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Energy transfer in scattering by rotating potentials

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Abstract. Quantum mechanical scattering theory is studied for time-dependent Schrödinger operators, in particular for particles in a rotating potential. Under various assumptions about the decay rate at infinity we show uniform boundedness in time for the kinetic energy of scattering states, existence and completeness of wave operators, and existence of a conserved quantity under scattering. In a simple model we determine the energy transferred to a particle by collision with a rotating blade.

Keywords. Schrödinger operators; scattering theory; rotating potentials.

1. Introduction

This note is a preliminary report on the study of explicitly time-dependent periodic Schrödinger operators on $L^2(\mathbb{R}^\nu)$, $\nu \geq 2$

$$H(t) = H_0 + V_t\,, \quad H_0 = -\frac{1}{2m}\,\Delta \tag{1.1}$$

with 'rotating' potentials of the form

$$V_t(\mathbf{x}) = V(\mathcal{R}(t)^{-1}\mathbf{x}), \tag{1.2}$$

where V is some time-independent function decaying at infinity and $\mathcal{R}(t)$ is a rotation by ωt in the x_1, x_2-plane with period $2\pi/\omega$,

$$\begin{aligned}(\mathcal{R}(t)\mathbf{x})_1 &= \cos(\omega t)x_1 - \sin(\omega t)x_2,\\ (\mathcal{R}(t)\mathbf{x})_2 &= \sin(\omega t)x_1 + \cos(\omega t)x_2,\\ (\mathcal{R}(t)\mathbf{x})_k &= x_k, \qquad k = 3, \ldots, \nu.\end{aligned}$$

An important difference between time-independent and time-dependent perturbations is that the latter do not conserve the energy. See, e.g., [7,17,10] for studies of time periodic potentials. If one knows that the kinetic energy remains uniformly bounded (or increases at most logarithmically in time) then (cf. [5]) the machinery of time-dependent scattering theory [3] applies giving the existence and completeness of the wave operators. Some sufficient conditions for the boundedness of energy (in the sense of definition on p. 171 of

[5], see (2.8)) for repulsive potentials (in particular, of the form (1.2)) are given by Huang and Lavine [8,9], for smooth potentials e.g. by Nakamura [12].

In the context of classical mechanics a somewhat similar question of energy transfer and boundedness was recently discussed in a study of dynamics of black holes [6]. We also mention the work of Cooper and Strauss (see [1] and references therein) where the scattering off periodically moving obstacles and the boundedness of energy for scattering states have been considered for the wave equation in the framework of Lax–Phillips theory.

In § 2 we show boundedness of the kinetic energy on the ranges of wave operators for a wide class of potentials. In §§ 3 and 4 we study the time evolution in a rotating frame for potentials which need not be smooth. This transformation which has a well-known counterpart in classical mechanics (see, e.g., Example 2 in § 5.33 of [2] or § 39 of [11]) yields an explicit formula for the propagator $U(t, s)$. Methods of stationary scattering theory can then be applied to show existence and completeness of the wave operators. In the final section we discuss a simple model which describes the energy transfer between a quantum particle and a rotating blade.

2. Boundedness of kinetic energy

In this section we study bounds of the kinetic energy on incoming and outgoing scattering states. These bounds follow from suitable decay assumptions on the potential. If one knows that the scattering operator S is unitary (i.e., $\mathrm{Ran}\,\Omega^+ = \mathrm{Ran}\,\Omega^-$) or even that scattering is asymptotically complete then we will show that the kinetic energy is bounded uniformly in both time directions on all scattering states.

We begin with a rather abstract proposition. Then we show that certain classes of potentials satisfy the assumptions in the proposition. The concrete form of H_0 does not matter,

$$H_0 = \frac{1}{2m} p^2 = -\frac{1}{2m} \Delta, \qquad H_0 = \sqrt{p^2 c^2 + m^2 c^4}, \ \ldots$$

or Dirac operators (with some straightforward modifications) can be treated equally well. It is only the propagation properties in configuration space under the free time evolution analogous to (2.12) and (2.13) which matter in the applications. Throughout this section we assume for simplicity of presentation that the potentials V_t (which need not be of the special form (1.2)) are uniformly Kato-bounded with respect to a free Hamiltonian H_0, i.e., there are constants $a < 1$ and $b < \infty$ such that

$$\|V_t \Psi\| \le a\|H_0 \Psi\| + b\|\Psi\| \qquad \forall \Psi \in \mathcal{D}(H_0), \quad t \in \mathbb{R}, \tag{2.1}$$

$$\sup_{t \in \mathbb{R}} \|\partial_t V_t\| < \infty. \tag{2.2}$$

In particular, $H(t) = H_0 + V_t$ is self-adjoint on $\mathcal{D}(H(t)) = \mathcal{D}(H_0)$ for all $t \in \mathbb{R}$.

PROPOSITION 2.1

Let $H(t) = H_0 + V_t$ be a self-adjoint family of operators which satisfies (2.1), (2.2) *and generates a unitary propagator $U(t, s)$ with $U(t, s)\ \mathcal{D}(H_0) \subseteq \mathcal{D}(H_0)$ and*

$$U(s, s) = I, \qquad i \frac{\mathrm{d}}{\mathrm{d}t} U(t, s)\ \Psi_s = H(t)\ U(t, s)\ \Psi_s \tag{2.3}$$

for $\Psi_s \in \mathcal{D}(H_0)$. Let the perturbation V_t satisfy the following conditions: There is a total set $\mathcal{D}_0$ such that for any $\Phi \in \mathcal{D}_0$ there is a positive integrable function $h \in L^1(\mathbb{R})$ (depending on Φ) with

$$\|V_t\ \mathrm{e}^{-iH_0t}\ \Phi\| \leq \frac{h(t)}{1+|t|}, \tag{2.4}$$

$$\|\partial_t V_t\ \mathrm{e}^{-iH_0t}\ \Phi\| \leq h(t). \tag{2.5}$$

Then the wave operators

$$\Omega^{\pm} = \operatorname*{s-lim}_{t\to\pm\infty} U(t,0)^*\ \mathrm{e}^{-iH_0t} \tag{2.6}$$

exist and the kinetic energy is uniformly bounded in time on the ranges of $\Omega^{\pm}$ in the following sense: For $\Psi \in \Omega^{\pm}\mathcal{D}_0$, a total set in Ran $\Omega^{\pm}$,

$$\sup_{\pm t\geq 0} \|H_0^{1/2}\ U(t,0)\ \Psi\| \leq \text{const.} \tag{2.7}$$

This implies for every $\varepsilon > 0$ and $\Psi \in$ Ran $\Omega^{\pm}$ *that there exists a cut-off energy $E(\varepsilon,\Psi)$ such that*

$$\sup_{\pm t\geq 0} \|F(H_0 > E(\varepsilon,\Psi))\ \ U(t,0)\ \Psi\| < \varepsilon. \tag{2.8}$$

Here and below $F(\cdot)$ denotes the spectral projection for the indicated self-adjoint operator and the subset of the spectrum.

Remarks. The condition of boundedness (2.2) can be replaced by much weaker conditions of relative boundedness in the case of specific time evolutions like (3.7) for rotating potentials. The conditions (2.4) and (2.5) alone do not guarantee the existence of the unitary propagator satisfying (2.3); very general sufficient conditions are given in [18].

Proof. The wave operators $\Omega^{\pm}$ exist since the norm in (2.4) is integrable with respect to t. They are unitary as maps $\Omega^{\pm}\ :\ \mathcal{H} \to$ Ran $\Omega^{\pm}$. For any total set $\mathcal{D}_0 \subset \mathcal{H}$ the images $\Omega^{\pm}\mathcal{D}_0$ are total in Ran $\Omega^{\pm}$.

The uniform Kato boundedness of the potentials implies a form bound

$$|(\Psi,\ V_t\Psi)| \leq a'(\Psi,\ H_0\Psi) + b'\|\Psi\|^2$$

with $a' < 1$ for all $\Psi \in \mathcal{D}(H_0)$ and $t \in \mathbb{R}$. Thus, for $\Psi \in \mathcal{D}(H_0)$ we can use the obvious estimate

$$(\Psi,\ H_0\Psi) \leq \frac{1}{1-a'}\left\{|(\Psi,\ H(t)\Psi)| + b'\|\Psi\|^2\right\}.$$

To verify (2.7) it is sufficient to show a bound for

$$\sup_{\pm t\geq 0} |(U(t,0)\ \Psi,\ H(t)\ U(t,0)\ \Psi)| \tag{2.9}$$

for suitable $\Psi = \Omega^{\pm}\ \Phi \in \mathcal{D}(H_0)$. The time derivative of the scalar product exists and is of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(U(t,0)\ \Psi,\ H(t)\ U(t,0)\ \Psi) = (U(t,0)\ \Psi,\ \partial_t V_t\ U(t,0)\ \Psi). \tag{2.10}$$

The supremum in (2.9) is finite if

$$\|\partial_t V_t\ U(t,0)\ \Psi\| \leq \|\partial_t V_t\|\ \|U(t,0)\ \Psi - \mathrm{e}^{-iH_0t}\ \Phi\| + \|\partial_t V_t\ \mathrm{e}^{-iH_0t}\ \Phi\|$$

is integrable on $\pm\, t \in [0,\infty)$. By Assumption (2.5) this follows for the second term on the r.h.s. for a total set of states Ψ.

For $\Psi = \Omega^{\pm}\ \Phi$ we have $\lim_{t\to+\infty} U(t,0)^*\ \mathrm{e}^{-iH_0t}\ \Phi = \Psi$. Thus

$$\begin{aligned}\|U(t,0)\ \Psi - \mathrm{e}^{-iH_0t}\ \Phi\| &= \|\Psi - U(t,0)^*\ \mathrm{e}^{-iH_0t}\ \Phi\| \\ &\leq \int_s^\infty \mathrm{d}s \|V_s\ \mathrm{e}^{-iH_0s}\ \Phi\| = \int_t^\infty \mathrm{d}s \frac{h(s)}{1+|s|}\end{aligned}$$

for some integrable function $h \in L^1([0,+\infty))$ by Assumption (2.4). Using partial integration we conclude integrability:

$$\begin{aligned}\int_0^\infty \mathrm{d}t \int_t^\infty \mathrm{d}s \frac{h(s)}{1+s} &= t \int_t^\infty \mathrm{d}s \frac{h(s)}{1+s}\Big|_{t=0}^{t=\infty} + \int_0^\infty \mathrm{d}t \frac{t}{1+t} h(t) \\ &\leq \int_0^\infty \mathrm{d}s\, h(s) < \infty.\end{aligned}$$

Consequently, the time derivative (2.10) is integrable on $[0,\infty)$ and the supremum (2.9) is finite for a total set of $\Psi = \Omega^+\ \Phi$, $t \geq 0$. The uniform boundedness for $t \leq 0$ and $\Psi = \Omega^-\ \Phi$ is proved similarly. □

Next we will give sufficient conditions which guarantee that (2.4) and (2.5) are satisfied. For simplicity of presentation we use standard nonrelativistic kinematics (1.1), $H_0 = p^2/2m$. We will apply geometrical time-dependent methods. Then a convenient total set $\mathcal{D}_0 \subset \mathcal{H}$ consists of states with good localization in momentum space. Let $\widehat{\varphi}(\mathbf{p})$ denote the momentum space wave function of Φ and $B_{mv/3}(m\mathbf{v}) \subset \mathbb{R}^\nu$ the open ball of radius $mv/3$ with center $m\mathbf{v} \in \mathbb{R}^\nu$, $\mathbf{v} \neq 0$, $v = |\mathbf{v}|$. We choose the set $\mathcal{D}_0$ as

$$\begin{aligned}\mathcal{D}_0 := \{\Phi \in \mathcal{H} \mid \|\Phi\| = 1,\ \widehat{\varphi} \in C_0^\infty(\mathbb{R}^\nu),\ \exists \mathbf{v} \in \mathbb{R}^\nu, \mathbf{v} \neq 0, \\ \text{such that supp}\ \widehat{\varphi} \subseteq B_{mv/3}(m\mathbf{v})\}. \quad (2.11)\end{aligned}$$

Any state Ψ with $\widehat{\psi} \in C_0^\infty(\mathbb{R}^\nu)$, $0 \notin \text{supp}\ \widehat{\psi}$ can be written as a finite linear combination of vectors in $\mathcal{D}_0$. This set is dense in $L^2(\mathbb{R}^\nu) = \mathcal{H}$.

The states in $\mathcal{D}_0$ propagate mainly into regions where $\mathbf{x} \approx t\mathbf{p}/m \approx t\mathbf{v}$, $\mathbf{p} \in \text{supp}\,\widehat{\varphi}$. More precisely, one shows with a stationary phase estimate that propagation into 'classically forbidden' regions decays rapidly:

$$\|F(|\mathbf{x} - t\mathbf{v}| \geq \rho + |t|v/2)\,\mathrm{e}^{-itH_0}\Phi\| \leq C_N(1+\rho+|t|)^{-N},\ N \in \mathbb{N}, \rho \geq 0, \quad (2.12)$$

with a constant $C_N = C_N(\Phi) < \infty$ (see, e.g., §II of [4]). Similar estimates hold for other kinematics. We will use this bound for $\rho = 0$ here and with $\rho > 0$ in the last section.

While the estimate (2.12) follows from propagation of wave packets one has, in addition, the standard estimate of spreading in $\mathbb{R}^\nu$,

$$\sup_{x\in\mathbb{R}^\nu} |(\mathrm{e}^{-iH_0t}\ \Phi)(x)| \quad \leq \quad C(\Phi)\ (1+|t|)^{-\nu/2}, \quad (2.13)$$

where $C(\Phi) < \infty$ for $\Phi \in \mathcal{D}_0$.

Now we return to the rotating potentials (1.2) which are possible in $\nu \geq 2$ dimensions. We will give sufficient conditions for the two dimensional case which is the 'worst case': the falloff (2.13) is slowest and – compared to $\mathbb{R}^3$ – the potential does not decay in the direction parallel to the axis of rotation. We may use polar coordinates (r, ϕ) in the (x_1, x_2)-plane.

The potential can be decomposed into a rotationally invariant part

$$V_{\mathrm{inv}}(x) \quad := \quad \frac{\omega}{2\pi} \int_0^{2\pi/\omega} V(\mathcal{R}(t)^{-1}x)\, \mathrm{d}t$$

and the rest $V_{\mathrm{noninv}} = V - V_{\mathrm{inv}}$. The rotationally invariant part of the potential remains time-independent. It need not be bounded nor differentiable and it does not show up in (2.5). If for every $g \in C_0^\infty(\mathbb{R})$ there is an integrable $\widetilde{h} \in L^1([0,\infty))$ (e.g., $\widetilde{h}(\rho) = C(1+\rho)^{-1-\varepsilon}$) such that

$$\|V_{\mathrm{inv}}\ g(H_0)\ F(|x| > \rho)\| \quad \leq \quad \frac{\widetilde{h}(\rho)}{1+\rho}, \tag{2.14}$$

then (2.4) is satisfied for V_{inv}: For $\Phi \in \mathcal{D}_0$ choose $g \in C_0^\infty(\mathbb{R})$ such that $g(H_0)\Phi = \Phi$. Then

$$\begin{aligned}\|V_{\mathrm{inv}}\ \mathrm{e}^{-iH_0t}\ g(H_0)\ \Phi\| &\leq \|V_{\mathrm{inv}}\ g(H_0)\ F(|x| > |t|v/2)\|\ \|\Phi\| \\ &\quad + \|V_{\mathrm{inv}}\ g(H_0)\|\ \|F(|x| < |t|v/2)\ \mathrm{e}^{-iH_0t}\,\Phi\| \\ &\leq \frac{\widetilde{h}(|t|v/2)}{1+|t|v/2} + O(|t|^{-N}) = \frac{h(t)}{1+|t|}\end{aligned}$$

with $h \in L^1$ by (2.14) and (2.12).

Lemma 2.2. *Let V be Kato-bounded and let there exist an integrable function $h \in L^1([0,\infty))$ such that the potential V satisfies the condition*

$$\rho\ \|V\ F(|\mathbf{x}| > \rho)\| \leq h(\rho) \tag{2.15}$$

or one of the weaker conditions

$$\rho\ \|V\ (H_0+1)^{-1}\ F(|\mathbf{x}| > \rho)\| \leq h(\rho) \tag{2.16}$$

or for every $g \in C_0^\infty(\mathbb{R})$ there is an integrable $h = h_g$ with

$$\rho\ \|V\ g(H_0)\ F(|\mathbf{x}| > \rho)\| \leq h(\rho). \tag{2.17}$$

Then the rotating potential $V_t = V(\mathcal{R}(t)^{-1}\cdot)$ satisfies (2.4), *i.e., for every $\Phi \in \mathcal{D}_0$* (2.11) *there is an integrable $\widetilde{h}$ such that*

$$|t|\ \|V_t\ \mathrm{e}^{-itH_0}\ \Phi\| \leq \widetilde{h}(|t|).$$

If the partial (*distributional*) *azimuthal derivative* $(\partial_\phi V)(r,\phi)$ *yields a bounded multiplication operator $\partial_\phi V$ which satisfies*

$$\|\partial_\phi V\ F(|\mathbf{x}| > \rho)\| \leq h(\rho) \tag{2.18}$$

or the weaker

$$\|\partial_\phi V\ (H_0+1)^{-1}\ F(|\mathbf{x}|>\rho)\| \le h(\rho) \tag{2.19}$$

or for every $g \in C_0^\infty(\mathbb{R})$

$$\|\partial_\phi V\ g(H_0)\ F(|\mathbf{x}|>\rho)\| \le h(\rho) \tag{2.20}$$

for some integrable h *then* (2.5) *holds, i.e., for every* $\Phi \in \mathcal{D}_0$ *there is an integrable* $\widetilde{h}$ *with*

$$\|\partial_t V_t\ \mathrm{e}^{-itH_0}\ \Phi\| \le \widetilde{h}(|t|).$$

Remarks. If (2.15) holds then it implies (2.16) and (2.17) because the regularizing factors $(H_0+1)^{-1}$ or $g(H_0)$ act in configuration space as convolutions with a continuous rapidly decreasing function. Thus the required decay rate is preserved. But even if the operators on the l.h.s. of (2.15) are bounded the decay rate may be better in the regularized versions (2.16) or (2.17): think of a sequence of 'dipole' pairs of peaks with maxima and minima of equal amplitude but 'closer and thinner' pairs when they are localized farther away. Then $\|V\ F(|\mathbf{x}|>\rho)\|$ does not decay but the convolution causes falloff due to cancellations. The same applies to conditions (2.18)–(2.20).

A potential $V(r,\phi)$ which in an angular sector behaves like

$$V(r,\phi) = \frac{1}{r^2\ (\ln r)^2}\ \cos(r^\alpha\ \phi),\ \ r>2,\ \ \phi_1<\phi<\phi_2,$$

satisfies in this region (2.15) and (2.18) for exponents $0 \le \alpha \le 1$ but the latter is violated for $\alpha > 1$. A behavior like $\alpha = 1$ will show up in the next example.

Proof of Lemma 2.2. Since $\Phi \in \mathcal{D}_0$ has compact support in momentum space we may choose $g \in C_0^\infty(\mathbb{R})$ such that $g(H_0)\ \Phi = \Phi$. Due to rotational invariance of H_0 and $|\mathbf{x}|$ we have

$$\begin{aligned} \|V_t\ g(H_0)\ F(|\mathbf{x}|>\rho)\| &= \|V\ g(H_0)\ F(|\mathbf{x}|>\rho)\|, \\ \|\partial_t V_t\ g(H_0)\ F(|\mathbf{x}|>\rho)\| &= \omega\ \|\partial_\phi V\ g(H_0)\ F(|\mathbf{x}|>\rho)\|. \end{aligned}$$

To estimate (2.4) we use (2.17) and (2.12):

$$\begin{aligned} \|V_t\ \mathrm{e}^{-itH_0}\ \Phi\| &\le \|V\ g(H_0)\ F(|\mathbf{x}|>|t|\,v/2)\|\ \|\Phi\| \\ &\quad + \|V\ g(H_0)\|\ \|F(|\mathbf{x}|<|t|\,v/2)\ \mathrm{e}^{-itH_0}\ \Phi\| \\ &\le \frac{1}{1+|t|\,v/2}\, h(|t|\,v/2) + O(|t|^{-N}). \end{aligned}$$

Similarly, (2.20) and (2.12) yield (2.5).

In the case of regularization with a resolvent observe that $(H_0+1)^{-1}\ \Phi/\|(H_0+1)^{-1}\ \Phi\| \in \mathcal{D}_0$ has the same smoothness and support properties in momentum space as Φ. □

Another geometrical configuration is described by a strongly anisotropic potential localized near a hyperplane, in $\nu = 2$ dimensions near a line. For simplicity we assume that the support is bounded in the x_2-direction, a sufficiently rapid decay would give the same result. Moreover, we state the lemma for differentiable potentials in product form, the generalization to less regular ones as in the previous lemma is straightforward.

Lemma 2.3. *Let the potential* $V(x_1, x_2) = V^{(1)}(x_1)\, V^{(2)}(x_2) \in C^1(\mathbb{R}^2)$ *satisfy* $\operatorname{supp} V^{(2)} \subset [-d, d]$ *and the bound*

$$\rho^{1/2} \sup_{|x_1|\geq\rho} \left|V^{(1)}(x_1)\right| + \left(\frac{1}{1+\rho}\right)^{1/2} \sup_{|x_1|\geq\rho} \left|\frac{\mathrm{d}}{\mathrm{d}x_1} V^{(1)}(x_1)\right| \leq h(\rho) \tag{2.21}$$

for some integrable h. *Then* $V_t = V(\mathcal{R}(t)^{-1}\cdot)$ *satisfies conditions* (2.4) *and* (2.5) *for every* $\Phi \in \mathcal{D}_0$.

Proof. Up to rapidly decaying parts which do not affect the integrability the configuration space wave function is localized in a moving disk and satisfies for large $|t|$ the estimate

$$\left|\left(\mathrm{e}^{-itH_0}\,\Phi\right)(\mathbf{x})\right| \leq \frac{\mathrm{const}}{|t|}\ \chi_{B_{|t|\,v/2}(t\mathbf{v})}(\mathbf{x})$$

by (2.12) and (2.13). $\chi_{B_{|t|\,v/2}(t\mathbf{v})}$ denotes the characteristic function of $B_{|t|\,v/2}(t\mathbf{v})$. The k-th passage of a 'tail' of the rotating potential takes place around $t_k = k\pi/\omega$ and lasts less than $2\tau = \pi/\omega$ (for $|t| > 5d/v$). The area of intersection of the disk with the support of the potential is bounded by $\mathrm{d}v\,(|t_k| + \tau)$ and

$$|V(\mathbf{x})| \leq \sup\ |V^{(2)}|\ \frac{1}{v\,(|t_k| - \tau)/2}\, h(v\,(|t_k| - \tau)/2),$$
$$\mathbf{x} \in B_{|t|\,v/2}(t\mathbf{v}),\ |t| \geq |t_k| - \tau.$$

For given $\mathbf{v}$ and ω we obtain for one passage (up to rapidly decaying terms)

$$\begin{aligned} &\int_{t_k-\tau}^{t_k+\tau} \mathrm{d}t\ \left\|V_t\, \mathrm{e}^{-itH_0}\,\Phi\right\| \\ &\leq 2\tau\ \frac{1}{v\,(|t_k| + \tau)/2}\, h(v\,(|t_k| + \tau)/2)\ \frac{\mathrm{const}}{|t_k| - \tau}\ \{\mathrm{d}v(|t_k| + \tau)\}^{1/2} \\ &\leq \frac{\mathrm{const}}{|t_k| + \tau}\ h(\mathrm{const}\,|t_k|) \end{aligned} \tag{2.22}$$

for large enough $|k|$. Since $\|V_t\, \mathrm{e}^{-itH_0}\,\Phi\|$ is bounded on compact intervals the estimate (2.22) shows (2.4).

With $\partial_t V_t = \omega\,[\,x_2\partial_1 V - x_1\partial_2 V\,](\mathcal{R}(t)^{-1}\cdot)$ the first summand yields a bound on $B_{|t|\,v/2}(t\mathbf{v})$, $|t| \geq |t_k| - \tau$,

$$\omega\ \sup_{x_2} \left|x_2\, V^{(2)}(x_2)\right|\ [\,v\,(|t_k| - \tau)/2\,]^{1/2}\ h(v\,(|t_k| - \tau)/2)$$

by (2.21) while the second is bounded there by

$$\omega\ \sup_{x_2} \left|\frac{\mathrm{d}}{\mathrm{d}x_2}\, V^{(2)}(x_2)\right|\ \frac{3v\,(|t_k| - \tau)/2}{[v\,(|t_k| - \tau)/2]^{1/2}}\ h(v\,(|t_k| - \tau)/2).$$

Combining these estimates as above shows (2.5). □

Our third example demonstrates how dimensions strictly larger than two help if the potential decays in the other directions. For simplicity we assume $\nu = 3$ and compact support in the vertical direction (parallel to the axis of rotation) of a differentiable potential. Note that we do not need any falloff in the plane of rotation to show boundedness of the kinetic energy for asymptotically free scattering states. (The existence of wave operators follows easily for such potentials but one will need additional assumptions for asymptotic completeness.)

Lemma 2.4. *Let* $V \in C^1(\mathbb{R}^3)$ *have bounded* C^1*-norm and satisfy* supp $V \subset \{\mathbf{x} \in \mathbb{R}^3 \mid |x_3| \le d\}$. *Then* (2.4) *and* (2.5) *are satisfied.*

Proof. Let $\mathcal{D}_0$ be the total set of states with $\widehat{\varphi} \in C_0^\infty(\mathbb{R}^3)$, for which there exists a constant $b > 0$ such that either supp $\widehat{\varphi} \subset \{\mathbf{p} \in \mathbb{R}^3 \mid p_3 > mb\}$ or supp $\widehat{\varphi} \subset \{\mathbf{p} \in \mathbb{R}^3 \mid p_3 < -mb\}$. Then $\|F(|x_3| < |t|b/2)\ \mathrm{e}^{-itH_0}\ \Phi\| = O(|t|^{-N})$ and conditions (2.4) and (2.5) follow. □

To sum up the results of this section: If one knows (using any method) unitarity of the scattering operator or even asymptotic completeness and if the potential can be split into a sum of terms which satisfy any of the above sufficient conditions, then the kinetic energy is bounded uniformly in time in both time-directions simultaneously on the corresponding subspace of asymptotically free scattering states.

3. Evolution in a rotating frame

Here we study the time evolution in a rotating frame for potentials which no longer have to be smooth. This transformation yields an explicit formula for the propagator $U(t, s)$ in terms of the unitary group for some time-independent generator. This will allow to apply methods of stationary scattering theory to show existence and completeness of the wave operators in § 4.

Let $\mathcal{R}(t) \mapsto R(t)$ be the standard unitary representation of the one-parameter group $\mathcal{R}(t)$ in $L^2(\mathbb{R}^\nu)$, i.e., $(R(t)\psi)(x) = \psi(\mathcal{R}(t)^{-1}x)$. Let ωJ denote its generator, $R(t) = \exp\{-i\omega t J\}$. On a suitable domain the operator J is of the form $x_1(-i\partial/\partial x_2) - x_2(-i\partial/\partial x_1)$ or $-i\partial/\partial\phi$ if one uses cartesian or polar coordinates, respectively, in the x_1, x_2-plane.

For an observer in a rotating reference frame which turns around the orgin like the potential the latter becomes time-independent

$$V_t = R(t)\ V\ R(t)^* \longrightarrow R(t)^*\ V_t\ R(t) = V.$$

Let $t \mapsto \Psi(t) = U_{\text{inert}}(t, s)\,\Psi(s)$ be *any* time evolution in the given inertial frame with propagator U_{inert}. Then an observer in the rotating frame will see

$$R(t)^*\,\Psi(t) = R(t)^*\,U_{\text{inert}}(t, s)\,\Psi(s) = R(t)^*\,U_{\text{inert}}(t, s)\,R(s)\ R(s)^*\,\Psi(s)$$

with propagator

$$U_{\text{rot}}(t, s) = R(t)^*\,U_{\text{inert}}(t, s)\,R(s). \tag{3.1}$$

The free time evolution of a state then becomes

$$R(t)^*\ \mathrm{e}^{-itH_0}\ \Psi = \mathrm{e}^{it\omega J}\ \mathrm{e}^{-itH_0}\ \Psi, \tag{3.2}$$

where $e^{-itH_0}\,\Psi$ is the free time evolution in the inertial frame generated by H_0 as in (1.1) (or any other spherical free Hamiltonian like the relativistic one). Time zero (or $k\,2\pi/\omega$, $k \in \mathbb{Z}$) is singled out by the fact that the rotating and inertial frames coincide and the fixed potential $V_t|_{t=0} = V$ has been picked out of the family V_t for this reference time. Although the free time evolution is rotation invariant we have a different 'unperturbed' evolution which combines the unchanged free evolution with the rotation. Instead of a motion with constant velocity the unperturbed motion now is along spirals.

As the groups in (3.2) commute their product is again a unitary group with a self-adjoint generator denoted by H_ω

$$e^{i\omega J}\,e^{-itH_0} =: e^{-itH_\omega}.$$

Formally we have

$$H_\omega = H_0 - \omega J \tag{3.3}$$

but the domains differ. All three operators are essentially self-adjoint on each of the sets

$$\mathcal{D} := \{\Psi \in \mathcal{H} \mid \widehat{\psi} \in C_0^\infty(\mathbb{R}^\nu)\} \subset \mathcal{S}(\mathbb{R}^\nu) \subset \mathcal{D}(H_0) \cap \mathcal{D}(J), \tag{3.4}$$

where $\mathcal{D}$ is the set of states with smooth compactly supported wave functions in momentum space, $\mathcal{S}(\mathbb{R}^\nu)$ the Schwartz space of smooth rapidly decreasing functions (in configuration or momentum space) and $\mathcal{D}(A)$ denotes the domain of a self-adjoint operator A. All these sets are cores because they are dense in $L^2(\mathbb{R}^\nu)$ and invariant under each of the groups (see, e.g., ([13], Theorem VIII. 11)).

The operator (3.3) has been previously studied by Tip [15] in connection with the circular AC Stark effect. Let P_j, $j \in \mathbb{Z}$ denote the projection onto the eigenspace of J. Since H_0 and J commute, the subspaces $\mathcal{H}_j = P_j\mathcal{H}$ are invariant subspaces for H_ω such that

$$H_\omega = \bigoplus_{j\in\mathbb{Z}} H_{\omega,j} = \bigoplus_{j\in\mathbb{Z}} \left(H_{0j} - \omega j\right).$$

In the momentum representation H_{0j} is a real multiplication operator and consequently $H_{\omega,j}$ with domain $\mathcal{D}_j = (H_{\omega,j} - i)^{-1}\mathcal{H}_j \subset \mathcal{H}_j$ is self-adjoint on $\mathcal{H}_j$. Let now

$$\mathcal{D}(H_\omega) := \left\{ f = \bigoplus_j f_j \;\middle|\; f_j \in \mathcal{D}_j,\ \sum_j \|H_{\omega,j}\, f_j\|_j^2 < \infty \right\}$$

with $\|\cdot\|_j$ being the norm in $\mathcal{H}_j$. The operator H_ω with the domain $\mathcal{D}(H_\omega)$ can be easily shown to be self-adjoint. Its domain is rotational invariant $R(t)\,\mathcal{D}(H_\omega) = \mathcal{D}(H_\omega)$ and the operator commutes with rotations.

The set $\mathcal{D}(H_\omega)$ is strictly larger than $\mathcal{D}(H_0) \cap \mathcal{D}(J)$. Indeed, consider a state $\Psi_0 \in \mathcal{H}$ with $\|\Psi_0\| = 1$ which in the momentum representation is given by the function $\widehat{\psi}_0 \in C_0^\infty$. We assume that

$$\operatorname{supp} \widehat{\psi}_0 \subset \{\mathbf{p} \in \mathbb{R}^\nu \mid |\mathbf{p}| < 1/2\}$$

and $\widehat{\psi}_0(\mathbf{p})$ is rotational symmetric such that $\int p_1\,|\widehat{\psi}_0(\mathbf{p})|^2\,\mathrm{d}p = 0$. For $n \in \mathbb{N}$ and $\omega \neq 0$ consider the sequence of normalized pairwise orthogonal vectors in $\mathcal{D}$ (3.4)

$$\widehat{\psi}_{\omega,n}(\mathbf{p}) := \exp\left\{ in\frac{p_2}{2m\,\omega} \right\} \widehat{\psi}_0(\mathbf{p} - n\mathbf{e}_1),$$

with $\mathbf{e}_1$ being the unit vector in p_1 direction. These states are essentially localized in momentum space near $n\mathbf{e}_1$ and in configuration space near $(n/2m\omega)\mathbf{e}_2$. Simple calculations give

$$\begin{aligned}\|2m\, H_0 \Psi_{\omega,n}\|^2 &= \int_{\mathbb{R}^\nu} |\mathbf{p}|^4\, |\widehat{\psi}_0(\mathbf{p} - n\mathbf{e}_1)|^2\, \mathrm{d}p = \int_{\mathbb{R}^\nu} |\mathbf{p} + n\mathbf{e}_1|^4\, |\widehat{\psi}_0(\mathbf{p})|^2\, \mathrm{d}p \\ &= n^4 + O(n^2)\end{aligned}$$

because the term proportional to n^3 vanishes by symmetry. Further we estimate the norm of $\omega J\, \Psi_{\omega,n}$. In the momentum representation we have

$$(\omega J\, \widehat{\psi}_{\omega,n})(\mathbf{p}) = i\omega \frac{\partial}{\partial p_1}(p_2 \widehat{\psi}_{\omega,n})(\mathbf{p})) - i\omega \frac{\partial}{\partial p_2}(p_1 \widehat{\psi}_{\omega,n})(\mathbf{p})).$$

The first term is obviously bounded uniformly in n. The second term can be written in the form

$$\frac{n}{2m} \exp\left\{in\frac{p_2}{2m\,\omega}\right\} p_1 \widehat{\psi}_0(\mathbf{p} - n\mathbf{e}_1) - i\omega \exp\left\{in\frac{p_2}{2m\,\omega}\right\} p_1 \frac{\partial}{\partial p_2}\widehat{\psi}_0(\mathbf{p} - n\mathbf{e}_1). \tag{3.5}$$

For large n the first summand is the dominant contribution. Again, the square of the norm of (3.5) is $(n^2/2m)^2 + O(n^2)$. Now we turn to the estimate of $\|H_\omega \Psi_{\omega,n}\|$. The leading terms cancel in

$$\left(\frac{p_1}{2m} + i\omega \frac{\partial}{\partial p_2}\right) p_1 \widehat{\psi}_{\omega,n}(\mathbf{p})$$

and one obtains easily that $\|H_\omega \Psi_{\omega,n}\|^2 = O(n^2)$ or better.

Thus, we have shown that for large n the norms $\|H_0 \Psi_{\omega,n}\|$ and $\|J\, \Psi_{\omega,n}\|$ are of the order of magnitude $O(n^2)$ whereas the norm $\|H_\omega \Psi_{\omega,n}\|$ is of the order of magnitude $O(n)$. Choose an arbitrary sequence of coefficients $\{\alpha_n\}_{n\in\mathbb{N}_0}$ such that $\sum_{n=0}^\infty n^2|\alpha_n|^2 < \infty$ but $\sum_{n=0}^\infty n^4|\alpha_n|^2$ diverges. Let $\widehat{\Psi} = \sum_n \alpha_n \widehat{\Psi}_{\omega,n}$; by the preceding estimates it is contained in $\mathcal{D}(H_\omega)$ but neither in $\mathcal{D}(H_0)$ nor $\mathcal{D}(J)$. Thus $J\,(H_\omega - i)^{-1}$ is not a bounded operator! This means that there are quantum states for which the quantity $H_0 - \omega J$ is bounded but both the angular momentum and the kinetic energy are unbounded.

A similar calculation shows the corresponding statement for quadratic forms. $\widehat{\Psi} \in \mathcal{Q}(H_\omega)$, the form domain, for any square summable sequence of coefficients but $\widehat{\Psi} \notin \mathcal{Q}(H_0)$ and $\widehat{\Psi} \notin \mathcal{Q}(J)$ as soon as $\sum_{n=0}^\infty n^2|\alpha_n|^2$ diverges. This can happen, however, only for states with a bad localization in configuration and momentum space and a good correlation like $(p_1/2m) \sim \omega x_2$. In particular, the domains of self-adjointness of H_ω‘ $= H_0 - \omega J$’ are pairwise different for different values of ω. A further technical complication is the fact that H_ω is not bounded below.

For $\alpha \in \mathbb{R}$ we define

$$G^\alpha(\mathbf{x}) = (1 + |\mathbf{x}|^2)^\alpha, \qquad G^\alpha\, G^{-\alpha} = 1; \qquad \|G^\alpha\| = 1 \text{ if } \alpha \le 0. \tag{3.6}$$

We will need the following lemma, which is a variant of a result of Tip ([15], Lemmas 2.1 and 2.2).

Lemma 3.1. *Let $\Phi \in \mathcal{S}$, the Schwartz space of rapidly decreasing functions. Then for all $\alpha \geq 1$ $G^{-\alpha}\Phi \in \mathcal{D}(H_0)$ and for any arbitrarily small $\varepsilon > 0$*

$$\|H_0\, G^{-\alpha}\, \Phi\| \leq (1+\varepsilon)\, \|H_\omega\, \Phi\| + b(\varepsilon)\, \|\Phi\|,$$

with $b(\varepsilon)$ being non-negative.

If $(1+|\mathbf{x}|^2)$ V is bounded relative to H_0 with a bound less than one, then V is H_ω-bounded with a bound less than one too. Therefore, $H_\omega + V$ is self-adjoint on $\mathcal{D}(H_\omega + V) = \mathcal{D}(H_\omega)$.

Proof. For the first part, see [15]. For $\Psi \in \mathcal{S}(\mathbb{R}^\nu)$, a core for H_ω,

$$\begin{aligned}\|V\, \Psi\| &= \|V\, G^1\, G^{-1}\, \Psi\| \leq a\|H_0\, G^{-1}\, \Psi\| + b\|G^{-1}\, \Psi\| \\ &\leq a(1+\varepsilon)\|H_\omega\, \Psi\| + (a\, b(\varepsilon) + b)\|\Psi\|.\end{aligned}$$

□

The unitary propagator $\exp\{-i(t-s)(H_\omega + V)\}$ is *formally* related to the propagator U for the time-dependent Schrödinger equation (1.1) by

$$U(t,s) \;:=\; R(t)\; \exp\{-i(t-s)(H_\omega + V)\}\; R(s)^*. \tag{3.7}$$

If V is sufficiently smooth with respect to the angle ϕ then one can verify that $U(t,s)$ maps a core into $\mathcal{D}(H_0)$ and thus solves the Schrödinger equation with time-dependent Hamiltonian (1.1). However, even without the additional smoothness when it is not so clear in which sense the Schrödinger equation is satisfied due to domain problems one should use the propagator (3.7). It is justified by the discussion of rotating frames and (3.1) above. Next we will prove existence and completeness of the wave operators (2.6).

4. Wave and scattering operators

In the inertial frame we have chosen time $s = 0$ as reference time for the wave operators $\Omega^\pm = \Omega^\pm(H(t),\ H_0)$ in (2.6). For another reference time s one has

$$\Omega^\pm_{[s]} = \operatorname*{s-lim}_{t\to\pm\infty} U(t+s,\ s)^*\ \mathrm{e}^{-itH_0} = U(s,0)\ \Omega^\pm\ \mathrm{e}^{isH_0}. \tag{4.1}$$

There is no evident intertwining relation between Hamiltonians because of the explicit time dependence but due to periodicity we have it for monodromy operators:

$$U(2\pi\omega^{-1} + s,\ s)\ \Omega^\pm_{[s]} = \Omega^\pm_{[s]}\ \exp\{-i2\pi\omega^{-1}\, H_0\}.$$

See, however, (4.2) below. The corresponding scattering operators satisfy

$$S_{[s]} = (\Omega^+_{[s]})^*\ \Omega^-_{[s]} = \mathrm{e}^{-isH_0}\ S_{[0]}\ \mathrm{e}^{-isH_0}.$$

In general, they will depend on s because the scattering operator needs not commute with H_0.

We can combine the unitary families in (4.1) differently to obtain the evolutions in the rotating frame.

$$\begin{aligned}U(t+s,\ s)^*\ \mathrm{e}^{-itH_0} &= R(s)\ \mathrm{e}^{it(H_\omega+V)}\ R(t+s)^*\ \mathrm{e}^{-itH_0} \\ &= R(s)\ \mathrm{e}^{it(H_\omega+V)}\mathrm{e}^{-itH_\omega}\ R(s)^* = \mathrm{e}^{it(H_\omega+V_s)}\ \mathrm{e}^{-itH_\omega},\end{aligned}$$

where we have used $R(s)\, V\, R(s)^* = V_s$ in the last equality. Different wave operators are thus related by

$$\Omega^{\pm}_{[s]} = R(s)\ \Omega^{\pm}(H_\omega + V,\ H_\omega)\ R(s)^* = \Omega^{\pm}(H_\omega + V_s,\ H_\omega).$$

Instead of comparing the standard free time evolution with a perturbed one which has a time-dependent rotating potential one can study equivalently the more complicated unperturbed evolution in the rotating frame and its perturbation by a time-independent potential. If these wave operators exist we immediately get the intertwining relation

$$\mathrm{e}^{-i\tau(H_\omega+V_s)}\ \Omega^{\pm}(H_\omega + V_s,\ H_\omega) = \Omega^{\pm}(H_\omega + V_s,\ H_\omega)\ \mathrm{e}^{-i\tau H_\omega}, \quad \tau \in \mathbb{R}\,. \tag{4.2}$$

Now we can apply results of the standard scattering theory. We consider first the time-independent formulation in the rotating frame and we treat the physical case of dimension $\nu = 3$ as an example. The assumption on the decay of the potential is fulfilled if, e.g., $|V(\mathbf{x})| \sim |\mathbf{x}|^{-\beta}$ as $|\mathbf{x}| \to \infty$, $\beta > 7$.

Theorem 4.1. *Let the potential* V *satisfy* $(1 + |\mathbf{x}|^2)^2\, V \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. *Then the wave operators* $\Omega^{\pm}(H_\omega + V,\ H_\omega)$ *exist and are complete*, $\operatorname{Ran}\Omega^{\pm}(H_\omega + V,\ H_\omega) = \mathcal{H}_{\mathrm{ac}}(H_\omega + V)$.

Proof.

$$\begin{aligned} |V|^{1/2}\ (H_\omega + i)^{-1} &= |V\ G^2\,|^{1/2}\ G^{-1}\ (H_\omega + i)^{-1} \\ &= |V\ G^2\,|^{1/2}\ (H_0 + 1)^{-1} \cdot (H_0 + 1)\ G^{-1}\ (H_\omega + i)^{-1}. \end{aligned}$$

By Lemma 3.1 and since $\mathcal{S}$ is a core for H_ω we have that $(H_0+1)\ G^{-1}\ (H_\omega+i)^{-1}$ defines a bounded operator. Further we estimate

$$\|\ |V\ G^2\,|^{1/2}\ (H_0 + 1)^{-1}\|^2_{\mathrm{HS}} \le \text{const}\ \|V\ G^2\|_{L^1}$$

which is finite by assumption. Thus, $|V|^{1/2}\ (H_\omega + i)^{-1}$ is Hilbert–Schmidt.

We prove now that $|V|^{1/2}\ (H_\omega + V + i)^{-1}$ is also Hilbert–Schmidt. To this end we use the resolvent equation and write

$$\begin{aligned} |V|^{1/2}\ (H_\omega + V + i)^{-1} = {} & |V|^{1/2}\ (H_\omega + i)^{-1} \\ & - |V|^{1/2}\ (H_\omega + i)^{-1} V (H_\omega + V + i)^{-1}. \end{aligned}$$

Since V is H_ω-bounded with bound less than one, the operator $V\ (H_\omega + V + i)^{-1}$ is bounded, and thus, $|V|^{1/2}\ (H_\omega + V + i)^{-1}$ is Hilbert–Schmidt.

Since $|V|^{1/2}\ (H_\omega + i)^{-1}$ and $|V|^{1/2}\ (H_\omega + V + i)^{-1}$ are both Hilbert–Schmidt we apply the resolvent equation to obtain that $(H_\omega + V + i)^{-1} - (H_\omega + i)^{-1}$ is trace class. By the Kuroda–Birman theorem [14,16] existence and completeness of the wave operators follows. This completes the proof of Theorem 4.1. □

Obviously the same applies to wave operators for any other reference time s, i.e., if one replaces V by V_s. We state now the result in the setting of rotating potentials. The absolutely continuous spectral subspaces then correspond to the monodromy operators for one period $2\pi/\omega$.

COROLLARY 4.2

Let the potential satisfy $(1+|\mathbf{x}|^2)^\alpha\ V \in L^1(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$ *for some* $\alpha \geq 2$. *For any reference time s the wave operators* $\Omega^{\pm}_{[s]}$ *given by* (4.1) *exist and are complete in the sense that*

$$\text{Ran}\ \Omega^{\pm}_{[s]} = \mathcal{H}_{\text{ac}}(H_\omega + V_s) = R(s)\ \mathcal{H}_{\text{ac}}(H_\omega + V) = \mathcal{H}_{\text{ac}}(U(s+2\pi/\omega,\ s)).$$

The scattering operator $S_{[s]}$ *is unitary and* H_ω *is conserved under scattering*:

$$\mathrm{e}^{-i\tau H_\omega}\ S_{[s]} = S_{[s]}\ \mathrm{e}^{-i\tau H_\omega}, \quad \tau \in \mathbb{R}\,.$$

If, *in addition*, *the distributional azimuthal derivative of the potential is bounded and satisfies* $(1+|\mathbf{x}|^2)^\beta\ \partial_\phi V \in L^1(\mathbb{R}^3)\cap L^2(\mathbb{R}^3)$ *for some* $\beta > 1/2$ *then the kinetic energy is uniformly bounded.*

Remark. The boundednes of $\partial_\phi V$ has been assumed in § 2 for simplicity of presentation. This condition can be relaxed for rotating potentials e.g. to $\|\partial_\phi V\ (H_\omega + i)^{-1}\| < \infty$ or $\|(1+|\mathbf{x}|^2)\ \partial_\phi V\ (H_0+1)^{-1}\| < \infty$, cf. the proof of Proposition 2.1 and Lemma 3.1.

Proof. The first condition on the potential is the assumption of Theorem 4.1. It ensures that condition (2.16) of Lemma 2.2 is satisfied. The free resolvent is a bounded map $L^2(\mathbb{R}^3) \to L^\infty(\mathbb{R}^3)$. Therefore $(1+|\mathbf{x}|^2)^2\ V\ (H_0+1)^{-1}$ is a bounded operator on L^2. This implies boundedness of $\|V\ (H_0+1)^{-1}\ (1+|\mathbf{x}|^2)^2\|$ because the resolvent acts in configuration space as a convolution with a continuous rapidly decaying function. In particular, (2.16) follows. Similarly, the assumption in the last statement implies that (2.19) is satisfied as well. □

5. Scattering off a rotating blade

In this section we give a rough approximate description of energy transfer when a microscopic quantum particle hits a rotating macroscopic reflecting blade. During the scattering process the wave packet is assumed to be small compared to the size of the blade and the separation of the collision point from the axis of rotation. In addition, the speed of the collision point on the blade is small compared to the speed of the quantum particle (small ω) and the transmission through the blade by tunnelling is negligible.

As in § 3 we construct suitable states starting from a rotational symmetric Ψ_0 which has a smooth compactly supported momentum space wave function. It has zero angular momentum $J\ \Psi_0 = 0$. This time

$$\widehat{\psi}_{b,v}(\mathbf{p})\ :=\ \mathrm{e}^{-ibp_2}\ \widehat{\psi}_0(\mathbf{p}+mv\mathbf{e}_1)$$

describes a state which moves with velocity $-v$ in the $\mathbf{e}_1$-direction and is localized in configuration space near $x_2 = b$, $x_j \approx 0$ for $j \neq 2$. To ensure good propagation properties we assume that supp $\widehat{\psi}_0 \subset B_{mv/3}(\mathbf{0})$. Such a state has impact parameter b and is localized in angular momentum space near $-bmv$. In our units of measurement where Planck's constant $\hbar = 1$ we have $|-bmv| \gg 1$ for a macroscopic impact parameter and e.g. thermal velocities. Therefore the quantization of angular momentum is not relevant here.

The blade is represented by a strong potential with support near the hyperplane through the origin perpendicular to the $\mathbf{e}_1$-direction, e.g., in two dimensions near the line $x_1 = 0$, $|x_2| \le B$.

In the past the state has been essentially localized under the free time evolution far away from the support of the potential and it was 'incoming' from the right. Superimposing the rotation does not change the good separation from the potential if the parameters are suitably chosen, namely $B\omega$ small enough compared to v. We use this to show that in good approximation the wave operator can be calculated using a small finite negative time $-\sigma$ when the scattering sets in: $\Omega^-(H_\omega + V,\ H_\omega) \approx \exp\{i(-\sigma)(H_\omega + V)\}\ \exp\{-i(-\sigma)\,H_\omega\}$.

Let $\chi_{G(t)}$ denote the characteristic function in configuration space of a region $G(t) \subset \mathbb{R}^\nu$ and $\widetilde{\chi}_{G(t)}$ its convolution with a smooth function with integral one and support in a ball of radius one. Then $\nabla\widetilde{\chi}_{G(t)}$ and $\Delta\widetilde{\chi}_{G(t)}$ are uniformly bounded and have support in the union of balls $B_1(\partial G(t))$. The same holds for $1 - \widetilde{\chi}_{G(t)}$. The function $\widetilde{\chi}_{G(t)}$ is supported in $B_1(G(t))$ while the support of $1 - \widetilde{\chi}_{G(t)}$ is contained in $B_1(\mathbb{R}^\nu \setminus G(t))$. We choose the family $G(t)$ for negative times such that the main part of the state $\mathrm{e}^{-itH_\omega}\ \Psi_{b,v}$ is localized inside $G(t)$ and $\lim_{t\to-\infty}[\,1 - \widetilde{\chi}_{G(t)}\,]\ \mathrm{e}^{-itH_\omega}\ \Psi_{b,v} = 0$. Then

$$\begin{aligned}
&\Omega^-(H_\omega + V,\ H_\omega)\ \Psi_{b,v} - \mathrm{e}^{i(-\sigma)(H_\omega+V)}\ \widetilde{\chi}_{G(-\sigma)}\ \mathrm{e}^{-i(-\sigma)\,H_\omega}\ \Psi_{b,v} \\
&= \lim_{T\to-\infty}\left\{\mathrm{e}^{iT(H_\omega+V)}\widetilde{\chi}_{G(T)}\,\mathrm{e}^{-iTH_\omega} - \mathrm{e}^{i(-\sigma)(H_\omega+V)}\ \widetilde{\chi}_{G(-\sigma)}\mathrm{e}^{-i(-\sigma)\,H_\omega}\right\}\Psi_{b,v}.
\end{aligned} \tag{5.1}$$

If for $t \le -\sigma$ the condition $\operatorname{supp} V \cap \operatorname{supp}\widetilde{\chi}_{G(t)} = \emptyset$ is satisfied then the r.h.s. can be estimated by

$$\begin{aligned}
&\int_{-\infty}^{-\sigma}\mathrm{d}t\ \left\|\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{it(H_\omega+V)}\ \widetilde{\chi}_{G(t)}\ \mathrm{e}^{-itH_\omega}\ \Psi_{b,v}\right\| \\
&\le \int_{-\infty}^{-\sigma}\mathrm{d}t\ \left\{\left\|\,[\,H_\omega,\ \widetilde{\chi}_{G(t)}\,]\ \mathrm{e}^{-itH_\omega}\ \Psi_{b,v}\right\| + \left\|\,(\partial_t\widetilde{\chi}_{G(t)})\,\mathrm{e}^{-itH_\omega}\ \Psi_{b,v}\right\|\right\} \\
&\le \mathrm{const}\int_{-\infty}^{-\sigma}\mathrm{d}t\ \left\{\left\|F\{\mathbf{x} \in B_1(\partial G(t))\}\ \mathrm{e}^{-itH_\omega}\ \Psi_{b,v}\right\|\right. \\
&\qquad \left.+ \left\|F\{\mathbf{x} \in B_1(\partial G(t))\}\ \mathrm{e}^{-itH_\omega}\ \mathbf{p}\,\Psi_{b,v}\right\|\right\},
\end{aligned} \tag{5.2}$$

where the constant takes care of the suprema of the first and second derivatives of $\widetilde{\chi}_{G(t)}$ which are independent of t and $F\{\mathbf{x} \in M\}$ is the multiplication operator in configuration space with the characteristic function of M. Since

$$\|F\{\mathbf{x} \in B_1(\partial G(t))\}\ \mathrm{e}^{-itH_\omega}\ \Psi_{b,v}\| = \|F\{\mathbf{x} \in \mathcal{R}(t)\,B_1(\partial G(t))\}\ \mathrm{e}^{-itH_0}\ \Psi_{b,v}\|,$$

we can apply the propagation estimate (2.12) for the free time evolution.

We choose

$$G(t) = \mathcal{R}(t)^{-1}\ B_{\rho+1+|t|v/2}(b\mathbf{e}_2 - tv\mathbf{e}_1).$$

Then $\|F(\mathbf{x} \in B_1(\partial G(t))\ \mathrm{e}^{-itH_\omega}\ \Psi_{b,v}\| \le \mathrm{const}\ (1 + \rho + |t|)^{-2}$. The same estimate applies to the term with $\mathbf{p}\ \Psi_{b,v}$. The integral (5.2) is as small as desired by choosing ρ large enough. The support of the potential is separated by 1 from the support of $\widetilde{\chi}_{G(t)}$ for all small enough ω and times $t < -\sigma := -2(\rho+3)/v$. The approximation of the incoming wave operator as given on the l.h.s. of (5.1) is as good as needed. Moreover,

$$\sup_{-\sigma < t < 0} \left\| \left(e^{it(H_\omega+V)} \, \widetilde{\chi}_{G(t)} \, e^{-itH_\omega} - e^{it(H_\omega+V)} \, e^{-itH_\omega} \right) \Psi_{b,v} \right\| = o(\rho)$$

is small as well. An analogous estimate can be given for the outgoing wave operator on suitably selected states and we obtain for the scattering operator $S = S_{[0]}$:

$$\begin{aligned} S\,\Psi_{b,v} &\approx e^{i\sigma H_\omega} \, \widetilde{\chi}_{G(-\sigma)} \, e^{-i2\sigma(H_\omega+V)} \, \widetilde{\chi}_{G(-\sigma)} \, e^{i\sigma H_\omega} \, \Psi_{b,v} \\ &\approx e^{i\sigma H_\omega} \, e^{-i2\sigma(H_\omega+V)} \, e^{i\sigma H_\omega} \, \Psi_{b,v}. \end{aligned} \tag{5.3}$$

The approximation (5.3) shows that the potential may be changed arbitrarily far away from $G(t)$. In particular, it may be replaced by a simpler potential barrier in the x_1-direction which is independent of the other coordinates. Since the time interval $[-\sigma, \sigma]$ is bounded we ignore ωt for small ω and we may replace the high potential barrier by a Dirichlet boundary condition at $x_1 = 0$. For this Hamiltonian – denoted by H_D – the eigenfunctions on $\mathbb{R}^\nu_+ = \{\mathbf{x} \in \mathbb{R}^\nu \mid x_1 \geq 0\}$ are

$$e^{i\mathbf{px}} - e^{i(\mathbf{px}-2p_1x_1)}, \quad x_1 \geq 0.$$

In this approximation S acts as a reflection at the hyperplane $x_1 = 0$ in the rotating frame.

We know from Corollary 4.2 that $H_\omega = H_0 - \omega J$ is conserved under scattering but the angular momentum of $\Psi_{b,v}$ changes sign under reflection $-bmv \to bmv$. Consequently, the kinetic energy changes to

$$\begin{aligned} S\,H_0\,\Psi_{b,v} = S\,(H_\omega + \omega J)\,\Psi_{b,v} &\approx (H_\omega - \omega J)\,S\,\Psi_{b,v} = (H_0 - 2\omega J)\,S\,\Psi_{b,v} \\ &\approx (H_0 - 2\omega bmv)\;S\;\Psi_{b,v}. \end{aligned}$$

The energy increases for $\omega < 0$ when the relevant part of the blade moves towards the particle. The behavior for quantum particles is the same as for classical elastic balls.

For simplicity we have assumed an orthogonal collision. The energy transfer is the same for other angles as long as the impact parameter b remains unchanged. It determines the classical angular momentum. We will give a better approximation with detailed error bounds in a forthcoming paper.

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References

[1] Cooper J and Strauss W A, Energy boundedness and decay of waves reflecting off a moving obstacle, *Indiana Univ. Math. J.* **25** (1976) 671–690; Scattering of waves by periodically moving bodies, *J. Funct. Anal.* **47** (1982) 180–229; Time-periodic scattering of symmetric hyperbolic systems, *J. Math. Anal. Appl.* **122** (1987) 444–452

[2] Dubrovin B A, Fomenko A T and Novikov S P, Modern Geometry – Methods and Applications. Part I: The Geometry of Surfaces, Transformation Groups, and Fields (New York: Springer) (1991)

[3] Enss V, Two- and three-body quantum scattering: completeness revisited, in Partial Differential Equations, Proc. Symp., Holzhau 1988, Teubner - Texte Math. (1989) vol. 112, pp. 108–120; Long-range scattering of two- and three-body quantum systems, in: Journées Equations aux dérivées partielles (Palaiseau: Publ. Ecole Polytechnique), (1989) pp. 1–31

[4] Enss V, Propagation properties of quantum scattering states, *J. Funct. Anal.* **52** (1983) 219–251

[5] Enss V and Veselić K, Bound states and propagating states for time-dependent Hamiltonians, *Ann. Inst. Henri Poincaré, Phys. Théor.* **39** (1983) 159–191

[6] Hawking S W, Stability of AdS and phase transitions, *Class. Quantum Gravit.* **17** (2000) 1093–1099

[7] Howland J S, Scattering theory for Hamiltonians periodic in time, *Indiana Univ. Math. J.* **28** (1970) 471–494

[8] Huang M J and Lavine R B, Boundedness of kinetic energy for time-dependent Hamiltonians, *Indiana Univ. Math. J.* **38** (1989) 189–210

[9] Huang M J, Absence of bound states in a class of time-dependent Hamiltonians, *J. Math. Anal. Appl.* **183** (1994) 471–476

[10] Kitada H and Yajima K, A scattering theory for time-dependent long-range potentials, *Duke Math. J.* **49** (1982) 341–376; Remarks on our paper 'a scattering theory for time-dependent long-range potentials', *Duke Math. J.* **50** (1983) 1005–1016

[11] Landau L D and Lifshits E M, Mechanics, Course of Theoretical Physics (Oxford: Pergamon) (1960) vol. 1

[12] Nakamura S, Structure of the scattering operator for time-periodic Schrödinger equations, *J. Math. Soc. J.* **38** (1986) 261–273; Integral kernels of the scattering matrices for time-periodic Schrödinger equations, *J. Funct. Anal.* **76** (1988) 176–192

[13] Reed M and Simon B, Methods of Modern Mathematical Physics, I: Functional Analysis (New York: Academic Press) (1980)

[14] Reed M and Simon B, Methods of Modern Mathematical Physics, III: Scattering Theory (New York: Academic Press) (1979)

[15] Tip A, Atoms in circularly polarised fields: the dilatation-analytic approach, *J. Phys.* **16** (1983) 3237–3259

[16] Yafaev D R, Mathematical Scattering Theory. General Theory, Amer. Math. Soc. Transl. of Math. Monographs (RI: Providence) (1992) vol. 105

[17] Yajima K, Scattering theory for Schrödinger equations with potentials periodic in time, *J. Math. Soc. J.* **29** (1977) 729–743

[18] Yajima K, Existence of solutions for Schrödinger evolution equations, *Commun. Math. Phys.* **110** (1987) 415–426