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The universal eigenvalue bounds of Payne–Pólya–Weinberger, Hile–Protter, and H C Yang

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Abstract. In this paper we present a unified and simplified approach to the universal eigenvalue inequalities of Payne–Pólya–Weinberger, Hile–Protter, and Yang. We then generalize these results to inhomogeneous membranes and Schrödinger's equation with a nonnegative potential. We also show that Yang's inequality is always better than Hile–Protter's (and hence also better than Payne–Pólya–Weinberger's). In fact, Yang's weaker inequality (which deserves to be better known),

$$\lambda_{k+1} < \left(1 + \frac{4}{n}\right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i,$$

is also strictly better than Hile–Protter's. Finally, we treat Yang's (and related) inequalities for minimal submanifolds of a sphere and domains contained in a sphere by our methods.

Keywords. Eigenvalues of the Laplacian; universal inequalities for eigenvalues; eigenvalue ratios; the Payne–Pólya–Weinberger inequality

1. Introduction

In this paper we consider the eigenvalue problem for the Laplacian (and certain generalizations, as discussed in §§ 4 and 5) on a bounded domain (= connected open set) $\Omega \subset \mathbb{R}^n$ given by

$$-\Delta u = \lambda u \qquad \text{in } \Omega, \tag{1.1}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{1.2}$$

This is the so-called *fixed membrane problem* (in two dimensions the eigenvalues λ are proportional to the squares of the characteristic vibrational frequencies of a uniformly stretched homogeneous membrane in the shape of Ω with fixed edges). It is well-known that the spectrum of this problem is precisely $\{\lambda_i\}_{i=1}^{\infty}$ where

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \nearrow \infty. \tag{1.3}$$

Here each λ_i is an eigenvalue of finite multiplicity which is repeated according to its multiplicity. We let $\{u_i\}_{i=1}^{\infty}$ be an associated orthonormal basis of real eigenfunctions. We can take $u_1 > 0$ on Ω, which we do henceforth. Since the u_i's are taken to be real-valued, we can go forward under the assumption that $L^2(\Omega)$ represents the real Hilbert space of real-valued L^2 functions on Ω. Thus we can dispense with all complex-conjugations in our inner products.

In 1956 Payne, Pólya, and Weinberger [27] (henceforth PPW; see also [26]) proved the following *universal inequalities* for the λ_i's in the case when $n = 2$:

$$\lambda_{k+1} - \lambda_k \le \frac{2}{k}(\lambda_1 + \lambda_2 + \cdots + \lambda_k) \quad \text{for } k = 1, 2, \ldots . \tag{1.4}$$

By a straightforward application of their procedure to the case of general n one arrives at

$$\lambda_{k+1} - \lambda_k \le \frac{4}{nk} \sum_{i=1}^{k} \lambda_i \quad \text{for } k = 1, 2, \ldots , \tag{1.5}$$

which we shall refer to in this paper as the *PPW inequality*. This generalized inequality was first hinted at explicitly by Thompson [32], but certainly it is implicit in the work of PPW. Inequality (1.5) is called a universal inequality because it applies to all domains $\Omega \subset \mathbb{R}^n$ 'universally'.

A stronger inequality was derived in 1980 by Hile and Protter [18] (henceforth HP), who used the same basic techniques as PPW to prove

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \ge \frac{nk}{4} \quad \text{for } k = 1, 2, \ldots . \tag{1.6}$$

Here the left-hand side is to be interpreted as infinity if $\lambda_{k+1} = \lambda_k$. We shall refer to inequality (1.6) as the *HP inequality*. Note that (1.6) implies (1.5), since we can replace the λ_i in the denominator of (1.6) by λ_k to obtain (1.5).

More recently, Yang [33] derived the inequality

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_{k+1} - \left(1 + \frac{4}{n} \right) \lambda_i \right) \le 0 \quad \text{for } k = 1, 2, \ldots \tag{1.7}$$

in 1991. This inequality will be referred to henceforth as *Yang's inequality*, or sometimes as *Yang's first inequality* to distinguish it from a simpler inequality implied by it (to be called Yang's second inequality). Inequality (1.7) is an implicit bound for λ_{k+1}, but we can derive an explicit bound from it by observing that its left-hand side is just a quadratic in λ_{k+1}:

$$k\lambda_{k+1}^2 - \left(2 + \frac{4}{n} \right) \left(\sum_{i=1}^{k} \lambda_i \right) \lambda_{k+1} + \left(1 + \frac{4}{n} \right) \sum_{i=1}^{k} \lambda_i^2 . \tag{1.8}$$

Thus we must have $\lambda_{k+1} \le$ [larger root] or

$$\begin{aligned} \lambda_{k+1} \le \frac{1}{2k} \Bigg\{ & \left(2 + \frac{4}{n} \right) \left(\sum_{i=1}^{k} \lambda_i \right) \\ & + \left[\left(2 + \frac{4}{n} \right)^2 \left(\sum_{i=1}^{k} \lambda_i \right)^2 - 4k \left(1 + \frac{4}{n} \right) \sum_{i=1}^{k} \lambda_i^2 \right]^{\frac{1}{2}} \Bigg\} \end{aligned} \tag{1.9}$$

for $k = 1, 2, \ldots$. If we now eliminate $k \sum_{i=1}^{k} \lambda_i^2$ in favor of $\left(\sum_{i=1}^{k} \lambda_i \right)^2$ using $k \sum_{i=1}^{k} \lambda_i^2 \ge \left(\sum_{i=1}^{k} \lambda_i \right)^2$ (an easy consequence of the Cauchy–Schwarz inequality) and observe that

$\left(2+\frac{4}{n}\right)^2 - 4\left(1+\frac{4}{n}\right) = \left(\left(1+\frac{4}{n}\right)+1\right)^2 - 4\left(1+\frac{4}{n}\right) = \left(\left(1+\frac{4}{n}\right)-1\right)^2 = \left(\frac{4}{n}\right)^2,$
we arrive at *Yang's second inequality*

$$\lambda_{k+1} \le \frac{1}{k}\left(1+\frac{4}{n}\right)\sum_{i=1}^{k}\lambda_i \quad \text{for } k = 1, 2, \ldots . \tag{1.10}$$

This inequality is clearly stronger than the PPW inequality, since it results from replacing the λ_k on the left-hand side of (1.5) by the average of the first k eigenvalues and λ_k is certainly larger than or equal to $(\lambda_1 + \ldots + \lambda_k)/k$ (in fact, strictly larger for $k > 1$ since we know by (1.3) that then $\lambda_1 < \lambda_k$). Thus, we conclude that both of Yang's inequalities are stronger than the PPW inequality (1.5).

That the smaller root of (1.8) is of little interest follows easily from the expression for (1.8) found on the left-hand side of (1.7), which we now denote by

$$H_k(x) \equiv \sum_{i=1}^{k}(x-\lambda_i)\left(x-\left(1+\frac{4}{n}\right)\lambda_i\right) \quad \text{for } k \ge 1, \tag{1.11}$$

to emphasize its dependence on x as a variable, and on the index k (we also define $H_0(x) \equiv 0$). Now (1.7) reads $H_k(\lambda_{k+1}) \le 0$ and since $H_k(x)$ is a quadratic in x it follows that [smaller root] $\le \lambda_{k+1} \le$ [larger root]. However, if we substitute $x = \lambda_k$ in H_k, the last term in the sum in (1.11) drops out and we find

$$H_k(\lambda_k) = H_{k-1}(\lambda_k) \le 0, \tag{1.12}$$

showing that the smaller root of H_k is always less than or equal to λ_k. Since $\lambda_{k+1} \ge \lambda_k$ always, this makes the smaller root irrelevant for our considerations here. Observations in this direction were made earlier by Yang [33] (see p. 7 of the 1995 version) and by Harrell and Stubbe [17] (see Proposition 6, parts (i) and (iii), on p. 1802), both of whom had somewhat different aims in view. We note that $\lambda_k \ge$ [smaller root of H_k] is irrelevant as well, since by the above this (implicit) inequality follows from $H_{k-1}(\lambda_k) \le 0$, which is nothing but Yang's first inequality with k shifted down by 1 (but this is not the point of view of [33] and [17]).

In this paper we give simplified proofs of the Hile–Protter and Yang inequalities (from either of which the PPW inequality may be recovered, as noted above). Moreover, given our simplified proof of the HP inequality, we show that Yang's inequality requires us to incorporate only one new element: the 'optimal' use of the Cauchy–Schwarz inequality (which we discuss in § 2).

While our two proofs will certainly suggest that Yang's inequality is stronger than the HP inequality, it is not entirely straightforward to prove this fact. In § 3 we give a proof based on convexity. In fact, we show that Yang's second inequality implies the HP inequality and thus that

$$\text{Yang 1} \Rightarrow \text{Yang 2} \Rightarrow \text{HP} \;\Rightarrow \text{PPW}. \tag{1.13}$$

These implications hold for each k, $k = 1, 2, 3, \ldots$.

In § 4 we consider other eigenvalue problems which extend problem (1.1)–(1.2) to more general operators than the Laplacian. In particular, we consider Schrödinger operators $-\Delta + V(\vec{x})$ with $V \ge 0$ on Ω and eigenvalue problems with a weight (e.g., the fixed membrane of variable density $\rho(\vec{x})$, with eigenvalue problem $-\Delta u = \lambda\rho(\vec{x})u$ in Ω, $u = 0$ on $\partial\Omega$). Indeed, with no extra effort we handle the problem that includes both these extensions simultaneously, i.e., a potential $V(\vec{x}) \geq 0$ and a variable density $\rho(\vec{x}) > 0$.

Finally in § 5 we use our approach to treat two further problems considered by Yang, showing that our simplified approach to Yang's inequalities works for them as well. These are the problems of a minimal hypersurface $M \subset \mathbb{S}^{n+1}$ (in fact, M can be any minimal submanifold) and of a domain $\Omega \subset \mathbb{S}^n$. That is, we consider inequalities for the eigenvalues of the Laplacian (= Laplace–Beltrami operator for the sphere) on these two sets (in the latter case our eigenvalues are for the boundary condition $u = 0$ on $\partial\Omega$). In § 6 we conclude with some remarks concerning extensions and further work.

2. Proofs of the Hile–Protter and Yang inequalities

The basic strategy is to use the Rayleigh–Ritz inequality

$$\lambda_{k+1} \leq \frac{\int_\Omega \varphi(-\Delta\varphi)}{\int_\Omega \varphi^2}, \tag{2.1}$$

which holds for a trial function φ which is nontrivial and orthogonal to $u_1, u_2, \ldots, u_k$. For suitable choices of φ, built up from $u_1, \ldots, u_k$, we can find bounds for λ_{k+1} in terms of the eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_k$.

In particular, we take

$$\varphi = \varphi_i = xu_i - \sum_{j=1}^{k} a_{ij}u_j \quad \text{for } 1 \leq i \leq k \tag{2.2}$$

where x represents the first cartesian coordinate x_1 (and later any cartesian coordinate x_ℓ) and the coefficients a_{ij}, $1 \leq i, j \leq k$, are chosen to make $\varphi_i \perp u_j$ for all $1 \leq i, j \leq k$. Thus the a_{ij} are the components of xu_i along u_j or

$$a_{ij} = \int_\Omega xu_iu_j = a_{ji}. \tag{2.3}$$

Furthermore, we find

$$\int_\Omega \varphi_i^2 = \int_\Omega xu_i\varphi_i = \int_\Omega x^2u_i^2 - \sum_{j=1}^{k} a_{ij}^2, \tag{2.4}$$

$$\begin{aligned} -\Delta\varphi_i &= -\Delta(xu_i) - \sum_{j=1}^{k} a_{ij}(-\Delta u_j) \\ &= \lambda_i xu_i - 2u_{ix} - \sum_{j=1}^{k} a_{ij}\lambda_j u_j, \end{aligned} \tag{2.5}$$

and hence (since $\varphi_i \perp u_j$ for all $1 \leq i, j \leq k$)

$$\begin{aligned} \int_\Omega \varphi_i(-\Delta\varphi_i) &= \lambda_i \int_\Omega xu_i\varphi_i - 2\int_\Omega \varphi_i u_{ix} \\ &= \lambda_i \int_\Omega \varphi_i^2 - 2\int_\Omega \varphi_i u_{ix}. \end{aligned} \tag{2.6}$$

Here and in the following we use the notation u_{ix} to denote the partial derivative of u_i with respect to the variable x. Now from the Rayleigh–Ritz inequality (2.1) we can conclude

$$(\lambda_{k+1} - \lambda_i) \int_\Omega \varphi_i^2 \leq -2 \int_\Omega \varphi_i u_{ix}, \tag{2.7}$$

which holds whether or not φ_i happens to vanish identically. In particular we can conclude that

$$\begin{aligned} 0 \leq -2 \int_\Omega \varphi_i u_{ix} &= -2 \int_\Omega \left[x u_i - \sum_{j=1}^{k} a_{ij} u_j \right] u_{ix} \\ &= -\int_\Omega x (u_i^2)_x + 2 \sum_{j=1}^{k} a_{ij} \int_\Omega u_{ix} u_j \\ &= \int_\Omega u_i^2 + 2 \sum_{j=1}^{k} a_{ij} b_{ij} \end{aligned} \tag{2.8}$$

after integrating by parts and introducing

$$b_{ij} \equiv \int_\Omega u_{ix} u_j. \tag{2.9}$$

We rewrite (2.7) as

$$\lambda_{k+1} - \lambda_i \leq \frac{-2 \int_\Omega \varphi_i u_{ix}}{\int_\Omega \varphi_i^2}, \tag{2.10}$$

where the right-hand side is to be interpreted as infinity if φ_i vanishes identically. We now use the Cauchy–Schwarz inequality on $-2 \int_\Omega \varphi_i u_{ix}$ to put (2.10) into a more manageable form. We have

$$\left(-2 \int_\Omega \varphi_i u_{ix} \right)^2 \leq 4 \left(\int_\Omega \varphi_i^2 \right) \left(\int_\Omega u_{ix}^2 \right) \tag{2.11}$$

or, since $-2 \int_\Omega \varphi_i u_{ix} \geq 0$,

$$\frac{-2 \int_\Omega \varphi_i u_{ix}}{\int_\Omega \varphi_i^2} \leq \frac{4 \int_\Omega u_{ix}^2}{-2 \int_\Omega \varphi_i u_{ix}}, \tag{2.12}$$

again with the understanding that if $\varphi_i \equiv 0$ both members are to be interpreted as infinity, and, moreover, that whenever $\int_\Omega \varphi_i u_{ix} = 0$ we interpret the right-hand side of (2.12) as infinity. Thus, combining (2.10) with (2.12) and then using (2.8), we find

$$\lambda_{k+1} - \lambda_i \leq \frac{4 \int_\Omega u_{ix}^2}{-2 \int_\Omega \varphi_i u_{ix}} = \frac{4 \int_\Omega u_{ix}^2}{1 + 2 \sum_{j=1}^{k} a_{ij} b_{ij}}. \tag{2.13}$$

It only remains to find b_{ij} in terms of a_{ij}. We have

$$\begin{aligned} 2b_{ij} &= 2\int_\Omega u_{ix}u_j \\ &= \int_\Omega [\Delta(xu_i) - x\Delta u_i]u_j \\ &= -\int_\Omega xu_i(-\Delta u_j) + \int_\Omega x(-\Delta u_i)u_j \\ &= (\lambda_i - \lambda_j)a_{ij} \end{aligned} \tag{2.14}$$

where we employed two integrations by parts on the first term of the integral in passing from the second to the third line (both boundary terms vanish due to the fact that each $u_i = 0$ on $\partial\Omega$). Note, in particular, that $b_{ji} = -b_{ij}$, i.e., that b_{ij} is antisymmetric. We therefore have

$$0 \le -2\int_\Omega \varphi_i u_{ix} = 1 + \sum_{j=1}^{k}(\lambda_i - \lambda_j)a_{ij}^2 \tag{2.15}$$

and hence from (2.13)

$$(\lambda_{k+1} - \lambda_i)\left[1 + \sum_{j=1}^{k}(\lambda_i - \lambda_j)a_{ij}^2\right] \le 4\int_\Omega u_{ix}^2. \tag{2.16}$$

Next we observe that everything done above for $x = x_1$ can be carried through for $x = x_\ell$, $1 \le \ell \le n$. Promoting x in this way and with the introduction of ℓ as an additional index, we find, in obvious notation, that we need to make the following replacements: $x \to x_\ell$, $\varphi_i \to \varphi_i^{(\ell)}$, $a_{ij} \to a_{ij}^{(\ell)}$, $b_{ij} \to b_{ij}^{(\ell)}$. In particular, (2.16) becomes

$$(\lambda_{k+1} - \lambda_i)\left[1 + \sum_{j=1}^{k}(\lambda_i - \lambda_j)(a_{ij}^{(\ell)})^2\right] \le 4\int_\Omega u_{ix_\ell}^2. \tag{2.17}$$

By summing (2.17) on ℓ for $\ell = 1, \ldots, n$ and using the fact that $\lambda_i = \int_\Omega |\nabla u_i|^2$ we find

$$(\lambda_{k+1} - \lambda_i)\left[n + \sum_{j=1}^{k}(\lambda_i - \lambda_j)A_{ij}\right] \le 4\int_\Omega |\nabla u_i| = 4\lambda_i, \tag{2.18}$$

where we have set

$$A_{ij} = \sum_{\ell=1}^{n}(a_{ij}^{(\ell)})^2. \tag{2.19}$$

We note that

$$A_{ij} = A_{ji} \ge 0. \tag{2.20}$$

To prove the Hile–Protter inequality, it simply remains to observe that we can eliminate the uncontrolled terms in A_{ij} by dividing (2.18) through by $\lambda_{k+1} - \lambda_i$ and then summing on i from 1 to k. The terms in $(\lambda_i - \lambda_j)A_{ij}$ then disappear, due to antisymmetry, and we are left with

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \geq \frac{nk}{4}, \tag{2.21}$$

the HP inequality (1.6).

To prove Yang's inequality we use the same basic strategy, but with one improvement. This is what we refer to as the 'optimal' use of the Cauchy–Schwarz inequality. Once this modification is effected, we have only to find a new way to exploit the antisymmetry of the combination of terms involving $a_{ij}b_{ij}$ (see (2.8)) and b_{ij}^2 (see (2.22) below) to eliminate them. This is quite straightforward, and leads immediately to Yang's inequality.

To proceed, we back up to our use of the Cauchy–Schwarz inequality in (2.11) and observe that the inner product $\int_\Omega \varphi_i u_{ix}$ is unaffected by subtracting from u_{ix} any function which is orthogonal to φ_i. In particular, since $\varphi_i \perp u_j$ for all $1 \leq i, j \leq k$, it is natural (and 'optimal' in this setting) to subtract from u_{ix} its components along the u_j's for $1 \leq j \leq k$. Now these components are none other than our b_{ij}'s, defined by $b_{ij} = \int_\Omega u_{ix} u_j$. Thus in place of (2.11) we write

$$\begin{aligned}
\left(-2\int_\Omega \varphi_i u_{ix}\right)^2 &= \left(-2\int_\Omega \varphi_i \left[u_{ix} - \sum_{j=1}^{k} b_{ij} u_j\right]\right)^2 \\
&\leq 4\left(\int_\Omega \varphi_i^2\right)\left(\int_\Omega \left[u_{ix} - \sum_{j=1}^{k} b_{ij} u_j\right]^2\right) \\
&= 4\left(\int_\Omega \varphi_i^2\right)\left[\int_\Omega u_{ix}^2 - \sum_{j=1}^{k} b_{ij}^2\right] \\
&= \left(\int_\Omega \varphi_i^2\right)\left[4\int_\Omega u_{ix}^2 - \sum_{j=1}^{k} (\lambda_i - \lambda_j)^2 a_{ij}^2\right].
\end{aligned} \tag{2.22}$$

The net effect of following through with this adjustment is that we arrive at an analog of (2.16) with its right-hand side replaced by

$$4\left(\int_\Omega u_{ix}^2 - \sum_{j=1}^{k} b_{ij}^2\right) = 4\int_\Omega u_{ix}^2 - \sum_{j=1}^{k} (\lambda_i - \lambda_j)^2 a_{ij}^2,$$

and thus a stronger inequality. We have in place of (2.16)

$$(\lambda_{k+1} - \lambda_i)\left[1 + \sum_{j=1}^{k} (\lambda_i - \lambda_j) a_{ij}^2\right] \leq 4\int_\Omega u_{ix}^2 - \sum_{j=1}^{k} (\lambda_i - \lambda_j)^2 a_{ij}^2 \tag{2.23}$$

or

$$(\lambda_{k+1} - \lambda_i) + \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_j)(\lambda_i - \lambda_j) a_{ij}^2 \leq 4\int_\Omega u_{ix}^2. \tag{2.24}$$

If we now promote x to x_ℓ and sum on ℓ from 1 to n as before we arrive at

$$n(\lambda_{k+1}-\lambda_i)+\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_j)(\lambda_i-\lambda_j)A_{ij}\le 4\int_\Omega|\nabla u_i|^2=4\lambda_i \tag{2.25}$$

with A_{ij} defined (and symmetric) as above. This time, to eliminate the uncontrolled terms in A_{ij} using antisymmetry we find ourselves needing to multiply in an extra factor of $\lambda_{k+1}-\lambda_i$ (rather than dividing by it as above), before summing on i from 1 to k. Doing this, we arrive at

$$n\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2\le 4\sum_{i=1}^{k}\lambda_i(\lambda_{k+1}-\lambda_i) \tag{2.26}$$

or

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_{k+1}-\left(1+\frac{4}{n}\right)\lambda_i\right)\le 0, \tag{2.27}$$

which is Yang's first inequality (1.7).

It seems reasonable to expect, given that Yang's inequality is based on the strengthened inequality (2.23) (as opposed to (2.16)), that Yang's inequality is stronger than Hile and Protter's. However, because of the difference in the way we applied the final step, using antisymmetry to eliminate unwanted terms, this is not obvious. In the next section we use a convexity argument to show that, in fact, Yang's weaker second inequality is still enough to imply the Hile–Protter inequality.

For the record, we also note an easy proof of Yang's second inequality ('Yang 2') based on our work above. If we simply average (2.18) and (2.25) we obtain

$$n(\lambda_{k+1}-\lambda_i)+\sum_{j=1}^{k}(\lambda_{k+1}-(\lambda_i+\lambda_j)/2)(\lambda_i-\lambda_j)A_{ij}\le 4\int_\Omega|\nabla u_i|^2=4\lambda_i. \tag{2.28}$$

Since the term in the summation here is antisymmetric in i and j, we can just sum on i from 1 to k directly to obtain

$$\lambda_{k+1}\le\left(1+\frac{4}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i\quad\text{for } k=1,2,\ldots \tag{2.29}$$

which is Yang's second inequality (1.10). Note that this derivation suggests that Yang 2 is in a sense midway between Yang 1 and the HP inequality. For a related point of view supporting this position, see Harrell–Stubbe [17] and Ashbaugh–Hermi [8].

Remarks.

(1) Hile and Protter's original proof of their inequality [18] is much more involved than that given here. Their proof also appears with little change in [29] (see also [28,30]).
(2) The HP inequality has been dressed in operator theory garb and derived in the context of operators and their commutators (using eigenprojections and traces) by Hook [20],

Harrell and Michel [15,16], and Harrell and Stubbe [17]. Indeed, Hook even proves it with strict inequality. More will be said about the possibility of making all our inequalities strict at the end of the next section. Hook's work is also discussed in Protter's articles [28–30].

(3) The HP inequality has been generalized to higher order elliptic operators, typically powers of or polynomials in the Laplacian, beginning with work of Hile and Yeh [19] in 1984 and Chen [11] in 1985. Chen, alone and with Qian, has a whole series of papers on this subject, many of which are listed in the references to [5] (and hence we forgo repeating them here; but see also Qian and Chen [31]). The works of Hook [20] and Harrell and Michel [16] also deal with higher order operators. In addition, there are various papers that generalize the PPW, HP, and related inequalities to the eigenvalues of the Laplacian (= Laplace–Beltrami operator) on a Riemannian manifold. These include Cheng [12], Maeda [25], Harrell and Michel [15,16], Lee [21], and Yang [33], as well as several additional papers (by Li [23], Yang and Yau [34], Leung [22], Harrell [14], and Anghel [2]) listed in [5] (see the remarks near the end of § 2 of [5] for a relatively complete survey of the literature of PPW-related bounds as of 1993). Maeda's early paper [25] seems to have been entirely overlooked in the literature until now.

(4) The proof of the HP inequality given above first appeared in [5]. We presented it again here for comparison with our new proof of Yang's inequality. As noted here, one only needs to incorporate our 'optimal' use of the Cauchy–Schwarz inequality and see how to make use of antisymmetry to free us of unwanted terms to promote this proof to a full proof of Yang's inequality.

(5) Harrell and Stubbe [17] have also given a proof of Yang's inequality. Their proof is from the operator viewpoint and uses a clever identity for sorting terms by symmetry or antisymmetry. They also give related inequalities based on these ideas. However, their proof does not seem as simple and straightforward as the one given here.

(6) Similar ideas to those used in our proof of Yang's inequality above (and specifically our 'optimal' use of the Cauchy–Schwarz inequality) were used in [6]. However, there the interest was solely in the first three eigenvalues, λ_1, λ_2, and λ_3, and, in particular, the last part of the general argument, elimination of the A_{ij}'s via antisymmetry, was unknown. While Yang's original argument [33] used the Cauchy–Schwarz inequality in what turns out to be an 'optimal' way, his proof was much more involved and less transparent than ours. In particular, it is not clear from Yang's proof that the Cauchy–Schwarz inequality is being used in an optimal way in the sense in which we introduced this notion above.

3. Yang's inequalities imply the Hile–Protter inequality

In this section we prove that Yang's second inequality (and therefore his first as well) implies the Hile–Protter inequality. After that, we examine the extent to which the various inequalities discussed here can be improved to strict inequalities.

To make the connection between Yang's second inequality and the Hile–Protter inequality, we begin by casting the Hile–Protter inequality in a new form (which goes back to Hile and Protter [18]). Another way to view the HP inequality is in terms of the function

$$F(s) \equiv \sum_{i=1}^{k} \frac{\lambda_i}{s - \lambda_i}. \tag{3.1}$$

This function has poles at the λ_i, $1 \leq i \leq k$, and is strictly decreasing between (and beyond) its poles. In particular, it is strictly decreasing for $s > \lambda_k$, varying from ∞ at $s = \lambda_k$ to 0 at $s = \infty$, and hence there is exactly one value $\sigma > \lambda_k$ at which $F(\sigma) = nk/4$. The Hile–Protter inequality may now be interpreted as the inequality

$$\lambda_{k+1} \leq \sigma. \tag{3.2}$$

To continue with our convexity argument we set

$$f(x) = \frac{x}{s - x} \tag{3.3}$$

and consider this function for $x < s$ with s positive. Since $f(x) = s/(s - x) - 1$, we find $f'(x) = s/(s - x)^2$ and $f''(x) = 2s/(s - x)^3$, showing that $f(x)$ is strictly convex for $x \in (-\infty, s)$. Observing that

$$F(s) = \sum_{i=1}^{k} f(\lambda_i), \tag{3.4}$$

assuming $s \geq \lambda_k$, and using the convexity of f we find

$$\begin{aligned} F(s) &= \sum_{i=1}^{k} f(\lambda_i) \geq kf\left(\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right) \\ &= k\frac{\frac{1}{k}\sum_{i=1}^{k}\lambda_i}{s - \frac{1}{k}\sum_{i=1}^{k}\lambda_i}, \end{aligned} \tag{3.5}$$

where we interpret the left-hand side as infinity if $s = \lambda_k$. Hence

$$F\left(\left(1 + \frac{4}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right) \geq \frac{nk}{4} \tag{3.6}$$

and it follows, by what we said above and the fact that by Yang's second inequality λ_k (indeed λ_{k+1}) is less than or equal to our choice of $s = \left(1 + \frac{4}{n}\right) 1/k \sum_{i=1}^{k} \lambda_i$, that

$$\left(1 + \frac{4}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i \leq \sigma. \tag{3.7}$$

This shows that the upper bound for λ_{k+1} given by Yang's second inequality is better than Hile and Protter's upper bound σ, and hence that both of Yang's inequalities are better than the HP inequality.

We now turn to the question of the strictness of the various inequalities. It turns out that each of the inequalities discussed here can be made strict, except perhaps Yang's first inequality, which we leave undecided. To begin at the beginning, we first go back to our proof of Yang's second inequality from our introduction and show how to make it strict. For this one only has to observe that in our use of the Cauchy–Schwarz inequality following (1.9) above we will have strict inequality so long as the vector $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is not proportional to $(1, 1, \ldots, 1)$, which is true as soon as $k \geq 2$ by virtue of $\lambda_2 > \lambda_1$. Or one can draw the same conclusion from either version of the error term (i.e., the right-hand side) in

$$k\sum_{i=1}^{k}\lambda_i^2-\left(\sum_{i=1}^{k}\lambda_i\right)^2=\frac{1}{2}\sum_{i=1}^{k}\sum_{j=1}^{k}(\lambda_i-\lambda_j)^2$$
$$=k\sum_{i=1}^{k}(\lambda_i-\langle\lambda\rangle_k)^2, \tag{3.8}$$

where $\langle\lambda\rangle_k = (\lambda_1+\lambda_2+\cdots+\lambda_k)/k$ denotes the average of the first k eigenvalues (which is definitely larger than λ_1 for $k \geq 2$). In any event we conclude that Yang's second inequality is strict for all $k \geq 2$.

The fact that Yang's second inequality is also strict for $k = 1$, that is, $\lambda_2/\lambda_1 < 1 + \frac{4}{n}$ follows from other known results. In particular, by a result of Brands [9] and its generalization to dimension n (see [4]), we have

$$\frac{\lambda_2+\lambda_3+\cdots+\lambda_{n+1}}{\lambda_1}\leq n+3+\frac{\lambda_1}{\lambda_2}, \tag{3.9}$$

and from this it follows that

$$\frac{\lambda_2}{\lambda_1}\leq\frac{n+3+\sqrt{n^2+10n+9}}{2n} \tag{3.10}$$

(see [18,4,6], and for the 2-dimensional case [9]; it turns out that (3.10) can itself be made strict, by results of Chiti [13] and Lorch [24], see also [3,4]). We shall show that the right-hand side of (3.10) is strictly smaller than $1 + \frac{4}{n}$ for all $n > 0$. To this end we employ the theorem of the arithmetic and geometric means as follows:

$$\sqrt{n^2+10n+9}=\sqrt{(n+1)(n+9)}$$
$$< n+5,$$

implying

$$n+3+\sqrt{n^2+10n+9}<2n+8,$$

and in turn

$$\frac{n+3+\sqrt{n^2+10n+9}}{2n}<1+\frac{4}{n},$$

which is the desired result.

Thus Yang's second inequality is strict for all k. Had we proved this result earlier (which was certainly possible) we could have avoided dealing with the borderline cases that came up between equation (3.4) and inequality (3.7) above. In particular, we can arrive at the HP inequality $\lambda_{k+1} \leq \sigma$ with strict inequality, by using (3.7) and the strictness of Yang's second inequality. Reduced to its essence the argument above runs as follows:

HP $\Leftrightarrow$ $F(\lambda_{k+1}) \geq nk/4$, but Yang 2 in its strict form and F strictly decreasing $\Rightarrow$ $F(\lambda_{k+1}) > F\left(\left(1+\frac{4}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i\right) \geq nk/4 = F(\sigma)$, which is the HP inequality in its strict form. Alternatively, we can conclude that $\lambda_{k+1} < (1+4/n)\,(1/k)\sum_{i=1}^{k}\lambda_i \leq \sigma$.

From the fact that the HP inequality implies the PPW inequality it is immediate that we can also write the PPW inequality as a strict inequality.

As a further remark we note that it is also possible, at least for $k \geq 2$, to introduce a strict inequality in (3.5) (and hence also in (3.6) and (3.7)) due to the strict convexity of F and the fact that $\lambda_1 < \lambda_2$. This also allows us to conclude that the HP inequality can be made strict for $k \geq 2$ (and the $k = 1$ case, $(\lambda_2/\lambda_1) < 1 + \frac{4}{n}$, can be handled as before). Moreover, we find

$$\lambda_{k+1} < \left(1 + \frac{4}{n}\right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i < \sigma \tag{3.11}$$

for all $k \geq 2$ (and when $k = 1$, $\lambda_2 < \left(1 + \frac{4}{n}\right) \lambda_1 = \sigma$). This shows, in fact, that for $k \geq 2$ Yang's second inequality (and hence his first as well) is always strictly better than the HP inequality. In addition, it is easy to see that for $k \geq 2$ Yang's first inequality is strictly better than his second and that the HP inequality is strictly better than the PPW inequality. The sense in which 'strictly better' is to be understood in the foregoing is as strict comparisons between the various upper bounds for λ_{k+1} for all realizable choices of $\lambda_1, \ldots, \lambda_k$. Thus, if we denote our upper bounds for λ_{k+1} by $G_k^{(\text{Yang1})}(\lambda_1, \ldots, \lambda_k)$, etc., we have

$$\begin{aligned} \lambda_{k+1} &\leq G_k^{(\text{Yang 1})}(\lambda_1, \ldots, \lambda_k) < G_k^{(\text{Yang 2})}(\lambda_1, \ldots, \lambda_k) \\ &< G_k^{(\text{HP})}(\lambda_1, \ldots, \lambda_k) < G_k^{(\text{PPW})}(\lambda_1, \ldots, \lambda_k) \end{aligned} \tag{3.12}$$

for $k \geq 2$. Note that

$$G_k^{(\text{Yang 2})}(\lambda_1, \ldots, \lambda_k) = \left(1 + \frac{4}{n}\right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i \tag{3.13}$$

and

$$G_k^{(\text{PPW})}(\lambda_1, \ldots, \lambda_k) = \lambda_k + \frac{4}{nk} \sum_{i=1}^{k} \lambda_i, \tag{3.14}$$

but that there do not exist simple expressions for the other two upper bounds in general ($G_k^{(\text{Yang 1})}$ appears, of course, as the right-hand side of (1.9)). Note, too, that for $k = 1$ all four bounds reduce to the identical bound $(1 + (4/n))\,\lambda_1$.

As a final remark, we note that a useful way of comparing the various upper bounds for λ_{k+1} is to assume the asymptotic behavior

$$\lambda_k \sim ck^{\alpha} \text{ as } k \to \infty \tag{3.15}$$

for some positive constants c and α and see what the bound tells us about the possible values of α (that nothing can be learned about c in this way follows from the homogeneity of our bounds in $(\lambda_1, \ldots, \lambda_k)$). Thus we seek to find which powers α are consistent with the given inequality. It turns out that we get upper bounds for α. From the PPW inequality we learn nothing: any $\alpha > 0$ is consistent and thus we find an upper bound for α of ∞. The Hile–Protter bound does not lend itself to this analysis so we pass over it here; in any event, both Yang inequalities are better. Yang's second inequality leads easily to the bound $\alpha \leq \frac{4}{n}$. To see this one need only know that (3.15) implies

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i \sim \frac{ck^{\alpha}}{1+\alpha} \text{ as } k \to \infty. \tag{3.16}$$

Finally, to analyse Yang's first inequality we also need

$$\frac{1}{k}\sum_{i=1}^{k}\lambda_i^2 \sim \frac{(ck^{\alpha})^2}{1+2\alpha} \text{ as } k \to \infty. \tag{3.17}$$

It is then readily found that for the discriminant in (1.9) to be nonnegative asymptotically as $k \to \infty$ we must have $\alpha \leq 2/n$, and that, furthermore, when this condition holds the main inequality holds asymptotically as well. Thus Yang's first inequality gives the better bound $\alpha \leq 2/n$. Since $2/n$ is actually the correct power for the Weyl asymptotics of λ_k we see that Yang's first inequality correctly captures this behavior, whereas none of the weaker inequalities discussed here does. The reader might also consult Yang's discussion [33] of the Weyl asymptotics vis-à-vis the various bounds.

4. Extensions

In this section we extend the results of the previous sections to cover the eigenvalue problem

$$-\Delta u + V(\vec{x})u = \lambda\rho(\vec{x})u \qquad \text{in } \Omega \subset \mathbb{R}^n, \tag{4.1}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{4.2}$$

Here Ω remains a bounded domain in $\mathbb{R}^n$, $V(\vec{x})$ represents a nonnegative potential, and $\rho(\vec{x})$ is a positive function (or density) continuous on $\bar{\Omega}$. This problem has eigenvalues and eigenfunctions as above, which we shall continue to denote by $\{\lambda_i\}_{i=1}^{\infty}$ and $\{u_i\}_{i=1}^{\infty}$ where $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \nearrow \infty$ with corresponding real orthonormal basis eigenfunctions u_i, $i = 1, 2, 3, \ldots$. Orthogonality now is with respect to the weighted inner product introduced below; in particular, we have $\int_{\Omega} \rho u_i u_j = \delta_{ij}$.

We shall prove the Yang inequalities ($\rho_{\max}$ and $\rho_{\min}$ denote the obvious quantities)

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\left(\lambda_{k+1}-\left(1+\frac{4}{n}\frac{\rho_{\max}}{\rho_{\min}}\right)\lambda_i\right) \leq 0 \tag{4.3}$$

and

$$\lambda_{k+1} < \left(1+\frac{4}{n}\frac{\rho_{\max}}{\rho_{\min}}\right)\frac{1}{k}\sum_{i=1}^{k}\lambda_i \tag{4.4}$$

and then comment on some of their implications and interrelationships as done in previous sections for their predecessors. The upshot is that all our previous bounds hold if we replace all occurrences of $4/n$ by $(4/n)(\rho_{\max}/\rho_{\min})$.

While we shall give our proof for the general problem (4.1)–(4.2), it should be noted that two separate special cases are of greatest interest. These are the Schrödinger operator problem, where $\rho \equiv 1$, and the 'vibrating membrane' of variable density $\rho(\vec{x})$, where

$V \equiv 0$. After giving the proof we shall return to these two special cases and make some further comments about them.

To proceed with our proof we set $H = -\Delta + V(\vec{x})$ and note that the appropriate inner product is given by $\langle f, g\rangle = \int_\Omega \rho f g$, i.e., we work in the real Hilbert space $L^2(\Omega, \rho)$ where ρ appears as a weight function. (If we were to work primarily in inner product notation we might well want to view our primary operator as $\tilde{H} = (1/\rho(\vec{x}))H$, but we choose not to do this here as we will mainly work using integral notation.) In particular, the Rayleigh–Ritz inequality now reads

$$\lambda_{k+1} \le \frac{\int_\Omega \varphi(H\varphi)}{\int_\Omega \rho\varphi^2} \tag{4.5}$$

where the real-valued trial function φ must be nontrivial and orthogonal to $u_1, u_2, \ldots, u_k$ (with respect to our weighted inner product).

Proceeding as before we take (with x representing a single cartesian variable)

$$\varphi_i = xu_i - \sum_{j=1}^{k} a_{ij} u_j \tag{4.6}$$

with $a_{ij} = \int_\Omega \rho x u_i u_j = a_{ij}$ for $1 \le i, j \le k$. This yields $\varphi_i \perp u_j$ for all $1 \le i, j \le k$ and hence

$$\int_\Omega \rho\varphi_i^2 = \int_\Omega \rho x u_i \varphi_i = \int_\Omega \rho x^2 u_i^2 - \sum_{j=1}^{k} a_{ij}^2. \tag{4.7}$$

Furthermore

$$\begin{aligned} H\varphi_i &= xHu_i - 2u_{ix} - \sum_{j=1}^{k} a_{ij} H u_j \\ &= \lambda_i \rho x u_i - 2u_{ix} - \sum_{j=1}^{k} a_{ij}\lambda_j \rho u_j \end{aligned} \tag{4.8}$$

and if we now multiply by φ_i and integrate over Ω we find

$$\begin{aligned} \int_\Omega \varphi_i(H\varphi_i) &= \lambda_i \int_\Omega \rho x u_i \varphi_i - 2\int_\Omega \varphi_i u_{ix} \\ &= \lambda_i \int_\Omega \rho\varphi_i^2 - 2\int_\Omega \varphi_i u_{ix} \end{aligned} \tag{4.9}$$

by virtue of the orthogonality $\varphi_i \perp u_j$ and (4.7). Using the Rayleigh–Ritz inequality we obtain

$$\lambda_{k+1} - \lambda_i \le \frac{-2\int_\Omega \varphi_i u_{ix}}{\int_\Omega \rho\varphi_i^2}. \tag{4.10}$$

We must now try to simplify $-2\int_\Omega \varphi_i u_{ix}$ and also apply the Cauchy–Schwarz inequality much as before. However we have the additional complication that, at least at some point, we must introduce a factor of ρ into one of the integrals that results (so as to be able to cancel with the denominator $\int_\Omega \rho\varphi_i^2$). We begin by computing

$$\begin{aligned} 0 \le -2\int_\Omega \varphi_i u_{ix} &= -2\int_\Omega \left[x u_i - \sum_{j=1}^k a_{ij} u_j \right] u_{ix} \\ &= -\int_\Omega x(u_i^2)_x + 2\sum_{j=1}^k a_{ij} \int_\Omega u_{ix} u_j \\ &= \int_\Omega u_i^2 + 2\sum_{j=1}^k a_{ij} b_{ij} \end{aligned} \tag{4.11}$$

after integrating by parts and introducing

$$b_{ij} \equiv \int_\Omega u_{ix} u_j = -b_{ji}. \tag{4.12}$$

Next we evaluate b_{ij} much as we did in § 2:

$$\begin{aligned} 2b_{ij} &= 2\int_\Omega u_{ix} u_j \\ &= \int_\Omega [\Delta(x u_i) - x\Delta u_i] u_j \\ &= -\int_\Omega x u_i(-\Delta u_j) + \int_\Omega x(-\Delta u_i) u_j \\ &= -\int_\Omega x u_i (H u_j) + \int_\Omega x(H u_i) u_j \\ &= (\lambda_i - \lambda_j)\int_\Omega \rho x u_i u_j \\ &= (\lambda_i - \lambda_j) a_{ij}. \end{aligned} \tag{4.13}$$

Thus

$$-2\int_\Omega \varphi_i u_{ix} = \int_\Omega u_i^2 + \sum_{j=1}^k (\lambda_i - \lambda_j) a_{ij}^2. \tag{4.14}$$

We remark that here $\int_\Omega u_i^2$ does not reduce to 1, since our normalization is now $\int_\Omega \rho u_i^2 = 1$.

We now turn to the use of the Cauchy–Schwarz inequality. Unlike in § 2, this time we shall go straight for the best result, i.e., the analog (4.3) of Yang's first inequality, and only afterwards survey the weaker derivative inequalities (the analogs of the PPW, HP, and Yang 2 inequalities in this setting) in their strong forms (that is, as strict inequalities). Thus we attempt to 'make optimal use of the Cauchy–Schwarz inequality' from the beginning. To this end we recall that φ_i is orthogonal to u_j (for $1 \le i, j \le k$) with respect to the L^2 inner product weighted by ρ, and hence to be able to subtract away the components of u_{ix} along the u_j's for $1 \le j \le k$ we need to introduce the weight ρ appropriately as follows:

$$-2\int_\Omega \varphi_i u_{ix} = -2\int_\Omega \rho^{\frac{1}{2}} \varphi_i \left[\rho^{-\frac{1}{2}} u_{ix} - \sum_{j=1}^k b_{ij} \rho^{\frac{1}{2}} u_j \right]. \tag{4.15}$$

We apply the Cauchy–Schwarz inequality to the integral based on the factored form of the integrand as shown on the right above. We have

$$\begin{aligned}\left(-2\int_\Omega \varphi_i u_{ix}\right)^2 &\le 4\left(\int_\Omega \rho\varphi_i^2\right)\left(\int_\Omega \left[\rho^{-\frac{1}{2}}u_{ix} - \sum_{j=1}^k b_{ij}\rho^{\frac{1}{2}}u_j\right]^2\right)\\ &= 4\left(\int_\Omega \rho\varphi_i^2\right)\left[\int_\Omega \rho^{-1}u_{ix}^2 - 2\sum_{j=1}^k b_{ij}\int_\Omega u_{ix}u_j + \sum_{j=1}^k b_{ij}^2\right]\\ &= 4\left(\int_\Omega \rho\varphi_i^2\right)\left[\int_\Omega \rho^{-1}u_{ix}^2 - \sum_{j=1}^k b_{ij}^2\right], \end{aligned} \tag{4.16}$$

where in passing to the second line we used the fact that $\int_\Omega \rho u_i u_j = \delta_{ij}$ for $i, j = 1, 2, \ldots$. Note that while our coefficients b_{ij} in (4.15) could be replaced by arbitrary coefficients, (4.16) shows that choosing the b_{ij}'s is the best we could do. That is, if we put d_{ij} in place of b_{ij} in (4.15), proceed as above to the second line of (4.16), and then choose the d_{ij}'s to minimize the right-hand side, we find $d_{ij} = b_{ij}$. Thus our coefficients b_{ij} are optimal in this sense. But we should also remark that the b_{ij}'s are the components of u_{ix} along the u_j's with respect to an *unweighted* L^2 inner product. This may appear surprising, but is natural in this context (just as it is natural that no ρ appear in the integral defining b_{ij}, in the numerator of the Rayleigh quotient (4.5), or in the computation in (4.11)).

Now since $-2\int_\Omega \varphi_i u_{ix} \ge 0$ (and with our usual conventions from § 2 on how to interpret our inequalities if $-2\int_\Omega \varphi_i u_{ix} = 0$), we find from the Rayleigh–Ritz inequality (4.10) and our subsequent simplifications (4.13), (4.14), and (4.16)

$$\lambda_{k+1} - \lambda_i \le \frac{-2\int_\Omega \varphi_i u_{ix}}{\int_\Omega \rho\varphi_i^2} \le \frac{4\int_\Omega \rho^{-1}u_{ix}^2 - \sum_{j=1}^k (\lambda_i - \lambda_j)^2 a_{ij}^2}{\int_\Omega u_i^2 + \sum_{j=1}^k (\lambda_i - \lambda_j)a_{ij}^2} \tag{4.17}$$

or

$$(\lambda_{k+1} - \lambda_i)\left[\int_\Omega u_i^2 + \sum_{j=1}^k (\lambda_i - \lambda_j)a_{ij}^2\right] \le 4\int_\Omega \rho^{-1}u_{ix}^2 - \sum_{j=1}^k (\lambda_i - \lambda_j)^2 a_{ij}^2. \tag{4.18}$$

By collecting all the terms in the a_{ij}'s on the left we arrive at

$$(\lambda_{k+1} - \lambda_i)\int_\Omega u_i^2 + \sum_{j=1}^k (\lambda_{k+1} - \lambda_j)(\lambda_i - \lambda_j)a_{ij}^2 \le 4\int_\Omega \rho^{-1}u_{ix}^2. \tag{4.19}$$

We now proceed exactly as in § 2 by promoting x to x_ℓ and summing on ℓ from 1 to n to get

$$n(\lambda_{k+1} - \lambda_i)\int_\Omega u_i^2 + \sum_{j=1}^k (\lambda_{k+1} - \lambda_j)(\lambda_i - \lambda_j)A_{ij} \le 4\int_\Omega \rho^{-1}|\nabla u_i|^2, \tag{4.20}$$

where we define

$$A_{ij} \equiv \sum_{\ell=1}^n (a_{ij}^{(\ell)})^2 = A_{ji} \ge 0 \tag{4.21}$$

with $a_{ij}^{(\ell)} \equiv \int_\Omega \rho x_\ell u_i u_j$. Then it follows as before that multiplying through by $\lambda_{k+1} - \lambda_i$ and summing on i from 1 to k will free us of the terms in the A_{ij}'s, leaving

$$n \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \int_\Omega u_i^2 \leq 4 \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \int_\Omega \rho^{-1} |\nabla u_i|^2. \tag{4.22}$$

At this point we are almost done. It would have been nice if ρ had appeared not as ρ^{-1} on the right but as ρ on the left, but this situation is easily remedied at the expense of one factor each of $\rho_{\max} \equiv \max_{\vec{x} \in \bar{\Omega}} \rho(\vec{x})$ and $\rho_{\min} \equiv \min_{\vec{x} \in \bar{\Omega}} \rho(\vec{x})$. Thus

$$n \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \int_\Omega \frac{\rho}{\rho_{\max}} u_i^2 \leq 4 \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \rho_{\min}^{-1} \int_\Omega |\nabla u_i|^2 \tag{4.23}$$

or, since $\int_\Omega |\nabla u_i|^2 \leq \lambda_i$ (recall that $V(\vec{x}) \geq 0$ on Ω) and $\int_\Omega \rho u_i^2 = 1$,

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_{k+1} - \left(1 + \frac{4}{n} \frac{\rho_{\max}}{\rho_{\min}} \right) \lambda_i \right) \leq 0, \tag{4.24}$$

which is our analog of Yang's first inequality in this more general setting. Thus we have proved the following theorem.

Theorem 4.1. *Consider the eigenvalue problem* (4.1)–(4.2) *where Ω is a bounded domain in $\mathbb{R}^n$, V is a nonnegative potential in $L^\infty(\Omega)$, and ρ is a weight function which is positive and continuous on $\bar{\Omega}$. Then for each $k = 1, 2, \ldots$ the eigenvalues satisfy the inequality* (4.24). *In particular, λ_{k+1} is less than or equal to the larger root of the quadratic appearing as the left-hand side of* (4.24) *or explicitly*

$$\lambda_{k+1} \leq \left(1 + \frac{2}{n} \frac{\rho_{\max}}{\rho_{\min}} \right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i + \left[\frac{4}{n^2} \frac{\rho_{\max}^2}{\rho_{\min}^2} \left(\frac{1}{k} \sum_{i=1}^{k} \lambda_i \right)^2 - \left(1 + \frac{4}{n} \frac{\rho_{\max}}{\rho_{\min}} \right) \frac{1}{k} \sum_{i=1}^{k} (\lambda_i - \langle \lambda \rangle_k)^2 \right]^{\frac{1}{2}} \tag{4.25}$$

(here $\langle \lambda \rangle_k$ denotes the average of the first k eigenvalues, $(1/k) \sum_{i=1}^{k} \lambda_i$*) and furthermore λ_{k+1} also satisfies the simpler inequality* (*the analog of Yang's second inequality in this setting*)

$$\lambda_{k+1} \leq \left(1 + \frac{4}{n} \frac{\rho_{\max}}{\rho_{\min}} \right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i. \tag{4.26}$$

Remarks.

(1) It is clear from our transition from (4.22) to (4.23) and (4.24) that all our inequalities will be strict unless $\rho \equiv$ const. and $V \equiv 0$. If $\rho \equiv$ const. and $V \equiv 0$, then we are back in the case dealt with in the two previous sections and we know that even in this case every inequality except perhaps (4.24) and (4.25) can be made strict as well.

(2) Clearly $\rho_{\max}$ can be replaced by any upper bound for ρ and $\rho_{\min}$ can be replaced by any positive lower bound for ρ and our inequalities all continue to hold. In addition, as our proof shows, ρ continuous on $\bar{\Omega}$ is a stronger assumption than we really need; it would be enough for ρ to have positive upper and lower bounds on Ω (in which case these would replace $\rho_{\max}$ and $\rho_{\min}$).

(3) By mimicking our earlier arguments we can also obtain the corresponding PPW and HP inequalities for problem (4.1)–(4.2). In their strong forms these are simply

$$\lambda_{k+1} - \lambda_k < \frac{4}{nk} \frac{\rho_{\max}}{\rho_{\min}} \sum_{i=1}^{k} \lambda_i \quad \text{for } k = 1, 2, \dots, \tag{4.27}$$

(the PPW analog) and

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} > \frac{nk}{4} \frac{\rho_{\min}}{\rho_{\max}} \quad \text{for } k = 1, 2, \dots \tag{4.28}$$

(the HP analog).

(4) Theorem 4.1 also lends itself to finding PPW, HP, and Yang type bounds for domains in the constant curvature spaces $\mathbb{S}^2$ and $\mathbb{H}^2$ since when $n = 2$ the eigenvalue problem for the Laplacian in any metric conformal to the Euclidean metric is equivalent to an inhomogeneous membrane problem, $-\Delta u = \lambda \rho(\vec{x}) u$, in Euclidean space. In such a setting $\rho(\vec{x}) = \sqrt{g}$ where g is the determinant of the metric tensor (g_{ij}) (and hence $\rho(\vec{x})\, d\vec{x}$ is the Riemannian volume element $\sqrt{g}\, d\vec{x}$).

In the usual model of $\mathbb{S}^2$ as $\{\vec{x} \in \mathbb{R}^3 \mid |\vec{x}| = 1\}$ we have

$$\rho_{\mathbb{S}^2}(\vec{x}) = (1 + \cos\theta)^2,$$

where θ represents the polar angle (angle from the north pole). Similarly, in the Poincaré disk model of $\mathbb{H}^2$ we have

$$\rho_{\mathbb{H}^2}(\vec{x}) = \frac{4}{(1 - |\vec{x}|^2)^2},$$

where $\{\vec{x} \in \mathbb{R}^2 \mid |\vec{x}| < 1\}$ is the Poincaré disk.

We can thus obtain PPW, HP, and Yang type bounds for the eigenvalues of $-\Delta$ on $\Omega \subset \mathbb{S}^2$ or $\mathbb{H}^2$ with Dirichlet boundary conditions which are of the Euclidean form except for additional factors of $\rho_{\max}/\rho_{\min}$ as given in Theorem 4.1 and in Remark 3 above. For example, for a domain $\Omega \subset \mathbb{S}^2$ having geodesic radius Θ (this is the radius of a circumscribing circle; it is convenient to choose the north pole to be the center of this circle) we find from the $k = 1$ case that

$$\frac{\lambda_2}{\lambda_1} \le 1 + 2\left(\frac{2}{1 + \cos\Theta}\right)^2. \tag{4.29}$$

Note that this bound blows up as the circumradius Θ goes to π (which is equivalent to the inradius of the complement of Ω going to 0). This must, in fact, happen for any bound on λ_2/λ_1 for $\Omega \subset \mathbb{S}^2$ since it is known, for example, that $\lambda_1 \to 0^+$ for a geodesic ball as the ball approaches the full sphere (see ([10], pp. 50–54)).

(5) For the case of a pure Schrödinger operator ($\rho \equiv 1$), the inequalities of this section reduce to those of the earlier sections, i.e., to the inequalites that we derived for the eigenvalues of the Laplacian on a domain in Euclidean space. Moreover, if the potential V is bounded below we can always translate V and the eigenvalues so that all our inequalities apply, so long as we apply them to the translated eigenvalues. In fact, to get the sharpest inequalities we should always translate so that $\inf V = 0$. Even if the potential is not bounded below, it may still be possible to obtain eigenvalue bounds from a point of view similar to that espoused above. The earliest work in this direction is contained in Allegretto's paper [1].

It might be noted that in returning to the eigenvalues λ_i by passing from (4.23) to (4.24) we replaced $\int_\Omega |\nabla u_i|^2$ by its upper bound λ_i. It is quite possible that this leads to a relatively weak bound since in doing this we give up a term in the potential V which could be significant. One way to avoid giving up so much (at the expense of having to keep track of additional quantities) is to define these 'kinetic energy' terms via $\tau_i \equiv \int_\Omega |\nabla u_i|^2$ and just leave the τ_i's in the inequalities. For certain potentials we may be able to prove that τ_i is less than or equal to some fixed fraction of the 'total energy' λ_i, which then would allow us to return to the λ_i's without giving up as much as we would by just using $\int_\Omega |\nabla u_i|^2 \leq \lambda_i$. These ideas occur in physics as the subject of 'virial theory' and have been explored in the present context by Harrell and Stubbe [17].

In addition to the works on universal eigenvalue inequalities cited earlier, there are a number of works dealing strictly with the ratio λ_2/λ_1. These include [3] and [35]. In particular, we remark that [3] proves that the best upper bound on λ_2/λ_1 in the Euclidean case is given by the value of λ_2/λ_1 for an n-ball, which can be given explicitly as a certain ratio of squares of zeros of Bessel functions (this result is known as the *Payne–Pólya–Weinberger conjecture*; see [26,27] and also [3,5]).

5. Yang's bounds for minimal hypersurfaces and domains in spheres

In the 1995 version of his preprint, Yang [33] also applies his methods to obtain results for

(A) a compact minimal hypersurface M in $\mathbb{S}^{n+1}$ (Theorem 3 on p. 3), and
(B) a domain $\Omega \subset \mathbb{S}^n$ with Dirichlet boundary conditions imposed on $\partial\Omega$ (Theorem 4 on p. 14).

With only slight modification, our simplified approach to Yang's inequalities also handles these cases. In particular, our overall strategy of making optimal use of the Cauchy–Schwarz inequality (within the context of our problem) and arranging to eliminate uncontrolled terms in our preliminary inequality by means of antisymmetry applies without change. Moreover, if we want to find the 'HP-analog' of our inequality, we have only to drop the terms in b_{ij}^2 coming from our optimal use of the Cauchy–Schwarz inequality. The 'PPW-analogs' then follow at a glance as well. Since the methods are so close to those used above, we forgo the details (see [33], for example, for the correct definition of the b_{ij}'s in this context (p. 10)) and skip directly to the results.

A. *M a compact minimal hypersurface in* $\mathbb{S}^{n+1}$

In this case since 0 is an eigenvalue of $-\Delta$ on M, we start the spectrum with $\lambda_0 = 0$. Proceeding with our standard approach, we arrive at

$$n(\lambda_{k+1} - \lambda_i) + \sum_{j=0}^{k} (\lambda_{k+1} - \lambda_j)(\lambda_i - \lambda_j) A_{ij} \le 4\lambda_i + n^2 \tag{5.1}$$

as our analog of (2.25). For future use, we note that if we had not made optimal use of the Cauchy–Schwarz inequality, the factor $(\lambda_{k+1} - \lambda_j)$ in the sum in (5.1) would be replaced by $(\lambda_{k+1} - \lambda_i)$ and (5.1) would become the analog of (2.18).

From (5.1) we multiply in $(\lambda_{k+1} - \lambda_i)$ and sum on i from 0 to k to get Yang's inequality (49):

$$n \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)^2 \le \sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i)(4\lambda_i + n^2) \tag{5.2}$$

or

$$\sum_{i=0}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_{k+1} - \left(1 + \frac{4}{n} \right) \lambda_i - n \right) \le 0. \tag{5.3}$$

From (5.3), which is a quadratic inequality, we can derive an explicit upper bound for λ_{k+1} much as before:

$$\lambda_{k+1} \le \left(1 + \frac{2}{n} \right) \frac{1}{k+1} \sum_{i=0}^{k} \lambda_i + \frac{n}{2} + \left[\left(\frac{2}{n} \frac{1}{k+1} \sum_{i=0}^{k} \lambda_i + \frac{n}{2} \right)^2 - \left(1 + \frac{4}{n} \right) \frac{1}{k+1} \sum_{j=0}^{k} \left(\lambda_j - \frac{1}{k+1} \sum_{i=0}^{k} \lambda_i \right)^2 \right]^{\frac{1}{2}}. \tag{5.4}$$

This is Yang's inequality (10), the analog of Yang 1 from earlier in this paper. From (5.4) the analog of Yang 2 now follows easily:

$$\lambda_{k+1} \le \left(1 + \frac{4}{n} \right) \frac{1}{k+1} \sum_{i=0}^{k} \lambda_i + n. \tag{5.5}$$

In addition, if we go back to (5.1) and replace $(\lambda_{k+1} - \lambda_j)$ by $(\lambda_{k+1} - \lambda_i)$ we can proceed to derive the analogous HP inequality

$$\sum_{i=0}^{k} \frac{4\lambda_i + n^2}{\lambda_{k+1} - \lambda_i} \ge n(k+1). \tag{5.6}$$

And if we replace the denominator here by $\lambda_{k+1} - \lambda_k$ we obtain the PPW analog

$$\lambda_{k+1} \le \lambda_k + \frac{1}{n(k+1)} \sum_{i=0}^{k} (4\lambda_i + n^2) = \lambda_k + \frac{4}{n(k+1)} \sum_{i=0}^{k} \lambda_i + n. \tag{5.7}$$

Note also that if we replace the λ_k on the right by $\frac{1}{k+1}\sum_{i=0}^{k}\lambda_i$ then we recover (5.5). Thus (5.5) is stronger than (5.7). Finally we note some further simpler inequalities

$$\lambda_{k+1} \le \left(1+\frac{4}{n}\right)\frac{k}{k+1}\lambda_k + n \le \left(1+\frac{4}{n}\right)\lambda_k + n, \tag{5.8}$$

the last member of which is an analog of a further inequality of Payne, Pólya, and Weinberger [26,27].

Remarks.

(1) An inequality of PPW-type for this problem (i.e., for a compact minimal hypersurface in a sphere) was first derived by Maeda [25] (see the last displayed inequality on his p. 32, before the parameter ω is introduced, and also the theorem that follows). Later Yang and Yau [34] derived a related bound (this bound was incorrect in detail, and was corrected by Leung [22]). However, the Maeda and the (corrected) Yang-Yau bound are not as good as our PPW-type bound (5.7) above (which goes back to Harrell and Michel [15] and Yang [33]). To slightly further complicate matters here, we also remark that Maeda only gave details for the case of a domain with boundary (our Problem B, discussed below) while saying 'The case $\partial M = \emptyset$ can be discussed analogously' and going on to state 'when $\partial M = \emptyset$ the same inequality is true replacing λ_i by μ_i' (we note in this connection that Maeda indexes his μ_i's from $i = 0$). Unfortunately, he seems to have missed the fact that the natural development of Problem A leads to division by $k + 1$ (see, for example, (5.5)–(5.7) above), rather than k (which is how his inequality is stated, except that our k is his n). Thus, while not incorrect, Maeda's (semi-)stated PPW-type bound for Problem A is not quite even that of Yang and Yau as corrected by Leung. And neither bound represents the 'natural' PPW-type bound as developed above.

(2) Leung [22] then derived an HP-type inequality for this problem. There is an optimization in a parameter t that appears in his argument (and in a sense in Yang and Yau's argument, since Leung's argument leads back to Yang and Yau's via the natural reduction) and this makes his bounds more complicated and less easily compared with those above. However, setting $t = 1$ should not do great injustice to his bounds, and lends itself more directly to comparison with our inequalities above. We find (for the PPW-type version of Leung's inequality)

$$\lambda_{k+1} \le \lambda_k + \frac{4}{n(k+1)}\sum_{i=0}^{k}\lambda_i + 2n \tag{5.9}$$

(this is Leung's (3.2) with $t = 1$). A comparison with (5.7) shows the presence of an extra n in this formula. Varying t cannot entirely remove this problem since the $2n$ above comes from $(1+t)n$ and we must always use $t > 0$ (and, indeed, $t \to 0^+$ causes another term in the bound to blow up, so the optimal t turns out to be $t_0 \in (0, 1)$). In fact, it is not hard to see that the corrected Maeda and Yang–Yau inequalities (which involve optimization in t) lie between (5.7) and (5.9).

Furthermore, using the methods employed here, we can fully dispose of Leung's inequality, i.e., we can show that both Yang inequalities ((5.4) and (5.5) above) are better than Leung's best bound (written in terms of our notation above) which he proved for all $k \geq 1$ (throughout the following discussion we take k to be a fixed positive integer):

$$\lambda_{k+1} \leq \sigma \equiv \min_{t>0}[\sigma_t + (1+t)n] \tag{5.10}$$

where σ_t for $0 < t < \infty$ is the root $s > \lambda_k$ of

$$F(s) \equiv \sum_{i=0}^{k} \frac{\lambda_i}{s-\lambda_i} = \frac{n(k+1)t}{(1+t)^2}. \tag{5.11}$$

This root is unique (for each $k \geq 1$) since for $s > \lambda_k$, $F(s)$ decreases monotonically from infinity to 0. Note that this is almost exactly the same function $F(s)$ introduced in (3.1) except that our sum now starts at $i = 0$ (indeed, it is exactly the same function since $\lambda_0 = 0$, but for our purposes here we prefer to view it as above since we shall want to use the convexity of $\lambda/(s-\lambda)$ in λ (x in (3.3)) for a value which is a convex combination of $k+1$ points). In particular, we note that it follows as before that $f(x) = x/(s-x)$ is convex for $x \in (-\infty, s)$ if $s > 0$ (which always holds for the cases under consideration here).

We begin by showing that for all $t > 0$

$$\sigma_t > \left(1 + \frac{4}{n}\right) \frac{1}{k+1} \sum_{j=0}^{k} \lambda_j. \tag{5.12}$$

There are two cases to consider:

Case 1. If

$$\left(1 + \frac{4}{n}\right) \frac{1}{k+1} \sum_{j=0}^{k} \lambda_j \leq \lambda_k$$

then there is really nothing to prove, since by the very definition of σ_t we have

$$\left(1 + \frac{4}{n}\right) \frac{1}{k+1} \sum_{j=0}^{k} \lambda_j \leq \lambda_k < \sigma_t.$$

Case 2. If

$$\left(1 + \frac{4}{n}\right) \frac{1}{k+1} \sum_{j=0}^{k} \lambda_j > \lambda_k,$$

then we can employ our previous argument (see, in particular, (3.5) and (3.6) above) as follows:

$$
\begin{aligned}
&F\left(\left(1+\frac{4}{n}\right)\frac{1}{k+1}\sum_{j=0}^{k}\lambda_j\right)\\
&\quad=\sum_{i=0}^{k}\frac{\lambda_i}{\left(1+\frac{4}{n}\right)\frac{1}{k+1}\sum_{j=0}^{k}\lambda_j-\lambda_i}\\
&\quad>(k+1)\frac{\frac{1}{k+1}\sum_{j=0}^{k}\lambda_j}{\left(1+\frac{4}{n}\right)\frac{1}{k+1}\sum_{j=0}^{k}\lambda_j-\frac{1}{k+1}\sum_{j=0}^{k}\lambda_j}\\
&\quad=\frac{n(k+1)}{4}\\
&\quad\geq\frac{n(k+1)t}{(1+t)^2}\qquad\text{for all } t>0 \text{ (with equality iff } t=1)\,. \qquad (5.13)
\end{aligned}
$$

Since $F(s)$ is strictly decreasing on (λ_k, ∞), (5.12) now follows from the definition of σ_t (the strict inequality in (5.13) follows from the facts that $f(x) = x/(s - x)$ is strictly convex in x and that our sum involves at least two distinct points since $k \geq 1$; one could also arrive at strict inequality by reducing F to a sum from 1 to k using $\lambda_0 = 0$ and then using convexity to bound the average of those k terms by a single term). Thus

$$\left(1+\frac{4}{n}\right)\frac{1}{k+1}\sum_{j=0}^{k}\lambda_j < \sigma_t \quad \text{for all } t > 0$$

and it follows from our version of Yang's second inequality in the present context (inequality (5.5)) that

$$\lambda_{k+1} \leq \left(1+\frac{4}{n}\right)\frac{1}{k+1}\sum_{j=0}^{k}\lambda_j + n < \sigma_t + n < \sigma_t + (1+t)n \qquad (5.14)$$

for all $t > 0$. Since it is also clear that $\sigma \equiv \inf_{t>0}[\sigma_t + (1+t)n]$ is attained at some $t_0 \in (0, 1)$ (as noted already by Leung [22], based on the fact that σ_t is decreasing on $(0, 1)$, increasing on $(1, \infty)$, and goes to infinity as $t \to 0^+$), it follows that Leung's best bound, $\lambda_{k+1} \leq \sigma$, is always strictly weaker than the second (and weaker) Yang inequality (5.5). Thus it is also strictly weaker than Yang's first inequality (5.4) as asserted above. Since, as already mentioned, the earlier bound of Yang and Yau is a consequence of Leung's inequality (5.10) above, we can conclude that the Yang inequalities (5.4) and (5.5) supersede all previous inequalities in this vein (see our remarks below and at the end of this section for the full justification of this comment).

(3) Harrell and Michel [15] did derive the 'correct' HP-type inequality for this problem. That is, they obtained (5.6).

(4) All the authors discussed in Remarks 1–3 above actually treated the case of an m-dimensional manifold M^m which is a minimally immersed submanifold of a sphere $\mathbb{S}^{n+1}$ with $1 \leq m \leq n$. The bounds they then derived were as above but with m replacing n. Yang's bounds above are for the case $m = n$ but all extend with no problem to general dimension m, $1 \leq m \leq n$ (indeed, they even apply when $m = n + 1$, i.e., $M = \mathbb{S}^{n+1}$, as can be seen from Part B below).

(5) Presumably the inspiration for all the work discussed above on minimally immersed submanifolds was Cheng's paper [12], which dealt with compact domains in a minimal hypersurface of $\mathbb{R}^{n+1}$. The key for Cheng is that the cartesian coordinate functions for $\mathbb{R}^{n+1}$ are harmonic on M (i.e., $\Delta_M x_\ell = 0$), while for the other authors it is that the cartesian coordinate functions for $\mathbb{R}^{n+2}$ are eigenfunctions of $-\Delta_M$ with eigenvalue $m = \dim M$ (i.e., $-\Delta_M x_\ell = m x_\ell$). Necessary facts about minimal submanifolds can be found in Chavel ([10], see pp. 309–314).

B. Ω *a domain in* $\mathbb{S}^n$ *with Dirichlet boundary conditions*

In this case 0 is not an eigenvalue (unless $\Omega = \mathbb{S}^n$, a case we can exclude since its eigenvalues are known explicitly) so we index the eigenvalues by $\{\lambda_i\}_{i=1}^\infty$ with $0 < \lambda_1 < \lambda_2 \le \lambda_3 \le \ldots \nearrow \infty$ (multiplicities included). Actually it turns out that our inequalities do cover the case where $\Omega = \mathbb{S}^n$, but then $\lambda_1 = 0$ (just observe that our derivation also applies to that case). Proceeding much as before we find

$$n(\lambda_{k+1} - \lambda_i) + \sum_{j=1}^{k} (\lambda_{k+1} - \lambda_j)(\lambda_i - \lambda_j) A_{ij} \le 4\lambda_i + n^2 \tag{5.15}$$

as our analog of (2.25) (or (5.1); in fact, it is (5.1) exactly if we change k to $k+1$ and reindex so that our index starts at 0). As above we can pass to

$$n \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)^2 \le \sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)(4\lambda_i + n^2) \tag{5.16}$$

or

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i) \left(\lambda_{k+1} - \left(1 + \frac{4}{n}\right) \lambda_i - n \right) \le 0, \tag{5.17}$$

and thus to

$$\lambda_{k+1} \le \left(1 + \frac{2}{n}\right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i + \frac{n}{2} + \left[\left(\frac{2}{n} \frac{1}{k} \sum_{i=1}^{k} \lambda_i + \frac{n}{2} \right)^2 - \left(1 + \frac{4}{n}\right) \frac{1}{k} \sum_{j=1}^{k} \left(\lambda_j - \frac{1}{k} \sum_{i=1}^{k} \lambda_i \right)^2 \right]^{\frac{1}{2}} \tag{5.18}$$

which is Yang's inequality (51), the analog of Yang 1 in this setting. The analog of Yang 2 is then

$$\lambda_{k+1} \le \left(1 + \frac{4}{n}\right) \frac{1}{k} \sum_{i=1}^{k} \lambda_i + n. \tag{5.19}$$

For the HP analog we adjust $(\lambda_{k+1} - \lambda_j)$ to $(\lambda_{k+1} - \lambda_i)$ in (5.15) and then divide through by $\lambda_{k+1} - \lambda_i$ and sum on i from 1 to k to obtain

$$\sum_{i=1}^{k} \frac{4\lambda_i + n^2}{\lambda_{k+1} - \lambda_i} \ge nk. \tag{5.20}$$

Much as before this leads on to

$$\lambda_{k+1} \leq \lambda_k + \frac{4}{nk}\sum_{i=1}^{k}\lambda_i + n \tag{5.21}$$

and

$$\lambda_{k+1} \leq \left(1 + \frac{4}{n}\right)\lambda_k + n \tag{5.22}$$

(or in the case of the sphere $\mathbb{S}^n$, $\lambda_{k+1} \leq \left(1 + \frac{4}{n}\right)\frac{k-1}{k}\lambda_k + n \leq \left(1 + \frac{4}{n}\right)\lambda_k + n$).

Remarks.

(1) Yang [33] (1995 version) was the first to derive bounds for the eigenvalues of $-\Delta$ on $\Omega \subset \mathbb{S}^n$ of this form. The presence of the extra n is actually good from a certain point of view, which is that as Ω approaches all of $\mathbb{S}^n$ (consider geodesic balls in $\mathbb{S}^n$, say, to be specific) we expect $\lambda_1 \to 0^+$ since $\lambda_1(\mathbb{S}^n) = 0$ (a proof of this fact for geodesic balls can be found in Chavel ([10], pp. 50–54), for example). In particular, (5.22) for $k = 1$ could not hold without the n (since domain monotonicity forces $\lambda_2 \geq n$). The only way for our 'Euclidean-based' bound (4.29) to handle this eventuality (without the n) is for $\rho_{\max}/\rho_{\min}$ to blow up in this limit, which indeed it does. Thus, inequalities (4.25) and (5.18) (or (4.26) and (5.19), etc.), both of which apply for $\Omega \subset \mathbb{S}^n$ if ρ is identified as in Remark 4 at the end of § 4 (see the expression for $\rho_{\mathbb{S}^2}(\vec{x})$), might be regarded as complementary to some extent (in a rough sense, one should do a better job for small domains, the other for large).

(2) Inequality (5.22) shows, in particular, that

$$\frac{\lambda_2 - n}{\lambda_1} \leq 1 + \frac{4}{n}. \tag{5.23}$$

It can be shown using the methods of [7] (see Theorem 3.1 and Remark 1 on p. 1071 in particular) that for geodesic balls larger than a hemisphere

$$\frac{\lambda_2 - n}{\lambda_1} < 1 + \frac{2}{n},$$

with equality at the hemisphere (the inequality reverses for geodesic balls smaller than a hemisphere). Since as we approach the full sphere $\mathbb{S}^n$, $\lambda_1 \to 0^+$ while $\lambda_2 \to n^+$ (see ([10], pp. 52–53)), the limiting value of $(\lambda_2 - n)/\lambda_1$ might be of some interest to work out (either for geodesic balls, or for arbitrary domains if that limit exists), but is in any case between 0 and $1 + (2/n)$ inclusive.

For both problems considered in this section, we can prove results analogous to our results in § 3 above. In particular there is an argument based on convexity and the function

$$\tilde{F}(s) \equiv \sum_i \frac{4\lambda_i + n^2}{s - \lambda_i} = \sum_i \tilde{f}(\lambda_i)$$

which shows that (here $\langle\lambda\rangle$ denotes $\frac{1}{k+1}\sum_{i=0}^{k}\lambda_i$ or $\frac{1}{k}\sum_{i=1}^{k}\lambda_i$, depending on the problem, and similarly for the summation interval in the definition of $\tilde{F}$ above))

$$\tilde{F}\left(\left(1+\frac{4}{n}\right)\langle\lambda\rangle+n\right)\geq\begin{cases}n(k+1) & \text{for Problem A}\\ nk & \text{for Problem B}\end{cases}$$

and hence that the analogs of Yang's second inequality are stronger than the corresponding Hile–Protter analogs. This shows that in this context again Yang 1⇒Yang 2⇒HP⇒PPW.

Similarly we can argue that the smaller roots of our quadratic Yang inequalities are irrelevant: we simply define

$$\tilde{H}_k(x)\equiv\sum_i(x-\lambda_i)\left(x-\left(1+\frac{4}{n}\right)\lambda_i-n\right)$$

and observe that $\tilde{H}_k(\lambda_k)=\tilde{H}_{k-1}(\lambda_k)\leq 0$ by the quadratic Yang inequality. This implies that $\lambda_k\geq$ [smaller root of $\tilde{H}_k$], and hence that $\lambda_{k+1}\geq\lambda_k$ is always as good as or better than $\lambda_{k+1}\geq$ [smaller root of $\tilde{H}_k$]. Similarly, $\lambda_k\geq$ [smaller root of $\tilde{H}_k$] is uninteresting since this is an implicit bound which is implied by $\tilde{H}_{k-1}(\lambda_k)\leq 0$, i.e., Yang's quadratic inequality at one index lower.

Another useful observation is that fundamentally the Yang inequalities are all the same. We have already noted that the inequalities of Parts A and B above are the same if we just reindex our eigenvalues to start from $i=1$ (or $i=0$) in both cases. However, since the usual conventions dictate different starting indices (largely because in Part A the first eigenvalue is always 0 while in Part B it is not, unless $\Omega=\mathbb{S}^n$) we have chosen to present both sets of inequalities, to give them maximum exposure (as did Yang). Finally, we remark that (5.17) is equivalent to (2.27) if we take each eigenvalue λ in (2.27) and replace it by $\lambda+(n^2/4)$. Thus all the inequalities in Part B above could be gotten from the inequalities of §§ 1–3 simply by making this formal substitution. A similar comment applies equally to Part A, once the necessary index shift is taken into account. This demonstrates the identity of our inequalities across problems in a certain wider sense (and can be used to motivate the forms of $\tilde{F}$ and $\tilde{H}$ that we introduced above). Perhaps there is some deeper identity behind this, or perhaps there is a way to identify the sphere problems of Parts A and B with Schrödinger problems in Euclidean space where the potential $V(\vec{x})$ turns out to be bounded below by $-n^2/4$ suggesting the replacement $\lambda\to\lambda+n^2/4$ mentioned above (cf. Remark 5 following Theorem 4.1 above).

6. Concluding remarks

In future works we shall address problems having a more general divergence-form operator. In particular, we shall treat the cases of bounded domains in the constant curvature spaces $\mathbb{H}^n$ and $\mathbb{S}^n$ where the differential operator is the Laplace–Beltrami operator. By an extension of the methods presented above, we obtain relatively strong bounds generalizing our 2-dimensional results mentioned in Remark 4 at the end of § 4 above.

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