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Magnetic bottles for the Neumann problem: The case of dimension 3

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Abstract. The main object of this paper is to analyze the recent results obtained on the Neumann realization of the Schrödinger operator in the case of dimension 3 by Lu and Pan. After presenting a short treatment of their spectral analysis of key-models, we show briefly how to implement the techniques of Helffer–Morame in order to give some localization of the ground state. This leaves open the question of the localization by curvature effect which was solved in the case of dimension 2 in our previous work and will be analysed in the case of dimension 3 in a future paper.

Keywords. Spectral theory; Schrödinger operators; magnetic fields; superconductivity.

1. Introduction

Motivated by the superconductivity (see [He3] or [LuPa4] for a review on this aspect), we would like to analyse the ground state energy (that is the lowest eigenvalue) of the Neumann realization of the Schrödinger operator with magnetic potential in an open set Ω in $\mathbb{R}^3$:

$$P_{h,A} = \sum_{j=1}^{3} (hD_{x_j} - A_j(x))^2, \tag{1.1}$$

where $A = (A_1, A_2, A_3)$ is a magnetic potential. If $\mu^{(1)}(h)$ is the lowest eigenvalue and $u^{(1)}(x, h)$ is a corresponding normalized eigenstate, we would like to discuss the asymptotic of $\mu^{(1)}(h)$ and the localization of $u^{(1)}$ as $h \to 0$. By localization, we mean that, for a given set $\omega \subset \Omega$ (possibly depending on h), we shall analyse the asymptotic behavior as $h \to 0$ of the L^2-norm of $u^{(1)}$ in ω. These problems were treated in the case of dimension 2 by Bauman–Phillips–Tang [BaPhTa], Bernoff–Sternberg [BeSt], Lu–Pan [LuPa1,LuPa2,LuPa3], Del Pino–Fellmer–Sternberg [PiFeSt] and Helffer–Morame [HeMo2] in a rather complete way. We would like to present here in a somewhat simpler way some more recent work by Lu–Pan [LuPa5] concerning the dimension 3 and show briefly how the techniques of Helffer–Morame [HeMo2] can be implemented in order to improve their results on the localization of the ground state. Further developments will appear in [HeMo3].

2. Constant magnetic field: The case of dimension 2

2.1 *Main results for the models*

We first consider the operator:

$$P_{h,A^0} := (hD_{x_1} - A^0_1)^2 + (hD_{x_2} - A^0_2)^2,$$

with $A^0 = (-(B/2)\ x_2,\ (B/2)\ x_1)$, $h > 0$ and $B \neq 0$, and analyse the spectrum of its realization in $\mathbb{R}^2$ or of its Neumann realization.

We observe that by homogeneity, one can reduce the analysis to $h = 1$ and $B = 1$. It is well-known that in the case of $\mathbb{R}^2$ the spectrum is a point spectrum and that the eigenvalues are given by $(2n + 1)$ with $n \in \mathbb{N}$, each eigenvalue being of infinite multiplicity. One way to see this is to show the unitary equivalence (via a gauge transformation, a partial Fourier transform and a translation) with the harmonic oscillator $(-\partial_t^2 + t^2)$ but seen as an unbounded operator on $L^2(\mathbb{R}^2_{t,s})$.

The case of the Neumann realization in $\mathbb{R}^2_+$ is a little more delicate. A unitary transformation leads this time to the analysis of the Neumann realization of the operator of $H := D_t^2 + (t-s)^2$ on $L^2(\mathbb{R}_+ \times \mathbb{R})$ which is reduced to the analysis of the family $(s \in \mathbb{R})$ of operators : $H(s) := D_t^2 + (t-s)^2$ defined on $L^2(\mathbb{R}_+)$.

If $\mu(s)$ is defined as the lowest eigenvalue of the Neumann realization of $H(s)$ in $\mathbb{R}^+$, one easily shows (cf [ReSi]) that the infimum of the spectrum of the Neumann realization of $P_{h,A}$ is given by $|B|h\Theta_0$ with

$$\Theta_0 := \inf_{s\in\mathbb{R}} \mu(s).$$

2.2 *An important one-dimensional family*

Let us just list some properties of $\mu(s)$ and refer to [Bo] and [DaHe] for proofs or details. The Neumann eigenvalue $\mu(s)$ satisfies the following properties:

1. $\lim_{s\to-\infty} \mu(s) = +\infty$;
2. μ is decreasing for $s < 0$;
3. $\mu(0) = 1$;
4. $\lim_{s\to+\infty} \mu(s) = 1$;
5. μ admits in $]0, +\infty[$ a unique minimum $\Theta_0 < 1$ at some $s_0 > 0$. Moreover this minimum is non degenerate.

Let us just mention that the proof of some of these properties is based on the following identity, relating the first eigenfunction u_s and $\mu(s)$:

$$\|u_s\|^2\mu'(s) = (s^2 - \mu(s))(u_s(0))^2. \tag{2.1}$$

It is interesting to compare the properties of $\mu(s)$ with the properties of the Dirichlet eigenvalue $\lambda(s)$ of $H(s)$.

The Dirichlet eigenvalue $\lambda(s)$ satisfies the following properties:

1. $\mu(s) \leq \lambda(s)$;
2. $\lim_{s\to-\infty} \lambda(s) = +\infty$;

3. λ is decreasing for $s \in \mathbb{R}$;
4. $\lambda(0) = 3$;
5. $\lim_{s\to+\infty} \lambda(s) = 1$.

2.3 *Applications*

As an application of the analysis of the models, one gets (cf [LuPa3]) via a partition of unity for the lower bound and a suitable construction of quasimodes for the upper bound, the following general result (in the two-dimensional case):

Theorem 2.1. *If $\mu^{(1)}(h)$ (resp. $\lambda^{(1)}(h)$) is the lowest eigenvalue of the Neumann (resp. Dirichlet) realization of $P_{h,A}$ in Ω, then we have*

$$\lim_{h\to 0} \frac{\mu^{(1)}(h)}{h} = \inf\left(\inf_{x\in\Omega} |B(x)|, \Theta_0 \inf_{x\in\partial\Omega} |B(x)| \right)$$
$$\lim_{h\to 0} \frac{\lambda^{(1)}(h)}{h} = \inf_{x\in\Omega} |B(x)|. \tag{2.2}$$

We emphasize that there is no assumption that the magnetic field is constant. In the case of the constant magnetic field, one can actually have more precise results for $\mu^{(1)}$. Some of them will be described in §4.

Remark 2.2. The fact that the bottom of the spectrum of the Neumann realization in $\mathbb{R}^2_+$ is strictly less than the corresponding spectrum in $\mathbb{R}^2$ is basic in the analysis of the superconductivity.

3. Constant magnetic field: The case of dimension 3

3.1 *Introduction*

As in the case of dimension 2, where we first understood the model with constant magnetic field in $\mathbb{R}^2$ and $\mathbb{R}^2_+$, we shall discuss the case of $\mathbb{R}^3$ and $\mathbb{R}^3_+$. We start (after some gauge transformation) of the Schrödinger operator with constant magnetic field in dimension 3.

$$P(h, \vec{B}) := h^2 D_{x_1}^2 + (hD_{x_2} - B_3 x_1)^2 + (hD_{x_3} + B_2 x_1 - B_1 x_2)^2,$$

and consider the Neumann realization in $\{x_1 > 0\}$. After scaling, we can assume that $h = 1$ and $\|B\| = 1$. Here the norm of B is defined by

$$\|B\| = \sqrt{B_1^2 + B_2^2 + B_3^2}.$$

After some rotation in the (x_2, x_3) variables, we can assume that the new magnetic field is $(\beta_1, \beta_2, 0)$ and we are reduced to the problem of analysing:

$$P(\beta_1, \beta_2) := D_{x_1}^2 + D_{x_2}^2 + (D_{x_3} + \beta_2 x_1 - \beta_1 x_2)^2,$$

in $\{x_1 > 0\}$, where

$$\beta_1^2 + \beta_2^2 = 1.$$

We introduce

$$\beta_2 = \cos\vartheta,\ \ \beta_1 = \sin\vartheta,$$

and we observe that, if ν is the external normal to $x_1 = 0$, we have

$$\vec{B}\cdot\nu = -\sin\vartheta. \tag{3.1}$$

3.2 *Reduction to a one-dimensional family*

By partial Fourier transform, we arrive at

$$L(\vartheta,\tau) = D_{x_1}^2 + D_{x_2}^2 + (\tau + \cos\vartheta\ x_1 - \sin\vartheta\ x_2)^2,$$

in $x_1 > 0$ and with Neumann condition on $x_1 = 0$.

The bottom the spectrum is given by

$$\sigma(\vartheta) := \inf\ \mathrm{Sp}\ (L(\vartheta, D_t)) = \inf_{\tau}\,(\inf\ \mathrm{Sp}\ (L(\vartheta,\tau)))\,.$$

By symmetry considerations, we also observe that

$$\sigma(\vartheta) = \sigma(-\vartheta) = \sigma(\pi-\vartheta). \tag{3.2}$$

Consequently, it is enough to look at the restriction to $\left[0, \frac{\pi}{2}\right]$.

We first observe the following lemma:

Lemma 3.1. *If* $\vartheta \in \left]0, \frac{\pi}{2}\right]$, *then* $\mathrm{Sp}\ (L(\vartheta,\tau))$ *is independent of* τ.

This is trivial by translation. This lemma shows that, in this case, it is enough to analyse $L(\vartheta, 0)$ that we shall denote sometimes more shortly by $L(\vartheta)$.

3.3 *Continuity of* $\sigma(\vartheta)$

Lemma 3.2. *The function* $\vartheta \mapsto \sigma(\vartheta)$ *is continuous on* $\left]0, \frac{\pi}{2}\right[$.

After a change of variable $y_1 = \cos\vartheta\ x_1$, $y_2 = \sin\vartheta\ x_2$, we arrive at a continuous family of operators with a fixed domain. Using the mini-max principle, the lemma is then easy to prove.

Lemma 3.3.

$$\sigma(0) = \Theta_0 < 1.$$

Proof. We first observe that

$$L(0, D_t) = D_{x_1}^2 + D_{x_2}^2 + (D_t - x_1)^2.$$

We then have to analyse the bottom of the spectrum of the family

$$L(0,\tau,\xi_2) := D_{x_1}^2 + \xi_2^2 + (x_1-\tau)^2.$$

This infimum is obtained as the infimum over $\tau \in \mathbb{R}$ of the spectrum of the family

$$L(0,\tau,0) = D_{x_1}^2 + (x_1-\tau)^2.$$

We can then use the results of subsection 2.2.

Lemma 3.4.

$$\sigma\left(\frac{\pi}{2}\right) = 1.$$

Proof. We start from

$$L\left(\frac{\pi}{2}, \tau\right) = D_{x_1}^2 + D_{x_2}^2 + (\tau - x_2)^2.$$

The bottom of the spectrum is the same as the bottom of the Neumann realization of $D_{x_1}^2 + D_{x_2}^2 + x_2^2$, in $x_1 > 0$. This is easily computed as equal to 1.

What is still missing is the continuity at 0 and $\pi/2$.

3.4 *Lower bounds*

Let us now consider the case when $\vartheta \in \left]0, \frac{\pi}{2}\right[$. According to Lemma 3.1, we can take $\tau = 0$ and we have to analyse:

$$L(\vartheta) = D_{x_1}^2 + D_{x_2}^2 + (x_1 \cos\vartheta - x_2 \sin\vartheta)^2.$$

Let us introduce a parameter $\rho \in [0, 1]$ and we then associate the following decomposition

$$\begin{aligned} L(\vartheta) := D_{x_1}^2 &+ \rho^2(x_1 \cos\vartheta - x_2 \sin\vartheta)^2 + D_{x_2}^2 \\ &+ (1-\rho^2)(x_1 \cos\vartheta - x_2 \sin\vartheta)^2. \end{aligned}$$

We will find a lower bound of the spectrum of $L(\vartheta)$ by considering the sum of the lower bounds of the spectra of the following two operators:

$$P_1(\rho, \vartheta) = D_{x_1}^2 + \rho^2(x_1 \cos\vartheta - x_2 \sin\vartheta)^2$$

and

$$P_2(\rho, \vartheta) = D_{x_2}^2 + (1-\rho^2)(x_1 \cos\vartheta - x_2 \sin\vartheta)^2.$$

Easy computations (scaling and results of subsection 2.2) lead to

$$\inf \sigma(P_1(\rho, \vartheta)) = \rho\Theta_0 \cos\vartheta$$

and

$$\inf \sigma(P_2(\rho, \vartheta)) = \sqrt{1-\rho^2}\,\sin\vartheta.$$

Choosing $\rho = \cos\vartheta$, we obtain

$$\sigma(\vartheta) \geq \Theta_0(\cos\vartheta)^2 + (\sin\vartheta)^2.$$

This clearly shows two properties. The first one is that

$$\underline{\lim}_{\vartheta \to \frac{\pi}{2}} \sigma(\vartheta) \geq 1, \tag{3.3}$$

and the second one is the following:

PROPOSITION 3.5

When $\|B\|$ is fixed the bottom of the spectrum is minimal when $\vartheta = 0$ that is, according to (3.1), *when the magnetic field $\vec{B}$ is parallel to the hyperplane $x_1 = 0$.*

Remark 3.6. One can improve the lower bound by looking more carefully at the function $\rho \mapsto \phi(\rho) := \rho\Theta_0 \cos\vartheta + \sqrt{1-\rho^2}\sin\vartheta$. The derivative is given by

$$\phi'(\rho) = \Theta_0 \cos\vartheta - \frac{\rho}{\sqrt{1-\rho^2}}\sin\vartheta.$$

If we let $\rho = \sin\alpha$, we get that, when $\tan\alpha = \Theta_0 \cot(\vartheta)$, then $\phi' = 0$ and this improves the previous lower bound.

3.5 *Analysis of the essential spectrum*

The analysis in [LuPa3] will appear more transparent if one first studies the essential spectrum.

PROPOSITION 3.7

If $\vartheta \in \left]0, \frac{\pi}{2}\right[$, then the essential spectrum of $L(\vartheta)$ is contained in $[1, +\infty[$.

Proof. Using a variant of Persson's criterion, we have to show that if supp u is in $\{x_1 > R\}$ or $\{|x_2| > R\}$, then we have

$$\langle L(\vartheta)\, u,\ u\rangle \geq (1-\epsilon(R))\|u\|_2^2,$$

with $\epsilon(R) \to 0$ as $R \to +\infty$.

In the case when the support of u avoids the boundary we can use the Dirichlet result. In the case when the support of u is in $\{|x_2| > R\}$, we use the same idea as in the previous subsection. Using the same the decomposition with $\rho = \cos\vartheta$ and the properties of μ, we first get

$$\langle P_1(\rho,\vartheta))u,\ u\rangle \geq (\cos\vartheta)^2\ \mu(R\cot(\vartheta))\|u\|^2.$$

We have also seen that

$$\langle P_2(\rho,\vartheta))u,\ u\rangle \geq (\sin\vartheta)^2\|u\|^2.$$

This gives the result according to the properties of μ recalled in subsection 2.2. Once we have this information on the essential spectrum, the existence, for $\vartheta \in \left[0, \frac{\pi}{2}\right]$, of an eigenvalue as a bottom of the spectrum is the immediate consequence of the construction of a normalized element in the form domain of $L(\vartheta)$ with energy strictly less than one. This is the object of the next subsection.

3.6 *A rough upper bound*

We essentially follow Lu–Pan's proof (adding some comments) and consider the Neumann realization of $L(\vartheta)$ in $\{x_1 > 0\}$. Our aim is to find a normalized test function in the form domain of $L(\vartheta)$ with energy strictly less than one. We use first the change of variables:

$$u_1 = x_1\ \cos\vartheta - x_2\ \sin\vartheta,\ u_2 = x_1\ \sin\vartheta + x_2\ \cos\vartheta,$$

whose inverse is given by

$$x_1 = u_1 \cos\vartheta + u_2 \sin\vartheta,\ x_2 = -u_1 \sin\vartheta + u_2 \cos\vartheta.$$

This gives the Neumann realization of

$$L'(\vartheta) = -\frac{d^2}{du_1^2} - \frac{d^2}{du_2^2} + u_1^2,$$

in $\{u_1 > -\tan\vartheta\ u_2\}$.

A new change of variable: $y_1 = -u_1,\ y_2 = \tan\vartheta\ u_2$, shows that this problem is equivalent to the Neumann realization of

$$L^{\text{new}} = -\frac{d^2}{dy_1^2} - \tan^2(\vartheta)\frac{d^2}{dy_2^2} + y_1^2,$$

in $\{y_2 > y_1\}$.

Following Lu–Pan, we now introduce

$$f(t) = \exp -\frac{t^2}{2},\ \text{and}\ F(t) = \int_{-\infty}^{t} \exp -s^2\, ds.$$

We observe that F is strictly positive:

$$\lim_{t\to+\infty} F(t) = \sqrt{\pi},$$

and that

$$F(t) \sim -\frac{1}{2t}\exp -t^2,\ \text{as}\ t \to -\infty.$$

We shall apply the minimax principle with a test function in the form

$$\Psi(y_1, y_2) = f(y_1)g(y_2),$$

with g to be determined in $L^2(\mathbb{R})$. Integrating Ψ^2 in the domain, we first have

$$\|\Psi\|^2 = \int_{-\infty}^{+\infty} g(y_2)^2 F(y_2)\, dy_2.$$

Let us now compute the corresponding energy of Ψ. We first get

$$E(\Psi) = \int g(y_2)^2 \left(\int_{-\infty}^{y_2} \left((f'(y_1))^2 + y_1^2 f(y_1)^2\right) dy_1\right) dy_2$$
$$+ (\tan\vartheta)^2 \int g'(y_2)F(y_2)dy_2.$$

After a few computations and integration by parts, we get

$$E(\Psi) = \|\Psi\|^2 + \Sigma(g),$$

where

$$\Sigma(g) := \int g'(y_2)\Big((\tan\vartheta)^2 g'(y_2)F(y_2) - g(y_2)F'(y_2)\Big)\,\mathrm{d}y_2.$$

We observe that we are done if we find some $g \in L^2$ such that $\Sigma(g)$ is strictly negative. Let us first see what is going on if we try to get that the sum is zero. A natural try is then to solve the equation

$$(\tan\vartheta)^2 g'(y_2)F(y_2) - g(y_2)F'(y_2) = 0,$$

which leads to

$$g := cg_\alpha,$$

where

$$\alpha = (\cot\vartheta)^2 \text{ and } g_\alpha = F^\alpha.$$

We can compute $\Sigma(g_\alpha)$ for more general α. We get

$$\Sigma(g_\alpha) = \alpha(\alpha\tan\vartheta^2 - 1)\int f^4 F^{2\alpha-1}\mathrm{d}y_2.$$

Let us first control that this integral is well-defined. There is no problem at $+\infty$ because F tends to a constant and f is exponentially decreasing. Near $-\infty$, F decreases like f^2 (see above), so this has the right behavior for $\alpha > 0$. Now the sign of the expression is negative if

$$0 < \alpha < \cot\vartheta^2.$$

But note that g_α is not in L^2 at $+\infty$. So we are obliged to introduce a cut-off function χ_n defined by

$$\chi_n(t) = \chi\left(\frac{t}{n}\right),$$

where χ is equal to 1 for $t \le 1$ and equal to 0 for $t \ge 2$. We now take

$$g = g_{n,\alpha} = \chi_n\, g_\alpha.$$

We observe that the corresponding $\|\Psi_{n,\alpha}\|^2$ increases like n as $n \to +\infty$. More precisely, we have

$$-C + n\pi^{\alpha+1} \le \|\Psi_{n,\alpha}\|^2 \le (2n)\pi^{\alpha+1} + C.$$

Let us compare $\Sigma(g_{\alpha,n})$ and $\Sigma(g_\alpha)$ as $n \to +\infty$. We have

$$g'_{\alpha,n}(t) = \frac{1}{n}\chi'\left(\frac{t}{n}\right)g_\alpha(t) + \chi\left(\frac{t}{n}\right)g'_\alpha(t).$$

The more problematic term is

$$\frac{1}{n^2}\int \chi'\left(\frac{t}{n}\right)^2 g_\alpha^2(t)F(t)\,\mathrm{d}t.$$

But this term is less than $C/n^2\|\Psi_{n,\alpha}\|^2$, that is of order $n \times \mathcal{O}\ (1/n^2) = \mathcal{O}\ (1/n)$. The other terms appearing in the computation of $\Sigma(g_{\alpha,n}) - \Sigma(g_\alpha)$ are $\mathcal{O}\ (1/n)$. Now, observing that $\Sigma(g_\alpha) < 0$, we get, for n large enough, that

$$E(\Psi_{\alpha,n}) \leq \Sigma(g_\alpha) + \frac{C}{n} + \|\Psi_{\alpha,n}\|^2 < \|\Psi_{\alpha,n}\|^2.$$

This shows the property. Let us observe that

$$E(\Psi_{\alpha,n})/\|\Psi_{\alpha,n}\|^2 = 1 - \mathcal{O}\left(\frac{1}{n}\right).$$

So there is no hope in using this function $\Psi_{\alpha,n}$ as a good test function. As a conclusion of this part, we have shown the following:

Lemma 3.8. *For* $\vartheta \in \left]0, \frac{\pi}{2}\right[$, *the bottom of the spectrum of* $L(\vartheta, 0)$ *is an isolated eigenvalue with finite multiplicity and satisfies*

$$\sigma(\vartheta) < 1,\ \forall \vartheta \in \left]0, \frac{\pi}{2}\right[. \tag{3.4}$$

As a consequence of this lemma and of (3.3), we also obtain the continuity of σ at $\frac{\pi}{2}$:

$$\lim_{\theta \to \frac{\pi}{2}^-} \sigma(\vartheta) = \sigma\left(\frac{\pi}{2}\right) = 1. \tag{3.5}$$

3.7 *Monotonicity*

Let $\sigma(\vartheta)$ be the lowest eigenvalue, whose existence was obtained in Lemma 3.8. For the monotonicity, it is better (following Lu–Pan) to first do the dilation leading to the new operator

$$\hat{L} = (\cos\vartheta)^2 D_{y_1}^2 + (\sin\vartheta)^2 D_{y_2}^2 + (y_1 - y_2)^2.$$

We then apply Feynmann–Heilman formula (which gives the expression of $\sigma'(\vartheta)$) at a possible minimum of $\sigma(\vartheta)$ in $]0, \pi/2[$. If u_ϑ is the strictly positive normalized ground state of $\hat{L}$, we have

$$\sigma'(\vartheta) = -(\sin\vartheta\cos\vartheta)\left(\|D_{y_1}u_\vartheta\|^2 - \|D_{y_2}u_\vartheta\|^2\right).$$

The size of this expression is not evident but let us assume that $\sigma'(\vartheta_0) = 0$ for some $\vartheta_0 \in]0, \pi/2[$. Then we obtain

$$\|D_{y_1}u_{\vartheta_0}\|^2 = \|D_{y_2}u_{\vartheta_0}\|^2.$$

We then obtain the contradiction with (3.4) by writing

$$1 > \sigma(\vartheta_0) = \langle \hat{L}(\vartheta_0)u_{\vartheta_0},\ u_{\vartheta_0}\rangle = \|D_{y_2}u_{\vartheta_0}\|^2 + \|(y_1 - y_2)u_{\vartheta_0}\|^2 \geq 1.$$

The last inequality is a consequence of the lower bound for the harmonic oscillator (in the y_2 variable). We have consequently obtained the following:

Lemma 3.9. *The function* $\vartheta \mapsto \sigma(\vartheta)$ *is strictly increasing on* $\left]0, \frac{\pi}{2}\right[$.

3.8 *Another rough upper bound*

This rough upper bound (due to Lu–Pan) gives an easy way to show the right continuity of $\sigma(\vartheta)$ at 0.

Lemma 3.10. *When* $\vartheta \in \left]0, \frac{\pi}{2}\right[$,

$$\inf \text{Sp } L(\vartheta) \ \leq \cos\vartheta \ \Theta_0 + \sin\vartheta. \tag{3.6}$$

Proof. Let us write

$$L(\vartheta) = D_{x_1}^2 + (\cos\vartheta \ x_1 - z)^2 + D_{x_2}^2 + (\sin\vartheta \ x_2 - z)^2 \\ + 2(x_1\cos\vartheta - z)(z - x_2\sin\vartheta).$$

We take as test function the product of the eigenvector attached to the lowest eigenvalue of $D_{x_1}^2 + (\cos\vartheta \ x_1 - z)^2$ and of the eigenvector attached to the lowest eigenvalue of $D_{x_2}^2 + (\sin\vartheta \ x_2 - z)^2$. This gives, using the results of subsection 2.2, that observing the cross-term does not contribute and with the choice of $z = s_0 \sqrt{\cos\vartheta}$, the upper bound announced in (3.6). □

3.9 *The behavior for* ϑ *small*

A more precise estimate when $\|\vartheta\|$ is small could be useful.

Theorem 3.11. *The function* $\sigma(\vartheta)$ *has the following expansion for* ϑ *small*:

$$\sigma(\vartheta) \sim \Theta_0 + \sum_{n\geq 1} \alpha_n |\vartheta|^n, \tag{3.7}$$

with $\alpha_1 = \sqrt{(\mu''(s_0)/2)} > 0$.

Proof. By using the scaling, $(y_1, y_2) := ((\cos\vartheta)^{1/2}x_1, -(\sin\vartheta/(\cos\vartheta)^{1/2})x_2)$, we get that

$$\text{Sp } (L(\vartheta)) = \cos\vartheta \ \text{Sp } (M(\epsilon)), \tag{3.8}$$

where $M(\epsilon)$ is the Neumann realization on $\mathbb{R}^+ \times \mathbb{R}$ of $\epsilon^2 D_{y_2}^2 + H(y_2)$, ϵ is equal to $\tan\vartheta$, and $H(y_2)$ is the Neumann realization of $D_{y_1}^2 + (y_1 - y_2)^2$.

We are then essentially in the situation of the Born–Oppenheimer problem (see Combes–Duclos–Seiler [CDS] or more recently Martinez [Ma]). So we can apply directly Theorem 4.1 in [Ma][1]. □

4. Localization of the ground state

In the case of dimension 2, under the assumption that $B(x) > 0$, the basic estimate at the interior was the inequality:

$$h \int B(x)|u(x)|^2 \mathrm{d}x \leq \int |(h\nabla - iA)u|^2 \mathrm{d}x, \ \forall u \in C_0^\infty(\Omega). \tag{4.1}$$

[1]The assumptions are 'essentially satisfied'. The fact that there is a boundary condition is not a problem, because the operator in the y_1 variable is treated 'abstractly'.

This was not enough for understanding the Neumann problem. One should more carefully analyse the case of $\mathbb{R}^2_+$ and, in the case of the constant magnetic field, one should also analyse more complicate models (like for example the case of the disk). The most spectacular result was

Theorem 4.1. *When the magnetic field is constant, the ground state of the Neumann realization of P_{h,A^0} in an open regular bounded set $\Omega \subset \mathbb{R}^2$ is localized in the neighborhood of the points of the boundary of maximal curvature.*

In the case of dimension 3, estimate (4.1) should be replaced by the weaker estimate established in [HeMo1] (Theorem 3.1).

Theorem 4.2. *There exist C and $h_0 > 0$ such that, for all $h \in]0, h_0]$, we have*

$$h \int_\Omega (\|B(x)\| - Ch^{1/4})|u(x)|^2 \mathrm{d}x \le \int_\Omega |(h\nabla - iA)u|^2 \mathrm{d}x, \ \forall u \in C_0^\infty(\Omega). \tag{4.2}$$

If this result is essentially sufficient for analysing the Dirichlet problem in Ω, it is necessary to implement the analysis given in the first part in order to treat the Neumann problem. Near each point of the boundary x, we have to use the lower bound obtained for the model with constant magnetic field $B = B(x)$. For example, following the proof in [HeMo2] and using the same partition of unity, we get

Theorem 4.3.

$$h \int_\Omega W_h(x)|u(x)|^2 \mathrm{d}x \le \int |(h\nabla - iA)u|^2 \mathrm{d}x, \ \forall u \in H^1(\Omega), \tag{4.3}$$

where

$$\begin{aligned} W_h(x) &= \|B(x)\| - Ch^{1/4}, & \textit{if } \ d(x, \partial\Omega) \ge 2h^{3/8}, \\ &= \|B(s(x))\| \, \sigma(\vartheta(x)) - Ch^{1/4} & \textit{if } \ d(x, \partial\Omega) \le 2h^{3/8}. \end{aligned} \tag{4.4}$$

Here we recall that $\vartheta(x)$ satisfies

$$\|B(s(x))\| \cdot \sin \vartheta(x) = -B(s(x)) \cdot \nu(s(x)), \tag{4.5}$$

where $s(x)$ is, for x near $\partial\Omega$, the point in $\partial\Omega$ such that

$$d(x, \partial\Omega) = d(x, s(x)),$$

and we observe that, due to (3.2), $\sigma(\vartheta(x))$ is well-defined by (4.5). The first consequence (compare with Theorem 2.1) is

Theorem 4.4.

$$\lim_{h\to 0} \left(\mu^{(1)}(h)/h\right) = \inf\left(\inf \|B(x)\|, \inf_{x\in\partial\Omega} \|B(x)\| \, \sigma(\vartheta(x))\right). \tag{4.6}$$

The lower bound is a direct consequence of Theorem 4.3 and the proof of the upper bound is sketched in [LuPa5].

As in [HeMo2], this leads to two localization theorems (appearing in a weaker form in [LuPa5]; see also [PiFeSt]).

Theorem 4.5. *If* $\inf \|B(x)\| < \inf_{x\in\partial\Omega} \|B(x)\|\, \sigma(\vartheta(x))$, *then any normalized ground state of the Neumann problem is localized near the points in* Ω *where* $\|B(x)\|$ *is minimum. Moreover, the lowest eigenvalue of the Dirichlet problem is exponentially near the lowest eigenvalue of the Neumann problem.*

If $\inf \|B(x)\| > \inf_{x\in\partial\Omega} \|B(x)\|\sigma(\vartheta(x))$, *then any normalized ground state is localized near the boundary*

$$\left\| \exp \frac{\alpha}{\sqrt{h}} \mathrm{d}(x, \partial\Omega)\, u_h^{(1)} \right\| \leq C. \tag{4.7}$$

The last theorem gives a new localization.

Theorem 4.6. *If* $\inf \|B(x)\| > \inf_{x\in\partial\Omega} \|B(x)\|\sigma(\vartheta(x))$, *then any normalized ground-state is exponentially localized inside the boundary near the points where* $\|B(x)\|\sigma(\vartheta(x))$ *is minimum.*

In particular, if the magnetic field is constant, then the ground state is localized near the points of the boundary where the magnetic field is parallel to the tangent space.

These two results are based on the so-called Agmon estimates (see [Ag], [HeNo], [He1], [He2], [HeSj], [HeMo1], [HeMo2] and [PiFeSt]). The proof of the first one is actually completely similar to the proof given in [HeMo2], once we have Theorem 4.2. For the second result, we use Theorem 4.3. As for example more recently detailed in [HePa], we can show that the decay inside a fixed tubular neighborhood of the boundary is controlled, for any $\eta > 0$, by $C_\eta \exp(\eta/\sqrt{h})\ \exp -(\phi(s(x))/\sqrt{h})$, where we can for example take

$$\phi(s) = \alpha[\|B(s)\|\sigma(\vartheta(s)) - \inf_{s\in\partial\Omega} \|B(s)\|\sigma(\vartheta(s))]^{3/2}, \tag{4.8}$$

for some $\alpha > 0$ small enough.

Application

An interesting case is the case when B is constant and when Ω is the ellipsoid:

$$\alpha x_1^2 + \beta x_2^2 + \gamma x_3^2 \leq 1.$$

The set of points where $B = (B_1, B_2, B_3)$ is parallel is obtained by intersecting the boundary of the ellipsoid with the plane : $\alpha x_1 B_1 + \beta x_2 B_2 + \gamma x_3 B_3 = 0$.

More generally, if the surface is strictly convex and if the magnetic field is constant, it is possible to show that the set of points of the boundary where B is parallel to the tangent space is a C^∞ line. This theorem does not explain all the situation. In the case with constant magnetic field it would be nice to show the role of some curvature in the localization as in the case of dimension 2.

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