epsilon^h\, l^d]. \tag{36}$$

Remark 15. Actually the Wegner estimate is valid for any energy interval lying below $\inf \sigma((H_0 + V_1)^{l,j})$ which we have set to zero. In the case of a periodic background operator H_0 and ergodic V_ω (this means all the single site potentials $u = u_k$ have the same shape and moreover the coupling constants q_k, $k \in \mathbb{N}$ are i.i.d.) this value does not depend on $l \in \mathbb{N}$ and $j \in \mathbb{Z}^d$ if we impose periodic b.c. on the boundary of the cube $\Lambda_l(j)$.

Proof. We have for $E_n \le -\delta < 0$ and $H_\omega\psi_n = E_n\psi_n$,

$$-\langle\psi_n, V_\omega^2\psi_n\rangle = \langle\psi_n, (H_0 + V_\omega^1 - E_n)\psi_n\rangle \ge \langle\psi_n, (H_0 + V_1 - E_n)\psi_n\rangle \ge \delta,$$

which implies similarly as in Lemma 11

$$
\begin{aligned}
-\sum_{j\in A^{ac}} \frac{\partial E_n(q)}{\partial q_j} &\geq -\frac{1}{q_+ - q_c} \sum_{j\in A^{ac}} r_j \, \langle \psi_n, u_j(\cdot - \xi_j)\psi_n\rangle \\
&= -\frac{\langle \psi_n, V^2_\omega \psi_n\rangle}{q_+ - q_c} \geq \frac{\delta}{q_+ - q_c}. \qquad (37)
\end{aligned}
$$

Consider first the case $\emptyset \neq A \subset \Lambda^+$ and estimate

$$
\begin{aligned}
&\int_{\mathbb{R}^L} \prod_{k\in\Lambda^+} \mathrm{d}\,\mu(q_k)\chi_{\{A^{ac}(\omega)=A\}}(\omega) \sum_{n\in\mathbb{N}} \int_{-2t}^{2t} \rho'(E_n(q) - E + t) \\
&\leq \frac{q_+ - q_c}{\delta} \int_{\mathbb{R}^L} \prod_{k\in\Lambda^+} \mathrm{d}\,\mu(q_k)\chi_{\{A^{ac}(\omega)=A\}}(\omega) \\
&\quad \times \sum_{n\in\mathbb{N}} \int_{-2t}^{2t} \mathrm{d}\,t \left[-\sum_{j\in A^{ac}} \frac{\partial \rho(E_n(q) - E + t)}{\partial q_j} \right]. \qquad (38)
\end{aligned}
$$

As we know that all sites $j \in A^{ac}$ correspond to coupling constants q_j with values in the absolutely continuous region of the conditional density f_j we may estimate as in the proof of Theorem 8:

$$
\begin{aligned}
&-\sum_{n\in\mathbb{N}} \int_{\mathbb{R}} \mathrm{d}\,\mu(q_j)\chi_{\{A^{ac}(\omega)=A\}}(\omega) \frac{\partial \rho(E_n(q) - E + t)}{\partial q_j} \\
&= -\sum_{n\in\mathbb{N}} \int_{q_c}^{q_+} f_j(q_j)\,\mathrm{d}\,q_j \frac{\partial \rho(E_n(q) - E + t)}{\partial q_j} \\
&\leq f_\infty \sum_{n\in\mathbb{N}} \left[\rho(E_n(q, q_j = q_c) - E + t) - \rho((E_n(q, q_j = q_+) - E + t) \right] \\
&\leq f_\infty \min\left[C\,\tilde{\Theta}, C(h)\epsilon^{h-1} \right]. \qquad (39)
\end{aligned}
$$

We have to say something how we deal with the special case $A = \emptyset$. In this situation $V^2_\omega \equiv 0$ and $H_\omega = H_0 + V^1_\omega \geq H_0 + V_1 \geq 0$. Thus there are no eigenvalues in the considered energy interval for this potential configuration. Finally we use the decomposition (31) to finish the proof:

$$
\begin{aligned}
&\mathbb{E}\left(\mathrm{Tr}\, P^{l,j}_\omega([E - \epsilon, E + \epsilon]) \right) \\
&\leq \sum_{A\subset\Lambda^+} \int_{\mathbb{R}^L} \prod_{k\in\Lambda^+} \mathrm{d}\,\mu(q_k)\, \chi_{\{A^{ac}(\omega)=A\}}(\omega) \\
&\quad \times \sum_{j\in A^{ac}} \frac{(q_+ - q_c)}{\delta}\, 4\epsilon f_\infty \min\left[C\,\tilde{\Theta}, C(h)\epsilon^{h-1} \right] \\
&\leq \frac{(q_+ - q_c)}{\delta}\, 4\epsilon f_\infty \min\left[C\,\Theta\,\tilde{\Theta}, C(h)\epsilon^{h-1}\,\Theta \right]. \qquad (40)
\end{aligned}
$$

q.e.d.

6. Comments on the spectral type

The interesting feature of many generalized alloy type potentials is the presence of different spectral types. This is the case for sparse potentials [31,25,17], i.e. if the average concentration of the points in Ξ decays to 0 as one goes out to infinity on the configuration space. The same is true for models with decaying randomness [26,32].

For several examples of this type it has been proven that there exist energy regions with pure absolutely continuous and others with pure point spectrum. In some cases one even knows that there exists an energy value called *mobility edge* that separates the intervals with pure point and absolutely continuous spectrum, see e.g. [26,32]. In fact, ergodic alloy type models are expected to have the same type of mixed spectrum at last for space dimensions $d \geq 3$. However, at the moment a proof of extended states in the ergodic continuous alloy type or the discrete Anderson model seems out of reach.

In a subsequent work [2] we will extend the scattering methods from [17] to several examples presented in this paper to prove existence of absolutely continuous spectrum. On the other hand we will give an adaptation of the multiscale analysis for sparse alloy type potentials. This is the main tool to prove existence of pure point spectrum in an energy interval I lying at the bottom of the spectrum of H_ω. More precisely, to show that there exists a subset $\Omega' \subset \Omega$ of full measure and a $\Omega'' \subset \Omega'$ of positive measure such that

$$\sigma_c(H_\omega) = \emptyset \; \forall \omega \in \Omega' \text{ and } \sigma(H_\omega) \neq \emptyset \; \forall \omega \in \Omega''. \tag{41}$$

Besides the multiscale analysis an initial scale estimate for the decay of the resolvent is necessary for the localization proof. This will be established in [2] for several generalized and sparse alloy type potentials, as well as upper bounds for the infimum of the spectrum of such operators.

Acknowledgements

The authors would like to thank S Böcker for suggestions concerning an earlier version of this paper.

References

[1] Aizenman M and Molchanov S, Localization at large disorder and at extreme energies: an elementary derivation, *Comm. Math. Phys.* **157(2)** (1993) 245–278

[2] Böcker S, Kirsch W and Veselić I, Localization and scattering theory for some sparse potentials, in preparation

[3] Boutet de Monvel A and Stollmann P, Dynamical localization for continuum random surface models, to appear in *Arch. Math.*

[4] Buschmann D and Stolz G, Two-parameter spectral averaging and localization for non-monotonic random Schrödinger operators, *Trans. Am. Math. Soc.* **353(2)** (2001) 635–653

[5] Carmona R, Klein A and Martinelli F, Anderson localization for bernoulli and other singular potentials, *Commun. Math. Phys.* **108** (1987) 41–66

[6] Chahrour A, Densité intégrée d'états surfaciques et fonction généralisée de déplacement spectral pour un opérateur de Schrödinger surfacique ergodique, *Helv. Phys. Acta* **72(2)** (1999) 93–122

[7] Chahrour A, On the spectrum of the Schrödinger operator with periodic surface potential, *Lett. Math. Phys.* **52(3)** (2000) 197–209

[8] Chahrour A and Sahbani J, On the spectral and scattering theory of the Schrödinger operator with surface potential, *Rev. Math. Phys.* **12(4)** (2000) 561–573

[9] Combes J-M and Hislop P, Localization for some continuous, random Hamiltionians in d-dimensions. *J. Funct. Anal.* **124** (1994) 149–180

[10] Combes J-M, Hislop P D, Klopp F and Nakamura S, The Wegner estimate and the integrated density of states for some random operators, preprint, http://www.ma.utexas.edu/mp_arc/ (2001)

[11] Combes J-M, Hislop P D and Nakamura S, The L^p-theory of the spectral shift function, the Wegner estimate, and the integrated density of states for some random Schrödinger operators, *Commun. Math. Phys.* **70(218)** (2001) 113–130

[12] Damanik D, Sims R and Stolz G, Localization for one dimensional, continuum, Bernoulli-Anderson models, preprint no. 00-404, http://www.ma.utexas.edu/mp_arc/ (2000)

[13] Englisch H, Kirsch W, Schröder M and Simon B, The density of surface states, *Phys. Rev. Lett.*, **61** (1988) 1261–1262

[14] Englisch H, Kirsch W, Schröder M and Simon B, Random hamiltonians ergodic in all but one direction, *Commun. Math. Phys.* **128** (1990) 613–625

[15] Fröhlich J and Spencer T, Absence of diffusion in the Anderson tight binding model for large disorder or low energy, *Commun. Math. Phys.* **88** (1983) 151–184

[16] Hislop P D and Klopp F, The integrated density of states for some random operators with nonsign definite potentials, http://www.ma.utexas.edu/mp_arc, preprint no. 01-139 (2001)

[17] Hundertmark D and Kirsch W, Spectral theory of sparse potentials, in: Stochastic processes, physics and geometry: new interplays, I (Leipzig, 1999) pp. 213–238. *Am. Math. Soc.* (Providence, RI) (2000)

[18] Hupfer T, Leschke H, Müller P and Warzel S, The absolute continuity of the integrated density of states for magnetic Schrödinger operators with certain unbounded random potentials, *Comm. Math. Phys.* **221(2)** (2001) 229–254

[19] Jakšić V and Last Y, Spectral structure of Anderson type Hamiltonians, *Invent. Math.* **141(3)** (2000) 561–577

[20] Jakšić V and Last Y, Surface states and spectra, *Comm. Math. Phys.* **218(3)** (2000) 459–477

[21] Jakšić V, Molchanov S and Pastur L, On the propagation properties of surface waves, in: Wave propagation in complex media (Minneapolis, MN, 1994) (New York: Springer) (1998) pp. 143–154

[22] Jakšić V and Molchanov S, Localisation of surface spectra, *Commun. Math. Phys.* **208(1)** (1999) 153–172

[23] Kirsch W, Random Schrödinger operators, in: Schrödinger Operators (eds) H Holden and A Jensen, *Lecture Notes in Phys.* (Berlin: Springer) (1989) vol. 345

[24] Kirsch W, Wegner estimates and Anderson localization for alloy-type potentials, *Math. Z.* **221** (1996) 507–512

[25] Kirsch W, Scattering theory for sparce random potentials, preprint no. 99-458, http://www.ma.utexas.edu/mp_arc/ (1999)

[26] Kirsch W, Krishna M and Obermeit J, Anderson model with decaying randomness: mobility edge, *Math. Z.* **235(3)** (2000) 421–433

[27] Kirsch W, Stollmann P and Stolz G, Anderson localization for random Schrödinger operators with long range interactions, *Comm. Math. Phys.* **195(3)** (1998) 495–507

[28] Kirsch W, Stollmann P and Stolz G, Localization for random perturbations of periodic Schrödinger operators, *Random Oper. Stochastic Equ.* **6(3)** (1998) 241–268; available at http://www.ma.utexas.edu/mp_arc, preprint no. 96-409

[29] Klopp F, Localization for some continuous random Schrödinger operators, *Commun. Math. Phys.* **167** (1995) 553–569

[30] Kostrykin V and Schrader R, The density of states and the spectral shift density of random Schrödinger operators, *Rev. Math. Phys.* **12(6)** (2000) 807–847

[31] Krishna M and Obermeit J, Localization and mobility edge for sparsely random potentials, unpublished (1998)

[32] Krishna M and Sinha K, Spectra of Anderson type models with decaying randomness, preprint: atXiv:math-ph/9911009, 8 November 1999

[33] Obermeit J, Das Anderson–Modell mit Fehlplätzen, Ph.D.Thesis (Ruhr-Universität Bochum, D-44780 Bochum) (1998)

[34] Reed M and Simon B, Methods of Modern Mathematical Physics II, Fourier Analysis, Self-Adjointness (San Diego: Academic Press) (1975)

[35] Reed M and Simon B Methods of Modern Mathematical Physics IV, Analysis of Operators (San Diego: Academic Press) (1978)

[36] Simon B, and Wolff T, Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians, *Comm. Pure Appl. Math.* **39** (1986) 75–90

[37] Stollmann P, Caught by disorder: A Course on Bound States in Random Media, volume 20 of *Progress in Math. Phy.* (Birkhäuser) (July, 2001)

[38] Veselić I, Wegner estimate for some indefinite Anderson-type Schrödinger operators with differentiable densities, preprint 2000, http://www.ma.utexas.edu/mp_arc/, to appear in *Lett. Math. Phys.* as: Wegner estimate and the density of states of some indefinite alloy type Schrödinger operators

[39] von Dreifus H and Klein A, A new proof of localization in the Anderson tight binding model, *Commun. Math. Phys.* **124** (1989) 285–299

[40] Zenk H, Anderson localization for a multidimensional model including long range potentials and displacements, Preprint-Reihe des Fachbereichs Mathematik at the Johannes Guteberg-Universität Mainz (1999)