Proc. Indian Acad. Sci. (Math. Sci.) Vol. 112, No. 1, February 2002, pp. 117–130.

On spectral properties of periodic polyharmonic matrix operators

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Abstract. We consider a matrix operator $H = (-\Delta)^l + V$ in R^n, where $n \geq 2$, $l \geq 1$, $4l > n + 1$, and V is the operator of multiplication by a periodic in x matrix $V(x)$. We study spectral properties of H in the high energy region. Asymptotic formulae for Bloch eigenvalues and the corresponding spectral projections are constructed. The Bethe–Sommerfeld conjecture, stating that the spectrum of H can have only a finite number of gaps, is proved.

Keywords. Periodic Schrödinger operator; Bloch eigenvalues and eigenfunctions; Bethe–Sommerfeld conjecture.

1. Introduction

We consider the operator

$$H = (-\Delta)^l + V \tag{1}$$

in $L_2(R^n)^m$, where $m \geq 1$, $n \geq 2$, $l \geq 1$, $4l > n + 1$, $(-\Delta)^l$ is a diagonal $m \times m$ matrix, its diagonal elements being the scalar polyharmonic operators, and V is the operator of multiplication by a periodic in x matrix $V(x)$. For the sake of simplicity, we assume that $V(x)$ has orthogonal periods $a_1, ..., a_n$, however all the results are also valid for nonorthogonal periods. Without loss of generality we consider that

$$V(x) = V_0 + \alpha\tilde{V}(x),$$

where V_0 is a diagonal matrix with entries not depending on x:

$$(V_0)_{qs} = v_{0q}\delta_{qs}, \quad q, s = 1, ..., m,$$
$$v_{01} \leq v_{02} \leq ... \leq v_{0m};$$

a parameter α belongs to [0, 1], and $\tilde{V}(x)$ satisfies the condition:

$$\int_Q \tilde{V}(x)\mathrm{d}x = 0,$$

Q being the elementary cell of the periods: $Q = [0, a_1] \times \cdots \times [0, a_n]$. We assume that $V(x)$ is infinitely differentiable.

The scalar case of $(-\Delta)^l + V$, $4l > n+1$, was studied in [Sk1,Sk2,K1,K2]. The Bethe–Sommerfeld conjecture, stating that the spectrum of H can have only a finite number of gaps, was proved in [Sk1,Sk2] for the case of a rational lattice and a bounded potential. It was proved in [K1,K2] for the case of a general lattice and a general class of potentials. The case of the Schrödinger operator $l = 1$, $n \geq 3$ is not considered here. The reader can find the proof of the conjecture for the scalar case $n = 3$, $l = 1$ in [Sk3,V,K3,K2,HM], and for the more general situation $n \leq 4$ in [HM]. Recently, the results for the scalar cases $6l > n + 2$ [PS1,PS2] and $8l > d + 3$ [PS3] have appeared.

The aim of this paper is to construct perturbation formulae for eigenvalues and spectral projections in the high energy region and to prove the Bethe–Sommerfeld conjecture for the matrix operator (1). In the case, when the entries of V_0 are different from each other, the whole construction is analogous to the scalar case, see [K1,K2]. The scheme in [K1,K2] is based on the fact that the Bloch eigenvalues of the operator $(-\Delta)^l$ in $L_2(R^n)$ are not degenerated for almost all values of quasimomentum. In the matrix case, when $v_{0s} = v_{0q}$ for at least one pair of $q, s, q \neq s$, the scalar scheme does not work, for the reason that the Bloch eigenvalues of the 'unperturbed' operator $H_0 = (-\Delta)^l + V_0$ are degenerated for all values of quasimomentum. Here we will show how this case can be treated.

It is well-known (see f.e. [G, ReSi]) that the spectral analysis of H can be reduced to the study of a family of operators $H(t)$, $t \in K$, where K is the elementary cell of the dual lattice

$$K = [0, 2\pi a_1^{-1}) \times [0, 2\pi a_2^{-1}) \times \cdots \times [0, 2\pi a_n^{-1}).$$

Vector t is called a *quasimomentum.* An operator $H(t)$, $t \in K$, acts in $L_2(Q)^m$. It is described by the formula (1) and the quasiperiodic conditions:

$$\vec{u}(x_1, ..., x_{r-1}, a_r, x_{r+1}, ..., x_n) = \exp(it_r a_r)\vec{u}(x_1, ..., x_{r-1}, 0, x_{r+1}, ..., x_n),$$
$$r = 1, 2, ..., n. \tag{2}$$

The derivatives with respect to x_r must also satisfy the analogous conditions. The operator $H(t)$, $t \in K$, has a discrete semibounded from below spectrum $\Lambda(t)$:

$$\Lambda(t) = \cup_{s=1}^{\infty} \lambda_s(t),\ \lambda_s(t) \to_{s\to\infty} \infty.$$

The spectrum Λ of the operator H is the union of the spectra of the operators $H(t)$: $\Lambda = \cup_{t\in K} \Lambda(t) = \cup_{s\in N, t\in K} \lambda_s(t)$. The functions $\lambda_s(t)$ are continuous, so Λ has a band structure:

$$\Lambda = \cup_{s=1}^{\infty} [q_s, Q_s],\ q_s = \min_{t\in K} \lambda_s(t),\ Q_s = \max_{t\in K} \lambda_s(t).$$

Absolute continuity of the spectrum was proven in [Th] for the case of the scalar Schrödinger operator, however the proof can be generalized for the present case (see f.e. [Ku]).

The eigenfunctions of $H(t)$ and H are simply related. Extending all the eigenfunctions of the operators $H(t)$ quasiperiodically (see (2)) to R^n, we obtain a complete system of generalized eigenfunctions of H.

We use the following representation of the potential:

$$\tilde{V}_{qs}(x) = \sum_{j\in Z^n, j\neq 0} v_{qs,j} \exp i(\vec{p}_j(0), x),\quad q, s = 1, \ldots, m, \tag{3}$$

where $\tilde{V}_{qs}(x)$ are matrix elements of $\tilde{V}(x)$, $v_{qs,j}$, $j \in Z^n$, are Fourier coefficients of $\tilde{V}_{qs}(x)$, $(\cdot,\cdot)$ is the inner product in R^n, $\vec{p}_j(0)$ are vectors of the dual lattice:

$$\vec{p}_j(0) = 2\pi(j_1 a_1^{-1}, j_2 a_2^{-1}, ..., j_n a_n^{-1}), \; j \in Z^n. \tag{4}$$

In the case $\tilde{V} = 0$, the eigenvalues and eigenfunctions of the operators $H_0(t)$, $t \in K$, are naturally indexed by pairs of indices (j, q), $j \in Z^n$, $q = 1, \dots, m$:

$$\lambda_{j,q}^0(t) = p_j^{2l}(t) + v_{0q}, \quad \psi_{j,q}^0(t,x) = \frac{1}{|Q|^{1/2}} \vec{e}_q^{\,0} \exp i(\vec{p}_j(t), x),$$

here $\vec{e}_q^{\,0}$ are vectors of the canonical basis in R^m: $(\vec{e}_q^{\,0})_s = \delta_{qs}$, $s = 1, \dots, m$, and $\vec{p}_j(t) = \vec{p}_j(0) + t$, $\vec{p}_j(0)$ being given by (4); $p_j^{2l}(t) = |\vec{p}_j(t)|^{2l}$, and the symbol $|Q|$ stands for the volume of Q.

The operator $H(t)$ admits the matrix representation in the basis $\left\{\psi_{j,q}^0\right\}_{j \in Z^n, q=1,\dots,m}$:

$$H_{(j,q)(i,s)} = p_j^{2l}(t)\delta_{ji}\delta_{qs} + v_{0q}\delta_{ji}\delta_{qs} + v_{qs,j-i}.$$

In physical literature, the important concept of an isoenergetic surface of a periodic operator is used (see f.e. [Ki,Ma]). It is said that a point t belongs to the isoenergetic surface $S(k)$ of the operator H, if the operator $H(t)$ has an eigenvalue equal to k^{2l}, i.e., if there is an s such that $\lambda_s(t) = k^{2l}$. Let us consider the isoenergetic surface of the operator $H_0 = (-\Delta)^l + V_0$. The isoenergetic surface $S_0(k)$ of H_0 is the set in K, such that every $t \in S_0(k)$ satisfies the equation $p_j^{2l}(t) + v_{0q} = k^{2l}$ at least for one pair (j, q). The surface can be obtained as follows: the spheres of the radiuses $\left(k^{2l} - v_{0q}\right)^{1/2l}$, $q = 1, ..., m$ centered at the origin of R^n, are divided into pieces by the dual lattice $\{\vec{p}_j(0)\}_{j \in Z^n}$, and, then, all these pieces are shifted into the elementary cell K of the dual lattice. Thus, we obtain m spheres 'packed into the bag' K. If $v_{0q} = v_{0s}$ for $q \neq s$, then the corresponding spheres coincide. Thus, the real number of spheres is equal to the number of *different* entries in V_0. Each point of a degenerated sphere corresponds to a multiple eigenvalue. The self-intersections of $S_0(k)$ are described by the equations: $p_j^{2l}(t) + v_{0q} = p_i^{2l}(t) + v_{0s}$, $j, i \in Z^n$, $j \neq i$, $q, s = 1, ..., m$.

Let $S_1(k) \subseteq S_0(k)$. We say that $S_1(k)$ has an asymptotically full measure on $S_0(k)$ if the relation

$$\frac{s(S_1(k))}{s(S_0(k))} \xrightarrow[k \to \infty]{} 1 \tag{5}$$

holds, where $s(\cdot)$ is the area of a set, counting the multiplicity of the spheres.

2. The main results

For any $t \in S_0(k)$ there is a pair (j, q) such that

$$p_j^{2l}(t) + v_{0q} = k^{2l}, \tag{6}$$

i.e., $H_0(t)$ has an eigenvalue equal to k^{2l}. The multiplicity of this eigenvalue is at least equal to the multiplicity m_q of v_{0q} as an entry in V_0. Let us consider the situation when it is greater. In fact, all other eigenvalues of H_0 can be represented by either $p_j^{2l}(t) + v_{0s}$, $v_{0s} \neq v_{0q}$, or $p_i^{2l}(t) + v_{0s}$, $i \neq j$, s being any number of $1, ..., m$ including q. The eigenvalues of the first type are separated from (6) by a distance not less than

$$d_0 = \min_{s,r: v_{0s} \neq v_{0r}} |v_{0s} - v_{0r}|. \tag{7}$$

An eigenvalue of the second type may coincide with (6). In other words, (6) has the multiplicity greater than m_s if and only if there are $i \neq j$ and $s = 1, \dots, m$, such that

$$p_j^{2l}(t) + v_{0q} = p_i^{2l}(t) + v_{0s}. \tag{8}$$

This condition describes a self-intersection of the isoenergetic surface $S_0(k)$. Obviously, the set of the points t satisfying (8) has the measure zero on $S_0(k)$.

We introduce a set $\chi_0(k, \delta)$, δ being a positive parameter, as $S_0(k) \setminus A_0(k, \delta)$, where $A_0(k, \delta)$ is the $(k^{-n+1-\delta})$-neighborhood of all the self-intersections (8) of $S_0(k)$. The properties of $\chi_0(k, \delta)$ are described by the following lemma.

Lemma 1. *If t belongs to $\chi_0(k, \delta)$, then there are a unique index j and a unique value v_{0q}, such that* (6) *holds. Moreover, for all $i \neq j$ and $s = 1, ..., m$:*

$$\left| p_j^{2l}(t) + v_{0q} - p_i^{2l}(t) - v_{0s} \right| > 2k^{2l-n-\delta}. \tag{9}$$

The set $\chi_0(k, \delta)$ has an asymptotically full measure on $S_0(k)$.

Lemma 1 is stable in the $(k^{-n+1-\mathbf{2}\delta})$-neighborhood of $\chi_0(k, \delta)$ in the sense that, those points, which are close to $\chi_0(k, \delta)$, satisfy equation (6) approximately and also satisfy (9). The following generalization of Lemma 1 holds:

Lemma 2. *If t belongs to the $(k^{-n+1-\mathbf{2}\delta})$-neighborhood of $\chi_0(k, \delta)$, then there are a unique index j and a unique value v_{0q}, such that*

$$|p_j^{2l}(t) + v_{0q} - k^{2l}| < 3lk^{2l-n-\mathbf{2}\delta} \tag{10}$$

and the inequality (9) *holds for all $i \neq j$ and $s = 1, \dots, m$. The set $\chi_0(k, \delta)$ has an asymptotically full measure on $S_0(k)$.*

Thus, if $t \in \chi_0(k, \delta)$, there is only one index $j = j(t)$ and there is only one value v_{0q} satisfying (6). When all the entries of V_0 are different from each other, the index $q = q(t)$ is defined uniquely by v_{0q}. If there are multiple entries in V_0, we define $q(t)$ as the smallest of all indices satisfying (6). By this rule, $q(t)$ is defined uniquely. When t is in the $(k^{-n+1-2\delta})$-neighborhood of $\chi_0(k, \delta)$, the indices $j(t)$ and $q(t)$ are uniquely defined by inequality (10). We denote by m_q the multiplicity of v_{0q}, i.e., $m_q : v_{0,q-1} < v_{0,q} = \cdots = v_{0,q+m_q-1} < v_{0,q+m_q}$.

COROLLARY 1

If $2l \leq n$, and t is in the $(k^{-n+1-2\delta})$-neighborhood of $\chi_0(k, \delta)$, then the operator $H_0(t)$ has a unique eigenvalue satisfying (10). *It has the multiplicity m_q. The distance from $p_j^{2l}(t) + v_{0q}$ to the nearest other eigenvalue of $H_0(t)$ is not less than $2k^{2l-n-\delta}$, for k large enough.*

In fact, according to the lemma, the eigenvalues of the second type are separated from $p_j^{2l}(t)+v_{0q}$ at least by the distance $2k^{2l-n-\delta}$. The eigenvalues of the first type are separated at least by the distance d_0 (see (7)), which does not depend on k and obviously greater than $2k^{2l-n-\delta}$ for k large enough.

COROLLARY 2

If $2l > n$, $0 < \delta < 2l - n$, and t is in the $(k^{-n+1-2\delta})$-neighborhood of $\chi_0(k, \delta)$, then the operator $H_0(t)$ has a unique eigenvalue satisfying (10). *It has the multiplicity m_q. The distance from $p_j^{2l}(t) + v_{0q} = k^{2l}$ to the nearest other eigenvalue of $H_0(t)$ is not less than d_0, for k large enough.*

This corollary is based on the fact that $d_0 < 2k^{2l-n-\delta}$ for $2l > n$, $0 < \delta < 2l - n$, and k large enough.

Let us consider the circle C_0 in the complex plane, which is centered at $z = k^{2l}$ and has the radius $k^{2l-n-\delta}$:

$$C_0 = \{z : |z - k^{2l}| = k^{2l-n-\delta}\}.$$

If $2l \leq n$ and t belongs to the $(k^{-n+1-2\delta})$-neighborhood of $\chi_0(k, \delta)$, then according to Corollary 1, the circle includes only one eigenvalue of $H_0(t)$. It has the multiplicity m_q. If $2l > n$, $0 < \delta < 2l - n$, then the circle includes all the eigenvalues given by the formula

$$p_j^{2l}(t) + v_{0s},\ s = 1, 2, \ldots, m,$$

since $|v_{0s} - v_{0q}| < k^{2l-n-\delta}$. All other eigenvalues are outside the circle. Counting multiplicity, we have exactly m eigenvalues of $H_0(t)$ inside the circle.

In the scalar situation $m = m_q = 1$, i.e., the circle has one simple eigenvalue of $H_0(t)$ inside. In the matrix situation we can have multiple eigenvalues, which makes it more difficult to describe their shift under the perturbation $\alpha\tilde{V}$.

If we can prove the convergence of a perturbation series for the resolvent of $H(t)$ on C_0, then the number of eigenvalues inside C_0 does not change with perturbation. The circle C_0 is chosen in such a way, that its distance from the eigenvalues of $H_0(t)$ is equal to $k^{2l-n-\delta}$, if $t \in \chi(k, \delta)$. When $2l > n$, $0 < \delta < 2l - n$, and t is in the $(k^{-n+1-2\delta})$-neighborhood of $\chi_0(k, \delta)$, the perturbation series converges on C_0, since the distance from C_0 to the unperturbed eigenvalues is greater than $\|\tilde{V}\|$ for k large enough. When $2l \leq n$ then, generally speaking, the perturbation series diverges for $t \in \chi_0(k, \delta)$, since the distance from C_0 to the eigenvalues is small (it goes to zero as $k \to \infty$). We construct a 'nonsingular' set $\chi_1 \subset \chi_0(k, \delta)$, such that it provides the convergence of the series in the case $2l \leq n$ and also improves the convergence in the case $2l > n$.

Lemma 3. *Let $0 < \beta < 1$, and $0 < 2\delta < (n-1)(1-\beta)$. Then, for any sufficiently large k, $k > k_0(\beta, \delta, a_1, \ldots a_n)$, there exists a set $\chi_1(k, \beta, \delta) \subset \chi_0(k, \delta) \subset S_0(k)$, such that for any point t in the $(k^{-n+1-2\delta})$-neighborhood of this set the following inequality holds*:

$$\min_{z \in C_0} \min_{\substack{i, i' \in Z^n, 0 < |i - i'| < k^{\beta}, \\ s, s' = 1, \ldots, m}} |p_i^{2l}(t) + v_{0s} - z||p_{i'}^{2l}(t) + v_{0s'} - z| > k^{2\gamma},$$

$$2\gamma = 4l - n - 1 - \beta(n-1) - 2\delta. \tag{11}$$

The set $\chi_1(k,\beta,\delta)$ has an asymptotically full measure on $S_0(k)$. Moreover,

$$\frac{s(S_0(k)\setminus \chi_1(k,\beta,\delta))}{s(S_0(k))} =_{k\to\infty} O(k^{-\delta/8}). \tag{12}$$

Further we assume: $0<\beta<\beta_0$, $\beta_0=\min\{1,(4l-n-1)(n-1)^{-1}\}$, and $0<\delta<\delta_0$, $\delta_0=\frac{1}{4}\min\{(n-1)(1-\beta),4l-n-1-\beta(n-1)\}$. Obviously, under these conditions γ is positive.

The set $\chi_1(k,\beta,\delta)$ provides the convergence of the series for the resolvent:

Theorem 1. *If t belongs to the $(k^{-n+1-2\delta})$-neighborhood of $\chi_1(k,\beta,\delta)$ in K, then the perturbation series for the resolvent*

$$(H(t)-z)^{-1}=(H_0(t)-z)^{-1}\sum_{r=0}^{\infty}\left(-\alpha\tilde V(H_0(t)-z)^{-1}\right)^r \tag{13}$$

converges in the class of bounded operators for any $z\in C_0$.

This theorem enables us to obtain asymptotic formulae for Bloch eigenvalues and the corresponding spectral projections. The next two theorems provide information about the location of eigenvalues.

Theorem 2. *If $2l\le n$ and t belongs to the $(k^{-n+1-2\delta})$-neighborhood of $\chi_1(k,\beta,\delta)$, then there are exactly m_q, $q=q(t)$, eigenvalues of the operator $H(t)$ in the interval $\Delta_0=[p_j^{2l}(t)+v_{0q}-k^{2l-n-\delta},p_j^{2l}(t)+v_{0q}+k^{2l-n-\delta}]$. In fact, they are located in the smaller interval $\Delta_1=[p_j^{2l}(t)+v_{0q}-k^{-2l+1+\zeta},p_j^{2l}(t)+v_{0q}+k^{-2l+1+\zeta}]$, $\zeta=\beta(n-1)+2\delta$. The corresponding spectral projection is given by the series*:

$$E(t)=E_0+\sum_{r=1}^{\infty}\alpha^r G_r(k,t), \tag{14}$$

where E_0 is the unperturbed spectral projection:

$$E_0=\sum_{s=q}^{q+m_q-1}E_{0s},\quad (E_{0s})_{(ip)(i'p')}=\delta_{ii'}\delta_{ji}\delta_{pp'}\delta_{ps}, \tag{15}$$

the operators G_r are given by the formula

$$G_r(k,t)=\frac{(-1)^{r+1}}{2\pi i}\oint_{C_0}((H_0(t)-z)^{-1}\tilde V)^r(H_0(t)-z)^{-1}\mathrm{d}z,\quad r=1,2,..., \tag{16}$$

and satisfy the estimates:

$$\| G_r(k,t)\| < c_V^r k^{-\gamma r},\quad r=1,2,..., \tag{17}$$

c_V not depending on k, $\|\cdot\|$ being the norm in the class of bounded operators.

Theorem 3. *If $2l>n$ and t belongs to the $(k^{-n+1-2\delta})$-neighborhood of $\chi_1(k,\beta,\delta)$, then there are exactly m eigenvalues of the operator $H(t)$ in the interval*

$\Delta_0 = [p_j^{2l}(t) + v_{0q} - k^{2l-n-\delta}, p_j^{2l}(t) + v_{0q} + k^{2l-n-\delta}]$. *In fact, they are located in the smaller intervals* $\Delta_s = [p_j^{2l} + v_{0s} - k^{-2l+1+\zeta}, p_j^{2l} + v_{0s} + k^{-2l+1+\zeta}]$, $\zeta = \beta(n-1) + 2\delta$, $s = 1, \dots, m$. *The corresponding spectral projection is given by the series*

$$E(t) = E_0 + \sum_{r=1}^{\infty} \alpha^r G_r(k, t), \tag{18}$$

where E_0 *is the unperturbed spectral projection*:

$$E_0 = \sum_{s=1}^{m} E_{0s}, \quad (E_{0s})_{(ip)(i'p')} = \delta_{ii'}\delta_{ji}\delta_{pp'}\delta_{ps}, \tag{19}$$

the operators G_r *are given by formula* (16) *and satisfy the estimates* (17).

Remark. It is possible to construct the perturbation series for each interval Δ_s separately, however it gives slower convergence.

The proof of the Bethe–Sommerfeld conjecture for the scalar polyharmonic operator in [K1,K2] is essentially based on the fact that there is a unique simple eigenvalue in the interval Δ_0. Under such circumstances it is possible to differentiate the eigenvalue with respect to t, to obtain a suitable estimate from below for the derivative and to prove, using the estimate, that any sufficiently large k^{2l} belongs to the spectrum of H. In the present case we do not have, generally speaking, the uniqueness of eigenvalue. However, it is still possible to get the estimates for the derivatives of eigenvalues with respect to t and to prove the following theorem.

Theorem 4. (*The proof of the Bethe–Sommerfeld conjecture*). *There is only a finite number of gaps in the spectrum of the operator* H.

3. The proofs

The proof of Lemma 2, up to minor technical details, coincides with the proof of Lemma 2.5 in [K2]. For the scalar case the set $\chi_1(k, \beta, \delta)$ is explicitly constructed in [K2] (Lemma 3.3) and it is proved (Lemma 3.6) that $\chi_1(k, \beta, \delta)$ has an asymptotically full measure in the sense (12). The proof of Lemma 2 is analogous to the proof in [K2].

The proof of Theorem 1. Let us rewrite the series (13) in the form:

$$(H(t) - z)^{-1} = (H_0(t) - z)^{-1} + \sum_{r=1}^{\infty} (H_0(t) - z)^{-1/2}(-\alpha A)^r (H_0(t) - z)^{-1/2},$$
$$A = (H_0(t) - z)^{-1/2} \tilde{V} (H_0(t) - z)^{-1/2}. \tag{20}$$

We prove that the perturbation series converges for any t, satisfying (9) and (11). Inequality (9) and the definition of C_0 yields the estimate:

$$\|(H_0 - z)^{-1/2}\| = \left(\min_{i \in Z^n,\ s=1,2,\dots,m} |p_i^{2l}(t) + v_{0s} - z| \right)^{-1/2} = k^{-(2l-n-\delta)/2}. \tag{21}$$

To prove the convergence of the series for a large k, it is enough to check that

$$\|A\| < c_V k^{-\gamma}, \; z \in C_0 \tag{22}$$

C_V not depending on k. The matrix elements of the operator A are given by the formula:

$$A_{(is)(i's')} = \frac{v_{ss',i-i'}}{\sqrt{\left(p_i^{2l}(t) + v_{0s} - z\right)\left(p_{i'}^{2l}(t) + v_{0s'} - z\right)}}, \tag{23}$$

if $i \neq i'$ and are equal to zero if $i = i'$. We represent A in the form $A = A' + A''$, where

$$A'_{(is)(i's')} = \begin{cases} A_{(is)(i's')}, & \text{if } 0 < |i - i'| < k^{\beta} \\ 0, & \text{otherwise} \end{cases}$$

and, correspondingly,

$$A''_{(is)(i's')} = \begin{cases} 0, & \text{if } |i - i'| < k^{\beta}, \\ A_{(is)(i's')}, & \text{otherwise.} \end{cases}$$

Considering the estimate (11), we see that the denominator on the right hand side of (23) can be bounded from below by k^{γ}, when $0 < |i - i'| < k^{\beta}$. Therefore,

$$\left|A'_{(is)(i's')}\right| < |v_{ss',i-i'}|k^{-\gamma}.$$

Taking into account that $v_{ss',i-i'}$ are the Fourier coefficients of infinitely differentiable functions, we obtain

$$\sup_{s,s'=1,\ldots,m} \sum_{i \in Z^n, |i-i'|<k^{\beta}} |v_{ss',i-i'}| < \tilde{c}_V$$

and, therefore,

$$\|A'(z)\| < c_V k^{-\gamma}, \; z \in C_0. \tag{24}$$

Using (21), we obtain

$$\left|A''_{(is)(i's')}\right| < |v_{ss',i-i'}|k^{-(2l-n-\delta)}.$$

Considering that V is infinitely differentiable, we obtain

$$\sup_{s,s'=1,\ldots,m} \sum_{i \in Z^n, |i-i'|>k^{\beta}} |v_{ss',i-i'}| < c_V k^{2l-n-\delta-\gamma}, \quad c_V = c_V(\delta, \beta, \gamma)$$

and, therefore,

$$\|A''(z)\| < c_V k^{-\gamma}, \; z \in C_0. \tag{25}$$

Adding (24) and (25), we obtain (22). *The theorem is proved.*

The proof of Theorem 2. The operator H_0 has m_q eigenvalues inside C_0. The fact that $(H(t)-z)^{-1}$ is bounded on C_0 for all α means that the number of eigenvalues inside C_0 does not change with the perturbation $\alpha\tilde{V}$, $\alpha \in [0,1]$. Since all eigenvalues are real, they are inside the interval Δ_0, which is cut out by the circle on the real axis. Integrating the resolvent over the contour, we get the corresponding spectral projection:

$$E(t) = -\frac{1}{2\pi i}\oint_{C_0}(H(t)-z)^{-1}\mathrm{d}z. \tag{26}$$

Substituting the series for the resolvent (13) in the last formula, we obtain (14)–(16). Considering the estimates (21) and (22), we get (17).

We prove that, in fact, the perturbation series for the resolvent converges on a smaller circle C_1, which has the radius $k^{-2l+1+\zeta}$ and the same center $z = k^{2l}$ as C_0. Let $z \in C_1$ and $z_0 \in C_0$. For all $i \neq j$:

$$|p_i^{2l}(t) + v_{0s} - z| > |p_i^{2l}(t) + v_{0s} - z_0|, \tag{27}$$

since the circle C_1 is smaller than C_0 and, therefore, further away from outside eigenvalues. Since (11) holds for z_0, we easily obtain

$$|p_i^{2l}(t) + v_{0s} - z||p_{i'}^{2l}(t) + v_{0s'} - z| > k^{2\gamma} \text{ when } i, i' \neq j,\ 0 < |i-i'| < k^{\beta}.$$

Suppose $i = j$, $i' \neq j$. Then, using (27) for i', we get

$$\begin{aligned}&|p_j^{2l}(t) + v_{0s} - z||p_{i'}^{2l}(t) + v_{0s'} - z|\\ &> \frac{|p_j^{2l}(t) + v_{0s} - z|}{|p_j^{2l}(t) + v_{0s} - z_0|}|p_j^{2l}(t) + v_{0s} - z_0||p_{i'}^{2l}(t) + v_{0s'} - z_0|.\end{aligned}$$

Using the definitions of C_0 and C_1, we obtain

$$\frac{|p_j^{2l}(t) + v_{0s} - z|}{|p_j^{2l}(t) + v_{0s} - z_0|} = \frac{k^{-2l+1+\zeta}}{k^{2l-n-\delta}} = k^{-2\gamma+\delta}.$$

Taking into account (11) for z_0, we arrive at the inequality:

$$|p_j^{2l}(t) + v_{0s} - z||p_{i'}^{2l}(t) + v_{0s'} - z| > k^{\delta}.$$

Thus, for any $z \in C_1$,

$$\min_{\substack{i,i' \in Z^n, 0<|i-i'|<k^{\beta},\\ s,s'=1,\dots,m}} |p_i^{2l}(t) + v_{0s} - z|\,|p_{i'}^{2l}(t) + v_{0s'} - z| > k^{\delta}. \tag{28}$$

Considering the above, we prove that the perturbation series for the resolvent converges on the circle C_1 for all $\alpha \in [0,1]$. This means that all eigenvalues are, in fact, inside the interval Δ_1, which is cut out by C_1 on the real axis. Note that the series for the resolvent and, therefore, for the spectral projection converges slower on C_1 than on C_0, since the estimate (22) valid on C_0 is replaced by a weaker estimate $\|A(z)\| < c_V k^{-\delta}$ on C_1. *The theorem is proved.*

The proof of Theorem 3. This is analogous to that of Theorem 2. The only difference is that we replace the circle C_0 by smaller circles C_s which cut out the intervals Δ_s from the real axis.

Let $t_0 \in \chi_1(k, \beta, \delta)$ and $t = t_0 + \tau\vec{\nu}$, where $|\tau| < k^{-n+1-2\delta}$, and $\vec{\nu}$ is a unit vector in R^n. The eigenvalues λ_s, lying inside C_0, analytically depends on τ. Let us estimate their derivatives.

Lemma 4. *If $t_0 \in \chi_1(k, \beta, \delta)$ and $t = t_0 + \tau\vec{\nu}$, $|\tau| < k^{-n+1-2\delta}$, $\vec{\nu}$ being a unit vector in R^n, then the derivative of any $\lambda_s(\tau)$ with respect to τ satisfies the asymptotic estimate*

$$\frac{\partial \lambda_s(\tau)}{\partial \tau} = 2p_j^{2l-2}(t)(\vec{p}_j(t), \vec{\nu}) + O(k^{2l-1-\gamma}), \tag{29}$$

when $k \to \infty$.

Proof. Let us assume for definiteness that $2l \leq n$. There are exactly m_q eigenvalues inside Δ_0, counting the multiplicity. Let $\phi_s(\tau)$ be a normalized eigenvector corresponding to a $\lambda_s(\tau)$.[1] Then,

$$\frac{\partial \lambda_s}{\partial \tau} = \left\langle \frac{\partial H_0}{\partial \tau}\phi_s(\tau), \phi_s(\tau) \right\rangle, \tag{30}$$

where $< \cdot, \cdot >$ is the inner product in $l_2(Z^n)^m$ and $\partial H_0/\partial \tau$ is a diagonal matrix:

$$\left(\frac{\partial H_0}{\partial \tau}\right)_{(ir)(i'r')} = 2p_i^{2l-2}(t)(\vec{p}_i(t), \vec{\nu})\delta_{ii'}\delta_{rr'}, \ \ i, i' \in Z^n, \ r, r' = 1, \dots, m. \tag{31}$$

We denote by $\vec{e}_{jr}^{\,0}$ $r = q, ..., q + m_q - 1$ the eigenvectors of $H_0(t)$, corresponding to $p_j^{2l} + v_{0q}$: $(\vec{e}_{jr}^{\,0})_{(ir')} = \delta_{ij}\delta_{r'r}$, $i \in Z^n$, $r' = 1, \dots, m$. They form a basis in $E_0 l_2(Z^n)^m$. Let us consider the vectors $\vec{e}_{jr}$, $r = q, ..., q + m_q - 1$, defined by the formula: $\vec{e}_{jr} = E(t)\vec{e}_{jr}^{\,0}$. They form a complete almost orthogonal system in $E(t)l_2(Z^n)^m$, since

$$E(t) = E_0 + O(k^{-\gamma}) \tag{32}$$

(in the sense of bounded operators) and $E_0\vec{e}_{jr}^{\,0} = \vec{e}_{jr}^{\,0}$, and, therefore,

$$\vec{e}_{jr} = \vec{e}_{jr}^{\,0} + O(k^{-\gamma}). \tag{33}$$

Hence, an eigenvector $\phi_s(\tau)$ can be represented as a linear combination of $\vec{e}_{jr}$:

$$\phi_s(\tau) = \sum_{r=q}^{q+m_q-1} c_r^{(s)}\vec{e}_{jr}.$$

Considering that the function $\phi_s(\tau)$ is normalized, we obtain

$$\sum_{r,r'=q}^{q+m_q-1} c_r^{(s)}\bar{c}_{r'}^{(s)}\langle \vec{e}_{jr}, \vec{e}_{jr'} \rangle = 1$$

[1]An eigenvector $\phi_s(\tau)$ is defined up to $\pm$ sign if $\lambda_s(\tau)$ is simple. It is allowed to be any normalized eigenvector if the multiplicity of $\lambda_s(\tau)$ is greater than 1 for all τ. We exclude from the consideration the points τ, where two or more nonidentical eigenvalues coincide, since there is only a finite number of such points.

and, by (33),

$$\sum_{r=q}^{q+m_q-1} |c_r^{(s)}|^2 = 1 + O(k^{-\gamma}). \tag{34}$$

Using again (33), we obtain

$$\phi_s(\tau) = \phi_s^0(\tau) + \delta\phi_s(\tau), \tag{35}$$

where $\phi_s^0(\tau) = \sum_{r=q}^{q+m_q-1} c_r^{(s)} \vec{e}_{jr}^{\,0}$, and $\delta\phi_s(\tau)$ satisfies the estimate:

$$\|\delta\phi_s(\tau)\| < c_V k^{-\gamma}, \tag{36}$$

$\|\cdot\|$ being the norm in $l_2(Z^n)^m$. Substituting (35) in (30), we obtain

$$\begin{aligned} \frac{\partial\lambda_s(\tau)}{\partial\tau} &= \left\langle \frac{\partial H_0}{\partial\tau}\phi_s^0(\tau), \phi_s^0(\tau)\right\rangle + \left\langle \frac{\partial H_0}{\partial\tau}\phi_s^0(\tau), \delta\phi_s(\tau)\right\rangle \\ &\quad + \left\langle \frac{\partial H_0}{\partial\tau}\delta\phi_s(\tau), \phi_s^0(\tau)\right\rangle + \left\langle \frac{\partial H_0}{\partial\tau}\delta\phi_s(\tau), \delta\phi_s(\tau)\right\rangle. \end{aligned} \tag{37}$$

Obviously,

$$\left\langle \frac{\partial H_0}{\partial\tau}\phi_s(\tau)^0, \phi_s^0(\tau)\right\rangle = \sum_{r,r'=q}^{q+m_q-1} c_r^{(s)} \bar{c}_{r'}^{(s)} \left\langle \frac{\partial H_0}{\partial\tau}\vec{e}_{jr}^{\,0}, \vec{e}_{jr'}^{\,0}\right\rangle.$$

Substituting (31), we get

$$\left\langle \frac{\partial H_0}{\partial\tau}\phi_s^0(\tau), \phi_s^0(\tau)\right\rangle = 2p_j^{2l-2}(t)(\vec{p}_j(t), \vec{\nu}) \sum_{r=q}^{q+m_q-1} |c_r^{(s)}|^2.$$

Taking into account (34), we obtain

$$\left\langle \frac{\partial H_0}{\partial\tau}\phi_s^0(\tau), \phi_s^0(\tau)\right\rangle = 2p_j^{2l-2}(t)(\vec{p}_j(t), \vec{\nu})\left(1 + O(k^{-\gamma})\right). \tag{38}$$

Let us estimate $\|(\partial H_0/\partial\tau)\,\delta\phi_s(\tau)\|$. Using (31), it is easy to show that

$$\left\| \frac{\partial H_0}{\partial\tau}\delta\phi_s \right\|^2 = 4 \sum_{i\in Z^n, r=1,\dots,m} p_i^{4l-4}(t)(\vec{p}_i(t), \vec{\nu})^2 |(\delta\phi_s(\tau))_{ir}|^2.$$

We estimate the terms of the series as follows:

$$\begin{aligned} &p_i^{4l-4}(t)(\vec{p}_i(t), \vec{\nu})^2 |(\delta\phi_s)_{ir}|^2 \le X_{ir} Y_{ir}, \\ &X_{ir} = p_i^{4l-2}(t)|(\delta\phi_s)_{ir}|^{2-(1/l)}, \\ &Y_{ir} = |(\delta\phi_s)_{ir}|^{1/l}. \end{aligned}$$

Using Hölder's inequality with $p = 2l/(2l-1)$, $q = 2l$, we obtain

$$\left\| \frac{\partial H_0}{\partial\tau}\delta\phi_s(\tau) \right\|^2 \le \|H_0\delta\phi_s(\tau)\|^{(2l-1)/l}\, \|\delta\phi_s(\tau)\|^{1/l}. \tag{39}$$

It is easy to obtain an estimate for $\|H_0\delta\phi_s(\tau)\|$. In fact, $\phi_s(\tau)$ satisfies the equation $H(t)\phi_s(\tau) = \lambda_s\phi_s(\tau)$ and $\phi_s^0(\tau)$ satisfies the equation $H_0(t)\phi_s^0(\tau) = (p_j^{2l}(t)+v_{0q})\phi_s^0(\tau)$. Therefore, the function $\delta\phi_s$ satisfies the equation

$$H_0(t)\delta\phi_s(\tau) + \tilde{V}\phi_s(\tau) = (p_j^{2l}(t) + v_{0q})\delta\phi_s(\tau) + (\lambda_s - p_j^{2l}(t) - v_{0q})\phi_s(\tau).$$

Hence,

$$H_0(t)\delta\phi_s(\tau) = (p_j^{2l}(t) + v_{0q})\delta\phi_s(\tau) + f, \quad \|f\| < C_V,$$

and $\|H_0\delta\phi_s(\tau)\| < k^{2l}\|\delta\phi_s(\tau)\| + C_V$. Noting that $2l - \gamma > 0$ and using (36), we get $\|H_0\delta\phi_s(\tau)\| < C_V k^{2l-\gamma}$. Substituting the last estimate in (39) and using again (36), we obtain

$$\left|\frac{\partial H_0}{\partial\tau}\delta\phi_s(\tau)\right| \le c_V k^{2l-1-\gamma}. \tag{40}$$

Using (36) and (40), we arrive at the inequality:

$$\left|\left\langle\frac{\partial H_0}{\partial\tau}\phi_s^0(\tau), \delta\phi_s(\tau)\right\rangle + \left\langle\frac{\partial H_0}{\partial\tau}\delta\phi_s(\tau), \phi_s^0(\tau)\right\rangle \right.$$
$$\left. + \left\langle\frac{\partial H_0}{\partial\tau}\delta\phi_s(\tau), \delta\phi_s(\tau)\right\rangle\right| < c_V k^{2l-1-\gamma}. \tag{41}$$

Combining (30), (37), (38) and (41), we get (29). The analogous considerations give the proof of the lemma for the case $2l > n$. *The lemma is proved.*

COROLLARY 3

Suppose that $\vec{\nu}$ is chosen in such a way that $|(\vec{p}_j(t), \vec{\nu})| > p_j(t)/2$. Then,

$$\frac{\partial\lambda_s}{\partial\tau} = 2p_j^{2l-2}(t)(\vec{p}_j(t), \vec{\nu})\left(1 + O(k^{-\gamma})\right). \tag{42}$$

In fact, it is easy to see that the first term in the asymptotic formula (29) is the main one, since it satisfies the inequality $\left|2p_j^{2l-2}(t)(\vec{p}_j(t), \vec{\nu})\right| > k^{2l-1}$. Therefore, (42) holds.

Remark. The right part of (42) does not essentially depend on s. This means that, when τ changes, the eigenvalues $\lambda_s(\tau)$ lying in Δ_0 move essentially as a group – all together.

The proof of Theorem 4. It is enough to prove that all k^{2l}, which are large enough, belong to the spectrum of H. Let $t_0 \in \chi_1(k, \beta, \delta)$. There is a pair (j, q) such that $p_j^{2l}(t_0) + v_{0q} = k^{2l}$. Let $\vec{\nu}$ be a unit vector in the direction of $\vec{p}_j(t_0)$. According to Theorems 2 and 3, for any $t = t_0 + \tau\nu$, $|\tau| < k^{-n+1-2\delta}$ and k large enough, there are exactly m_q eigenvalues of $H(t)$, which satisfy the estimate

$$|\lambda_s(\tau) - (p_j^{2l}(t) + v_{0q})| < c_V k^{-2l+1+\zeta}. \tag{43}$$

The functions $\lambda_s(\tau)$ analytically depend on τ. Using Corollary 3 and the special choice of vector $\vec{\nu}$, we obtain

$$\frac{\partial \lambda_s(\tau)}{\partial \tau} = 2p_j^{2l-1}(t)\,(1 + o(1))\,, \tag{44}$$

for any eigenvalue λ_s satisfying (43). We consider $\lambda_s(\tau)$ as a function of τ in the domain $[-\tau_0, \tau_0]$, $\tau_0 = k^{-n+1-2\delta}$. It follows from (44) that $\lambda_s(\tau)$ is a monotonic function. Since it is continuous, its range is the interval $[\lambda_s(-\tau_0), \lambda_s(\tau_0)]$. Taking into account (43), we obtain

$$\begin{aligned}\lambda_s(-\tau_0) &= p_j^{2l}(t_0 - \tau_0\vec{\nu}) + v_{0q} + O(k^{-2l+1+\zeta}) \\ &= (k_0 - \tau_0)^{2l} + v_{0q} + O(k^{-2l+1+\zeta}), \\ k_0 = p_j(t_0) &= \left(k^{2l} - v_{0q}\right)^{1/2l}.\end{aligned}$$

Similarly,

$$\begin{aligned}\lambda_s(\tau_0) &= p_j^{2l}(t_0 + \tau_0\vec{\nu}) + v_{0q} + O(k^{-2l+1+\zeta}) \\ &= (k_0 + \tau_0)^{2l} + v_{0q} + O(k^{-2l+1+\zeta}).\end{aligned}$$

It is easy to observe that $k_0^{2l} + v_{0q} = k^{2l}$ and $k_0^{2l-1}\tau_0 = k^{2l-n-2\delta} > k^{-2l+1+\zeta}$. Therefore the interval $[\lambda_s(-\tau_0), \lambda_s(\tau_0)]$ contains $[k^{2l} - k^{2l-1}\tau_0, k^{2l} + k^{2l-1}\tau_0]$ when k is large enough, and, obviously, contains k^{2l}. Thus, there is a value τ_s such that $\lambda_s(\tau_s) = k^{2l}$, i.e., the point $\lambda = k^{2l}$ belongs to the spectrum. *The theorem is proved.*

Acknowledgement

This research was partially supported by USNSF grant DMS-9803498.

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